COSMOLOGY Homework 10 Fall 2018

Due on December 8 by 14.00.
These are the last exercises.

1. **Inflaton perturbations.** The field equation for the inflaton perturbation is

$$\delta\ddot{\varphi}_{\boldsymbol{k}} + 3H\delta\dot{\varphi}_{\boldsymbol{k}} + \left[\left(\frac{k}{a}\right)^2 + V''(\bar{\varphi})\right]\delta\varphi_{\boldsymbol{k}} = 0\ .$$

Make the approximations $H = \text{const.}$ ($\Rightarrow a = e^{Ht}$) and $V''(\bar{\varphi}) = 0$.

a) Show that

$$\delta\varphi_{\boldsymbol{k}}(t) = C(\boldsymbol{k})\eta^{\frac{3}{2}}H_{\frac{3}{2}}(k\eta)\,,$$

where

$$H_{\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}e^{-ix}\left(1 + \frac{1}{ix}\right)$$

is a Hankel function with $\eta \equiv -H^{-1}e^{-Ht}$, is a solution.

b) Show that η is the conformal time, and that the solution agrees with the one given in the lecture notes,

$$w_k(t) = \left(i + \frac{k}{aH}\right)\exp\left(\frac{ik}{aH}\right)\ .$$

c) What happens to the perturbation (its amplitude, distance scale and oscillation time scale) as inflation proceeds?

2. **Inflaton perturbations as vacuum fluctuations.**

a) Starting from the canonical commutation relation for the quantised perturbation, show that with the correct the normalisation factor, $\delta\varphi_{\boldsymbol{k}}(t)$ of the previous problem is

$$\delta\varphi_{\boldsymbol{k}}(t) = L^{-3/2}\frac{H}{\sqrt{2k^3}}\left(i + \frac{k}{aH}\right)\exp\left(\frac{ik}{aH}\right)\ .$$

b) Show that on time and length scales much smaller than the Hubble scale, $t \ll H^{-1}$, $k \gg aH$, this becomes (up to a slowly varying phase factor) the Minkowski space mode function

$$w_k(t) = (aL)^{-3/2}\frac{1}{\sqrt{2E_k}}e^{-iE_kt}\ .$$

c) What is E_k?

3. **Running of the spectral index.** Assuming slow-roll, calculate $d\varepsilon/d(\ln k)$, $d\eta/d(\ln k)$ and then $dn/d(\ln k)$ in terms of the slow-roll parameters to first non-trivial order.