8. Quantum field theory on the lattice

8.1. Fundamentals

- Quantum field theory (QFT) can be defined using *Feynman's path integral* [R. Feynman, Rev. Mod. Phys. 20, 1948]:

$$Z = \int \left[\prod_x d\phi(x)\right] \exp[iS(\phi)] \tag{1}$$
$$S = \int d^4x \mathcal{L}(\phi, \partial_t\phi)$$

  Here $x = (x_0, x_1, x_2, x_3)$, and $g_{\mu\nu} =$ diag(+,-,-,-)

- Physical observables can be evaluated if we add a source $S \rightarrow S + J_x\phi_x$. For example, 1- and connected 2-point functions are:

$$\langle\phi_x\rangle = \frac{\partial}{i\partial J_x}\bigg|_{J=0} \log Z = \frac{1}{Z}\int [d\phi]\phi_x \exp[iS]$$

$$\langle\phi_x\phi_y\rangle = \frac{\partial}{i\partial J_x}\frac{\partial}{i\partial J_y} \log Z = \frac{1}{Z}\int [d\phi]\phi_x\phi_y \exp[iS] - \langle\phi_x\rangle^2$$

- These can be computed using perturbation theory, if the coupling constants in S are small (see any of the oodles of QFT textbooks).

- *However:* if

  – 4-volume is *finite*, and

  – 4-coordinate x is *discrete* ($x \in aZ^4$, a lattice spacing),

  the integral in (1) has finite dimensionality and can, in principle, be evaluated by brute force (= computers)

  $\mapsto$ gives fully *non-perturbative* results.

- Need to extrapolate: $V \to \infty$, $a \to 0$ in order to recover continuum physics.

- But: - the dimensionality of the integral is (typically) huge
  - $\exp iS$ is complex+unimodular: every configuration $\{\phi\}$ contributes with equal magnitude

  $\mapsto$ extremely slow convergence in numerical computations (useless in practice).

- This is (largely) solved by using *imaginary time formalism* (Euclidean spacetime).

- Imaginary time also admits non-zero temperature T.

Units:

Standard HEP units, where

$$c = \hbar = k_B = 1,$$

and thus

$$[length]^{-1} = [time]^{-1} = [mass] = [temperature] = [energy] = \text{GeV}.$$

Mass m = rest energy mc^2 = (Compton wavelength)$^{-1}$ $mc/\hbar$.

$$(1\,\text{GeV})^{-1} \approx 0.2\,\text{fm}$$

8.2. Path integral in imaginary time

Consider *scalar field theory* in Minkowski spacetime:

$$S = \int d^3x\, dt\, \mathcal{L}(\phi, \partial_t\phi) = \int d^3x\, dt\, \left[\tfrac{1}{2}\partial_\mu\phi\, \partial^\mu\phi - V(\phi)\right]$$

We obtain the (classical) *Hamiltonian* by Legendre transformation:

$$\begin{aligned} H &= \int d^3x\, dt\, \left[\pi\dot{\phi} - \mathcal{L}\right] \\ &= \int d^3x\, dt\, \left[\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\partial_i\phi)^2 + V(\phi)\right] \end{aligned}$$

Here π is canonical momentum for ϕ: $\pi = \delta\mathcal{L}/\delta\dot{\phi}$.

Quantum field theory: consider Hilbert space of states $|\phi\rangle$, $|\pi\rangle$, and *field operators* ($\phi \to \hat{\phi}$, $\pi \to \hat{\pi}$, $H \to \hat{H}$), with the usual properties:

- $\hat{\phi}(\bar{x})|\phi\rangle = \phi(\bar{x})|\phi\rangle$
- $\int [\prod_x d\phi(\bar{x})] |\phi\rangle\langle\phi| = 1 \qquad \int [\prod_x d\pi(\bar{x})/(2\pi)] |\pi\rangle\langle\pi| = 1$
- $[\hat{\phi}(\bar{x}), \hat{\phi}(\bar{x}')] = -i\delta^3(\bar{x} - \bar{x}')$
- $\langle\phi|\pi\rangle = \exp\left[i \int d^3x \pi(\bar{x})\phi(\bar{x})\right]$

Time evolution operator is $U(t) = e^{-i\hat{H}t}$

Feynman showed that the quantum theory defined by $\hat{H}$ and the Hilbert space is equivalent to the path integral (1). We shall now do this in imaginary time.

Let us now consider the quantum system in *imaginary time:*

$$t \to \tau = it\,, \qquad \exp[-i\hat{H}t] \to \exp[-\tau\hat{H}]$$

This makes the spacetime *Euclidean*: $g = (+---) \to (++++)$.
We shall keep the Hamiltonian $\hat{H}$ as-is, and also the Hilbert space.
Let us also discretize space (convenient for us but not necessary at this stage), and use finite volume:

$$\begin{aligned} \bar{x} &= a\bar{n}, \qquad n_i = 0 \ldots N_s \\ \int d^3x &\to a^3 \sum_x \\ \partial_i \phi &\to \frac{1}{a}[\phi(\bar{x} + \bar{e}_i a) - \phi(\bar{x})] \equiv \Delta_i \phi(\bar{x}) \end{aligned}$$

Thus, the Hamiltonian is:

$$\hat{H} = a^3 \sum_x \left[\tfrac{1}{2}[\hat{\pi}(x)]^2 + \tfrac{1}{2}[\Delta_i \hat{\phi}(\bar{x})]^2 + V[\hat{\phi}(\bar{x})] \right]$$

Let us now consider the amplitude $\langle \phi_B | e^{-(\tau_B - \tau_A)\hat{H}} | \phi_A \rangle$:
1) divide time in small intervals: $\tau_B - \tau_A = N_\tau a_\tau$. Obviously

$$\langle \phi_B | e^{-(\tau_B - \tau_A)\hat{H}} | \phi_A \rangle = \langle \phi_B | (e^{-a_\tau \hat{H}})^{N_\tau} | \phi_A \rangle .$$

2) insert 1-operators $\int [\prod_x d\phi_i] |\phi_i\rangle\langle\phi_i|$, $\int [\prod_x d\pi_i] |\pi_i\rangle\langle\pi_i|$

$$\langle \phi_B | e^{-\tau \hat{H}} | \phi_A \rangle =$$
$$\int \left[\prod_{i=2}^{N_\tau} \prod_x d\phi_i(x) \right] \left[\prod_{i=1}^{N_\tau} \prod_x d\pi_i(x) \right] \langle \phi_B | \pi_{N_\tau} \rangle \langle \pi_{N_\tau} | e^{-a_\tau \hat{H}} | \phi_{N_\tau} \rangle$$
$$\times \langle \phi_{N_\tau} | \pi_{N_\tau - 1} \rangle \langle \pi_{N_\tau - 1} | e^{-a_\tau \hat{H}} | \phi_{N_\tau - 1} \rangle \ldots \times \langle \phi_2 | \pi_1 \rangle \langle \pi_1 | e^{-a_\tau \hat{H}} | \phi_A \rangle$$

Thus, we need to calculate

$$\begin{aligned}\langle \pi_i | e^{-a_\tau \hat{H}} | \phi_i \rangle &= \exp\left[-a^3 a_\tau \sum_x \left(\tfrac{1}{2}\pi_i^2 + \tfrac{1}{2}(\Delta_i \phi_i)^2 + V[\phi_i] \right) \right] \\ &\times \langle \pi_i | \phi_i \rangle + O(a_\tau^2)\end{aligned}$$

and

$$\begin{aligned}\langle \phi_{i+1} | \pi_i \rangle \langle \pi_i | \phi_i \rangle &= e^{ia^3 \sum_x \pi_i(\bar{x}) \phi_{i+1}(\bar{x})} e^{-ia^3 \sum_x \pi_i(\bar{x}) \phi_i(\bar{x})} \\ &= \exp[ia^3 a_\tau \sum_x \pi_i(\bar{x}) \Delta_0 \phi_i(\bar{x})] \end{aligned} \qquad (2)$$

where we have defined $\Delta_0 \phi_i = \frac{1}{a_\tau}(\phi_{i+1} - \phi_i)$.

We can now integrate over $\pi_i(\bar{x})$:

$$\int [\prod_x \pi_i(\bar{x})] \exp[a^3 a_\tau \sum_x -\tfrac{1}{2}\pi_i^2 + (i\Delta_0 \phi_i)\pi_i]$$

$$= \left[\frac{2\pi}{a^3 a_\tau} \right]^{N_S^3/2} \times \exp[-a^3 a_\tau \sum_x \tfrac{1}{2}(\Delta_0 \phi_i(\bar{x}))^2]$$

Repeating this for $i = 1 \ldots N_\tau$, and identifying the time coordinate $x_0 = \tau = a_\tau i$, we finally obtain the path integral

$$\langle \phi_B | e^{-\tau \hat{H}} | \phi_A \rangle \propto \int [\prod_x d\phi] e^{-S_E}$$

where S_E is the *Euclidean* (imaginary time) action

$$\begin{aligned} S_E &= a^3 a_\tau \sum_x (\tfrac{1}{2}(\Delta_\mu \phi)^2 + V[\phi]) \\ &\to \int d^4x (\tfrac{1}{2}(\partial_\mu \phi)^2 + V[\phi]) \end{aligned}$$

and we have the boundary conditions $\phi(\tau_A) = \phi_A$, $\phi(\tau_B) = \phi_B$.
The path integral is precisely in the form of a *partition function* for a 4-dimensional *classical* statistical system, with the identification $S_E \leftrightarrow H/(k_B T)$.
For convenience, we make the system *periodic* in time by identifying $\phi_A = \phi_B$ and integrating over ϕ_A.

We can now make a connection between the correlation functions of the "statistical" theory and the Green's functions of the quantum field theory. First, note that we can interpret $T_{\phi_{i+1},\phi_i} = \langle\phi_{i+1}|e^{-a_\tau \hat{H}}|\phi_i\rangle$ as a *transfer matrix*. In terms of T the partition function is

$$Z = \int [d\phi] e^{-S_E} = \text{Tr}\,(T^{N_\tau})$$

Let us label the eigenvalues of T by $\lambda_0, \lambda_1 \ldots$, so that $\lambda_0 > \lambda_1 \geq \ldots$. Note that $\lambda_i = \exp -E_i$, where E_i are eigenvalues of $\hat{H}$. Thus, λ_0 corresponds to the state of lowest energy, *vacuum* $|0\rangle$. If we now take N_τ to be very large (while keeping a_τ constant; i.e. take $\Delta\tau$ to be large),

$$Z = \sum_i \lambda_i^{N_\tau} = \lambda_0^{N_\tau}[1 + O((\lambda_1/\lambda_0)^{N_\tau})]$$

For example, a 2-point function can be written as (let $i - j > 0$)

$$\langle\phi_i\phi_j\rangle = \frac{1}{Z}\int [d\phi]\phi_i\phi_j e^{-S_E} = \frac{1}{Z}\text{Tr}\,(T^{N_\tau - i + j}\hat{\phi}T^{i-j}\hat{\phi}).$$

Taking now $N_\tau \to \infty$, and recalling $a_\tau(i-j) = \tau_i - \tau_j$,

$$\langle\phi_i\phi_j\rangle = \langle 0|\hat{\phi}(T/\lambda_0)^{i-j}\hat{\phi}|0\rangle = \langle 0|\hat{\phi}(\tau_i)\hat{\phi}(\tau_j)|0\rangle,$$

where we have introduced time-dependent operators

$$\hat{\phi}(\tau) = e^{\tau \hat{H}} \hat{\phi} e^{-\tau \hat{H}}.$$

Allowing for both positive and negative time separations of τ_i and τ_j, we can identify

$$\langle \phi(\tau_i)\phi(\tau_j)\rangle = \langle 0|\mathcal{T}[\hat{\phi}(\tau_i)\hat{\phi}(\tau_j)]|0\rangle,$$

where $\mathcal{T}$ is the time ordering operator.

Mass spectrum:

• Green's functions in time τ:

$$\begin{aligned}
\langle 0|\Gamma(\tau)\Gamma^{\dagger}(0)|0\rangle &= \frac{1}{Z}\int [d\phi]\Gamma(\tau)\Gamma^{\dagger}(0)e^{-S} \\
&= \langle 0|e^{\hat{H}\tau}\Gamma(0)e^{-\hat{H}\tau}\Gamma^{\dagger}(0)|0\rangle \\
&= \langle 0|\Gamma(0)\sum_n |E_n\rangle\langle E_n|e^{-\hat{H}\tau}\Gamma^{\dagger}(0)|0\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_n e^{-E_n \tau} |\langle 0|\Gamma(0)|E_n\rangle|^2 \\
&\to e^{-E_0 \tau} |\langle 0|\Gamma(0)|E_0\rangle|^2 \qquad \text{as } \tau \to \infty
\end{aligned}
$$

where $|E_0\rangle$ is the *lowest* energy state with non-zero matrix element $\langle 0|\Gamma(0)|E_0\rangle$.

$\to$ measure masses (E_0) from the exponential fall-off of correlation functions.

8.3. Finite temperature

Connection *Euclidean QFT* $\leftrightarrow$ *classical statistical mechanics* was derived for zero-temperature quantum system. However, this can be readily generalized to finite temperature:
Quantum thermodynamics w. the Gibbs ensemble:

$$Z = \text{Tr}\, e^{-\hat{H}/T} = \int [d\phi]\, \langle \phi | e^{-\hat{H}/T} | \phi \rangle$$

Expression is of the same form as the one which gave us the Euclidean path integral for $T = 0$ theory! The difference here is

1) Finite + fixed "imaginary time" interval $1/T$
2) Periodic boundary condition: $\phi(1/T) = \phi(0)$.

Repeating the previous derivation, the partition function becomes

$$Z(T) = \int [d\phi]\, e^{-S_E} = \int [d\phi]\, \exp[-\int_0^{1/T} d\tau \int d^3x \mathcal{L}_{\mathcal{E}}]$$

Thus, a connection between:
– Quantum statistics in 3d: $Z = \text{Tr}\, e^{-\hat{H}/T}$

– Classical statistics in 4d: $Z = \int [d\phi] e^{-S}$

Euclidean P.I. is a very common tool for finite T field theory analysis [J. Kapusta, *Finite Temperature Field Theory*, Cambridge University Press]

8.4. Some terminology:

In numerical work, lattice is a finite box with finite lattice spacing a. In order to obtain continuum results, we should take 2 limits:

- $V \to \infty$ *thermodynamic limit*
- $a \to 0$ *continuum limit*

Both have to be controlled – expensive!

1) Perform simulations with fixed a, various V. Extrapolate $V \to \infty$.
2) Repeat 1) using different a's. Extrapolate $a \to 0$.
3) [In QCD, one often has to extrapolate $m_q \to m_{q,\text{phys.}}$.]

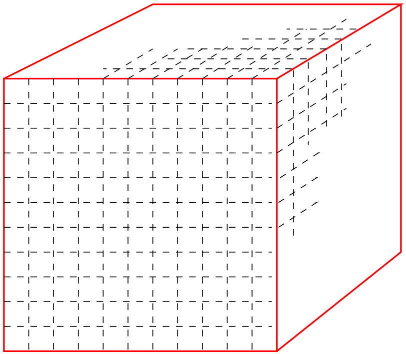
$T = 0$:

1) $V \rightarrow \infty$:

$N_\tau, N_s \rightarrow \infty$, a constant.

2) continuum:

$a \rightarrow 0$.



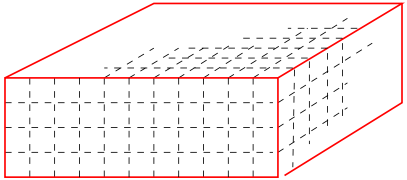
$T > 0$:

1) $V \rightarrow \infty$:

$N_s \rightarrow \infty$, N_τ, a constant.

2) continuum:

$a \rightarrow 0$, $\frac{1}{T} = aN_\tau$ constant.



8.5. Scalar field

Free scalar field on a finite d-dimensional lattice with periodic boundary conditions:

$$x_\mu = a n_\mu, \quad n_\mu \in Z$$

Action:

$$S = \sum_x a^d \left[\tfrac{1}{2} \sum_\mu \frac{1}{a^2} (\phi_{x+\mu} - \phi_x)^2 + \frac{1}{2} m^2 \phi^2 \right] = a^d \left[\tfrac{1}{2} \phi_x \Box_{x,y} \phi_y + \tfrac{1}{2} m^2 \phi_x^2 \right]$$

(implicit sum over x, y), and the lattice d'Alembert operator is

$$\Box_{x,y} \phi_y = -\Delta^2 \phi = \sum_\mu \frac{2\phi_x - \phi_{x+\hat{\mu}} - \phi_{x-\hat{\mu}}}{a^2}$$

The action is of form $S = \frac{1}{2} \phi_x M_{x,y} \phi_y$, and

$$Z = \int [d\phi] e^{-S} = (\mathrm{Det}\, M/2\pi)^{-1/2}$$

Fourier transforms:

$$\tilde{f}(k) = \sum_x a^d e^{-ikx} f(x)$$

Since $x = an$, $\tilde{f}(k + 2\pi n) = \tilde{f}(k)$, and we restrict k to *Brillouin zone*: $-\pi/2 < k_\mu \leq \pi/2$

Inverse transform:

$$f(x) = \int_{-\pi/a}^{\pi/a} \frac{d^d k}{(2\pi)^d} e^{ikx} \tilde{f}(k)$$

Note: often it is convenient to use dimensionless natural lattice units $x_\mu \in Z$, $-\pi < k_\mu \leq \pi$.

Lattice propagator:

The lattice propagator $G(x,y)$ is defined to be the inverse of operator $a^{-d}M = (\Box + m^2)$:

$$\sum_y a^d(\Box_{x,y} + m^2\delta_{x,y})G(y,z) = \delta_{x,z}$$

Take Fourier transform ($G(x,y) = G(x-y)$):

$$\left[\sum_\mu \frac{2}{a^2}(1 - \cos k_\mu a) + m^2\right] \tilde{G}(k) = 1$$

which gives the lattice propagator

$$\tilde{G}(k) = \frac{1}{\hat{k}^2 + m^2}, \quad \text{where} \quad \hat{k}_\mu = \frac{4}{a^2}\sin^2\frac{k_\mu a}{2}.$$

Continuum limit: when $a \to 0$, $\tilde{G}(k) = 1/(k^2 + m^2) + O(a^2)$.

Generating function for Green's functions:

$$S \to S(J) = \sum_x a^d \left[\tfrac{1}{2}\phi_x(\Box + m^2)\phi_x - J_x\phi_x\right]$$

Now

$$Z(J) = \int [d\phi] e^{-S(J)} = Z(0)\exp[\sum_{x,y} a^{2d}\tfrac{1}{2} J_x G(x,y) J_y]$$

N-point functions

$$\langle \phi_x \dots \phi_y \rangle = Z(0)^{-1} \frac{\delta}{\delta J_x} \cdots \frac{\delta}{\delta J_y} Z(J)\Big|_{J=0}$$

Interactions: just modify (for example)

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \frac{1}{4!}\lambda\phi^4$$

8.6. Gauge fields

• Gauge field lagrangian in (Euclidean) continuum:

$$\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} = \frac{1}{2}\text{Tr}\,(F_{\mu\nu}F_{\mu\nu})$$

• Field tensor

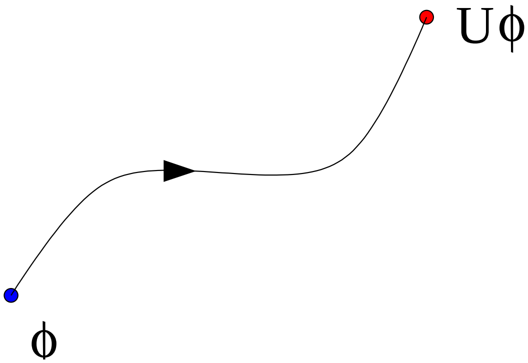
$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

where $A_\mu = A^a_\mu \lambda^a$, and λ^a are the group generators. We shall consider only unitary groups U(1) [QED] and SU(N) [QCD, Electroweak]. For SU(N), the generators are orthonormalized

$$\text{Tr}\,\lambda^a\lambda^b = \frac{1}{2}\delta^{ab}.$$

• Put gauge potential A_μ directly on a lattice? Difficult to maintain gauge invariance.

• Good starting point for gauge field on a lattice is to consider consider gauge fields acting in a *parallel transport:* As field $\phi \in$ SU(N) is "parallel transported" along a path p, parametrized by $x_\mu(s)$, $s \in [0,1]$, gauge fields rotate it: $\phi \to U(s)\phi$



• Define $U(s)$ differentially via

$$\frac{dU(s)}{ds} = \frac{dx_\mu}{ds} igA_\mu U(s),$$

which can be formally solved:

$$U(s) = P\exp\left[ig\int_p ds\frac{dx_\mu}{ds}A_\mu\right]$$

• Here $P =$ "path ordering": in the power series expansion of the exponential one always takes $A(x)$ in the order they are encountered along the path.

• For U(1) (or any other Abelian group) path ordering is irrelevant.

• Alternatively, if we divide the path into N finite intervals of length Δs, and $x^n = x(n\Delta s)$, $n = 0 \ldots N-1$:

$$U(s) \approx P \exp\left[ig \sum_n \Delta s \frac{dx_\mu^n}{ds} A_\mu(x^n)\right] = \prod_i \exp\left[ig\Delta s \frac{dx_\mu^n}{ds} A_\mu(x^n)\right]$$

Gauge transformations:

• Gauge transformation $\Lambda(x)$ is a (SU(N)) group element defined at every point. Gauge potential transforms as

$$A_\mu \to \Lambda A_\mu \Lambda^{-1} + \frac{i}{g} \Lambda \partial_\mu \Lambda^{-1}$$

• Field ϕ:

$$\phi(x) \to \Lambda(x)\phi(x)$$

• Path p:

$$U(p) \to \Lambda(x_1) U(p) \Lambda^{-1}(x_0),$$

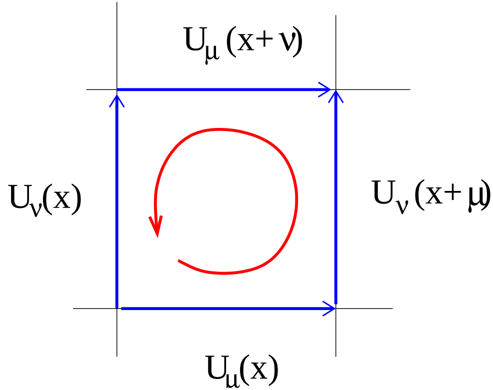
where x_0 and x_1 are the beginning and end of path.

- Closed loops: $U(C) \to \Lambda(x_0)U(C)\Lambda^{-1}(x_0)$
- Trace of a closed loop: Tr $U(C)$ is gauge invariant!

8.7. Lattice gauge fields

• Variables: parallel transporters from one lattice site to a neighbouring one, *Links:*

$$U_\mu(x) = P\exp\left[ig\int_x^{x+\mu} dx_\mu A_\mu\right] = \exp\left[igaA_\mu(x+\tfrac{1}{2}\mu)\right] + O(ga^3)$$



• The lattice action has to be gauge invariant. Only traces of closed loops are gauge invariant; the simplest one is *plaquette*

$$\mathrm{Tr}\, U_\Box = \mathrm{Tr}\; U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x)$$

• When a small (for SU(N)),

$$\mathrm{Re}\,\mathrm{Tr}\, U_\Box = N - \frac{a^4g^2}{4}F^a_{\mu\nu}F^a_{\mu\nu} + O(g^4a^6)$$

- $\rightarrow$ Wilson gauge action (K. Wilson, 1974):

$$S_g = \frac{2N}{g^2 a^{4-d}} \sum_{\Box} \left[1 - \frac{1}{N} \text{Re Tr}\, U_{\Box} \right]$$
$$= \int d^d x \frac{1}{2} \text{Tr}\, [F_{\mu\nu} F_{\mu\nu}] + O(a^2 g^2)$$

Partition function

$$Z = \int [\prod_{x,\mu} dU_\mu(x)] e^{-S_g}$$

Here the integral is over group elements U_μ (compact), *not* over algebra A_μ

Common notation:

$$\beta = \beta_G \equiv \frac{2N}{g^2} \quad \text{for SU(N),} \qquad \beta = \frac{1}{g^2} \quad \text{for U(1).}$$

8.7.1. Continuum limit for U(1) (in 4d)

- $U_\mu(x) = \exp iga A_\mu(x)$
- The Wilson action for U(1) is

$$S = \frac{1}{g^2} \sum_{x;\mu<\nu} \left(1 - \text{Re}\, U_\mu(x) U_\nu(x+\mu) U_\mu^\dagger(x+\nu) U_n^\dagger u(x)\right)$$
$$= \frac{1}{g^2} \sum_{x;\mu<\nu} (1 - \text{Re}\, \exp\left[iga(A_\mu(x) + A_\nu(x+\mu) - A_\mu(x+\nu) - A_\nu(x))\right])$$

Expanding $A_\nu(x+\mu) = A_\nu(x) + a\partial_\mu A_\nu(x) + \frac{1}{2}a^2\partial_\mu^2 A_\nu(x) + \ldots$

$$S = \frac{1}{g^2} \sum_{x;\mu<\nu} \left(1 - \text{Re}\, \exp\left[iga^2(\partial_\mu A_\nu - \partial_\nu A_\mu) + O(a^4)\right]\right)$$
$$= \frac{1}{g^2} \frac{1}{2} \sum_{x;\mu\neq\nu} \left[\frac{1}{2} g^2 a^2 F_{\mu\nu}^2 + O(g^2 a^6)\right]$$
$$= \frac{1}{4} \int d^4x F_{\mu\nu}^2$$

- Error in lagrangian $O(a^2)$
- Non-Abelian continuum limit comes in a similar way, but now one has to be careful with the commutators. Check it! (Campbell-Baker-Hausdorff formula $e^A e^B = e^{A+B-[A,B]+\ldots}$ might help, but it also comes directly from the expansion of the exponents.)

8.8. Updating the gauge fields:

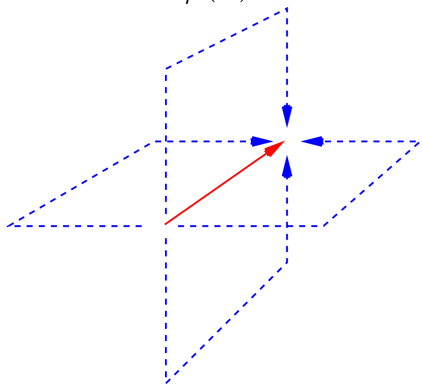
Let us consider U(1) theory, for simplicity. The action is

$$S = \beta \sum_{x;\mu<\nu} (1 - \text{Re}\, U_\Box(\mu,\nu;x))$$

$$U_\Box(\mu,\nu;x) = U_\mu(x) U_\nu(x+\mu) U_\mu^\dagger(x+\nu) U_\nu^\dagger(x),$$

where $U_\mu(x) = \exp[i\theta_\mu(x)]$, $0 \le \theta_\mu(x) < 2\pi$.

To update $U_\mu(x)$ we need the part of the action which depends on it, i.e. the plaquettes which contain the link $U_\mu(x)$. This is



$$S_\mu(x) \equiv -\beta \text{Re}\, U_\mu(x) P_\mu(x)$$

$$P_\mu(x) = \sum_{\nu;|\nu|\neq\mu} U_\nu(x+\mu) U_\mu^\dagger(x+\nu) U_n^\dagger u(x)$$

where P_μ is called the sum of *staples*, for obvious reasons.

The sum over ν above goes over both positive and negative directions; here $U_{-\nu}(x) = U_\nu^\dagger(x-\nu)$.

Now the link variable $U_\mu(x)$ can be updated with a suitable algorithm (heat bath, Metropolis, overrelaxation . . .).

For U(1), one staple is $\exp[i(\theta_\nu(x+\mu) - \theta_\mu(x+\nu) - \theta_n u(x))]$; thus, the sum of staples is a complex number: $P_\mu(x) = Ae^{-i\theta'}$, and the 'local action' becomes

$$S_\mu(x) = -\beta \text{Re}\, U_\mu(x) P_\mu(x) = -\beta A \cos(\theta_\mu(x) - \theta')$$

Thus, the local action is just like with the XY model discussed earlier! We can apply the same update algorithms as discussed there.

In the above discussion we used the standard ***compact*** formalism for U(1) gauge fields: $U_\mu = e^{igaA_\mu}$; S as given above. For U(1) (but not for general non-abelian gauge field) it is also possible to use ***non-compact*** formulation: let the link variable be

$$\alpha_\mu(x) = gaA_\mu(x), \qquad -\infty < \alpha_\mu(x) < \infty,$$

and the action is now

$$S^{\rm NC} = \frac{1}{g^2} \sum_{x;\mu<\nu} \frac{1}{2} [\alpha_\mu(x) + \alpha_\nu(x+\mu) - \alpha_\mu(x+\nu) - \alpha_\nu(x)]^2$$

The continuum limit for this action is the same as for the compact action (check it!). The difference in the continuum limit is in the higher order $O(a^2)$ terms. At finite a the physics can be quite different!
Note: switch $\frac{1}{2}[\cdot]^2 \mapsto (1 - \cos[\cdot])$, and you recover the compact action.
Non-compact U(1) gauge field theory is actually completely solvable (it is free theory, as the continuum theory). However, if we would couple the theory to matter this would not be the case.

8.9. Gauge transformations:

- Gauge transform $\Lambda(x) \in$ SU(N) lives on lattice *sites*
- Link variable: $U_\mu(x) \to U'_\mu(x) = \Lambda(x) U_\mu(x) \Lambda^\dagger(x+\mu)$

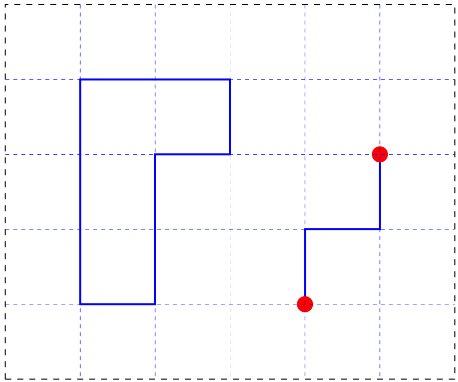
  $\mapsto$

  $A_\mu(x) \to \Lambda(x) A_\mu(x) \Lambda^\dagger(x) + \frac{i}{g} \Lambda(x) \partial_m u \Lambda^\dagger(x)$
- Trace of a closed loop is gauge invariant
- Fundamental matter: $\phi(x) \to \Lambda(x)\phi(x)$, where ϕ is a N-component complex vector: operator

  $\phi^\dagger(x) U_P(x \mapsto y) \phi(y) =$

  $\phi^\dagger(x) U_\mu(x) U_\nu(x+\mu) \ldots U_\rho(y-\rho) \phi(y)$

  is gauge invariant.



- We want *only* gauge invariant animals on a lattice: lattice action +

observables must consist of *closed loops* (gauge only quantities) or $\phi^\dagger U_P \phi$ - “strings” (matter).

- For example, the matter field kinetic term

$$(D_\mu \phi)^\dagger (D_\mu \phi),$$

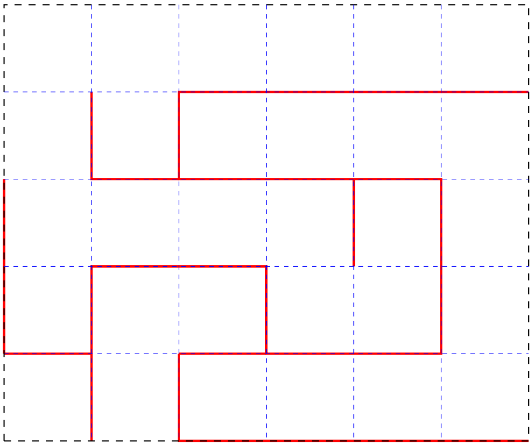
where $D_\mu = \partial_\mu + igA_\mu$, becomes

$$\frac{1}{a^2}\left[2d\phi^\dagger\phi - 2\sum_\mu \phi^\dagger(x)U_\mu(x)\phi(x+\mu)\right]$$

- *Elitzur’s theorem:* expectation values of gauge non-invariant object = 0: $\langle U_\mu(x)\rangle = 0$.

8.9.1. Gauge fixing on the lattice

- Something you don't want to do
- Necessary for perturbative calculations
- Most of the (Euclidean) continuum gauges go over to lattice
- Special gauge: *maximal tree*



- A tree which connects every point of the lattice, but does not have closed loops.
- Link matrices $U_\mu(x)$ are *fixed* to arbitrary values (for example all $= 0$) in the tree
- Expectation values of gauge invariant quantities do not change (proof: easy)
- (Almost) complete gauge fixing

8.10. Confinement and Wilson loop

- Potential between static charge and anticharge, separated by R:

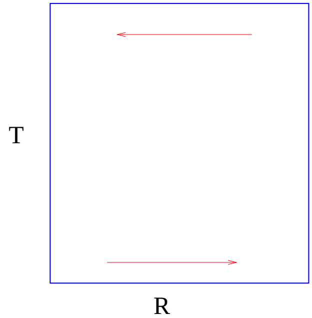
$$V(R \to \infty) \to \begin{cases} \infty : & \text{confinement} \\ \text{finite:} & \text{no confinement} \end{cases}$$

- Standard probe: Wilson loop W_{RT}. Let W be a rectangular path of size $R \times T$ along x_1 (say) and $x_0 = \tau$ directions.

$$W_{RT} = \text{Tr}\, P \exp ig \oint_W ds_\mu A_\mu = \text{Tr} \prod_{<xy>\in W} U_{xy}.$$

Now $-\log W_{RT}$ gives the "free energy" of a static charge-anticharge ("quark-antiquark") system separated by R and which evolves for time T:

$$-\log\langle W_{RT}\rangle = V(R)T \qquad \text{valid as } T \to \infty$$



- *Perimeter law:* $-\log\langle W_{RT}\rangle \sim m(2R+2T)$

  Free charges, m: “mass” of the charge due to gauge field

- *Area law:* $-\log\langle W_{RT}\rangle \sim \sigma RT$; $V(R) = \sigma R$

  Charges confined with linear potential, σ: string tension

- In general, $\frac{1}{T}\log\langle W\rangle \sim \sigma R +$ const. $+ c/R + \ldots$

Motivation:

- In temporal gauge $A_0 = 0 \quad \rightarrow \quad U_0 = 1$ we can define gauge field Hamiltonian $\hat{H}$ [Kogut-Susskind]

- Ψ: (trial) wave function of $q\bar{q}$ pair at $\bar{x}$, $\bar{y}$; spatial gauge transformations

  $\Psi_{ab} \rightarrow \Lambda_{ac}(\bar{x})\Lambda^{\dagger}_{bd}(\bar{y})\Psi_{cd}$

- $\hat{H}$ gauge invariant: $\langle \Psi' | e^{-T\hat{H}} | \Psi \rangle = 0$ unless Ψ, Ψ' have similar gauge transformation properties.

$$\begin{aligned} \langle \Psi | e^{-T\hat{H}} | \Psi \rangle &= \sum_n |\langle \Psi_n | \Psi \rangle|^2 e^{-E_n T} \\ &\rightarrow |\langle \Psi_0 | \Psi \rangle|^2 e^{-E_0 T} \qquad \text{when } T \rightarrow \infty \end{aligned}$$

- $E_0 = V(R)$ ground state energy of static charges ($R = |\bar{x} - \bar{y}|$)

- Trial wave function: $\Psi_{ab} = U_{ab}(x \mapsto y)\Psi_{\text{vac}}$, where Ψ_{vac} is the (gauge invariant) vacuum wave function

$$\langle\Psi|e^{-T\hat{H}}|\Psi\rangle = \frac{1}{Z}\int[dU]\,\text{Tr}\,[U^\dagger(T;\bar{x}\mapsto\bar{y})\,U(0;\bar{x}\mapsto\bar{y})]e^{-S}$$
$$= \langle W_{RT}\rangle$$

- W_{RT} gauge invariant: can be measured in *any* gauge:

$$-\frac{1}{T}\log\langle W_{RT}\rangle \to V(R)$$

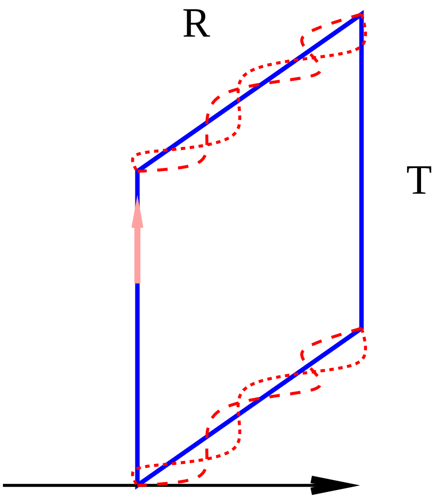
  as $T\to\infty$.

- If we can guess/construct $\Psi\approx\Psi_0$, T need not be very large.

8.10 $\frac{1}{2}$. Measuring the string tension from Monte Carlo

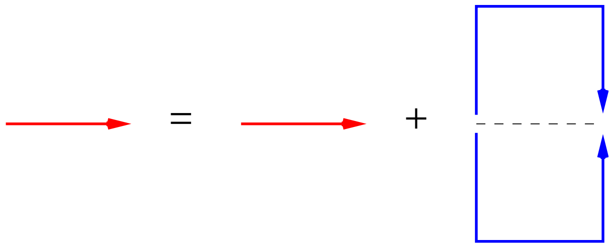
The Wilson loops become exponentially small ($\sim e^{-\sigma RT}$) as the size increases. However, the statistical noise is $\sim$ constant in magnitude independent of loop size! Thus, we need to increase the number of measurements exponentially as the loop size increases.

Smearing improves the situation: instead of using only straight const-T legs in the loop, take an averaged sum over paths around the straight one. The smearing is to mimic the wave function of the desired $q\bar{q}$ ground state. The hope is that now a smaller T-extent would be sufficient to get the asymptotic behaviour, i.e. look like the limit $T \to \infty$.



Smearing can be done recursively: let $i \perp t$, and

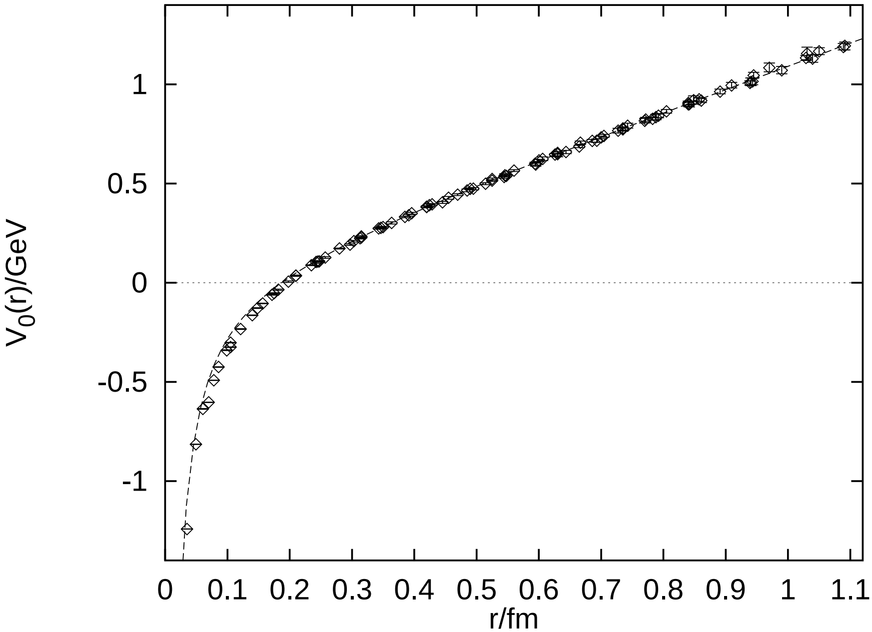
$$U_i(x) \leftarrow U_i(x) + c \sum_{j \perp i,t} U_j(x) U_i(x+\hat{j}) U_j^\dagger(x+\hat{i})$$



i.e. substitute link matrices on a plane $\perp$ to t by the weighted average of the link and the staples (which are also $\perp$ to t). This can be repeated a few times.

Wilson loop in pure gauge SU(3)

[Bali,Schilling, Wachter 1995] $48^3 \times 64$ quenched lattice, $\beta = 6.8$



For large loops

$$\langle W \rangle \sim e^{-V(r)T}$$

Phenomenological form

$$V(r) = V_0 + \sigma r - \frac{e}{r} + f\left[G_L(\vec{r}) - \frac{1}{r}\right]$$

8.11. Strong coupling expansion and the Wilson loop

- Perturbation theory (weak coupling, $g \ll 1$) is toothless: *no confinement*

- Strong coupling: $g \gg 1$. For simplicity, consider U(1) ($\beta = 1/g^2 \ll 1$). The link variable is $U_\mu = \exp i\theta_\mu$.

$$
\begin{aligned}
S &= \beta \sum_{\Box} (1 - \text{Re}\, e^{i\theta_\Box}) \\
Z &= \int_0^{2\pi} \left[\prod_{x,\mu} \frac{d\theta_\mu}{2\pi} \right] \prod_{\Box} \exp[\beta \cos \theta_\Box]
\end{aligned}
$$

Here $\theta_\Box = \theta_1 + \theta_2 - \theta_3 - \theta_4$ around the plaquette $\Box$.

- Expand in powers of β? OK, but more convenient is the following character expansion (here I_n are modified Bessel functions)

$$
e^{\beta \cos \theta} = \sum_{n=-\infty}^{\infty} I_n(\beta) e^{in\theta} = A \left[1 + \sum_{n=1}^{\infty} f_n \cos n\theta \right]
$$

where

$$
\begin{aligned}
A &= I_0(\beta) = 1 + \frac{1}{4}\beta^2 + \frac{1}{64}\beta^4 + \dots \\
f_n &= 2I_n(\beta)/I_0(\beta) = \frac{1}{2^{n-1}n!}\beta^n + O(\beta^{n+2}) \\
f_1 &= \beta - \frac{1}{8}\beta^3 + \frac{1}{48}\beta^5 + \dots
\end{aligned}
$$

Compare to the "character expansion" in the Ising model:

$$e^{\beta s_i s_j} = a(1 + b s_i s_j)$$

with $a = \cosh\beta$, $b = \tanh\beta$.

In Ising gauge theory, $\Box(i,j;x) = s_i(x)s_j(x+\hat{i})s_i(x+\hat{j})s_j(x) = \pm 1$, and the expansion of $e^{\beta\Box(i,j;x)}$ is exactly analogous.

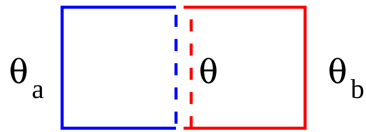
The partition function becomes

$$Z = \int [\frac{d\theta}{2\pi}] \, A^{N_\Box} \prod_\Box (1 + f_1 \cos\theta_\Box + f_2 \cos 2\theta_\Box + \ldots)$$

here $N_\Box = 6V$ is the number of plaquettes (in 4d).

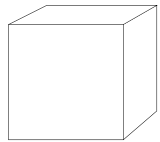
- Integration: $\int d\theta_a \cos n\theta_\Box = 0$, if $\theta_a \in \Box$.
  2 adjacent plaquettes:

$$\int \frac{d\phi}{2\pi} \cos(\theta + \theta_a) \cos(-\theta + \theta_b) = \tfrac{1}{2} \cos(\theta_a + \theta_b)$$



(also for $\cos n\theta$)

- Non-zero contributions only from *closed surfaces*: lowest non-trivial comes from 3-d cube:



$$Z = A^{N_\Box}[1 + 4V(\tfrac{1}{2} f_1)^6 + O(\beta^{10})]$$

- Each plaquette on the surface contributes $\frac{1}{2}f_i$

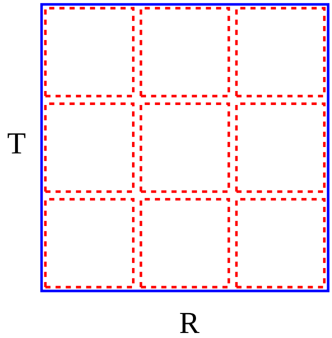
- Expansion of free energy $F = -\log Z$ contains only connected graphs – *cluster expansion*

Wilson loop:

To leading order, the Wilson loop is

$$\langle W_{RT}\rangle = \frac{A^{N_\square}}{Z}\int[\frac{d\theta}{2\pi}]\prod_{a\in W} e^{i\theta_a}\prod_{\square}[1+\tfrac{1}{2}f_1(e^{i\theta_\square}+e^{-i\theta_\square})+O(f_2)]$$

- First contribution comes when we "tile" the loop area with plaquettes:



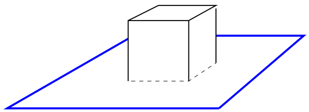
$$\langle W_{RT}\rangle \approx (\tfrac{1}{2}f_1)^{RT} = (\tfrac{1}{2}\beta)^{RT} + O(\beta^{RT+2})$$

- We see *confinement:*

$$-\log\langle W\rangle/T = \log(\tfrac{1}{2}f_1)R = \sigma R$$

- Next order: "bump"

$$\langle W_{RT}\rangle \approx (\tfrac{1}{2}f_1)^{RT}[1+4RT(\tfrac{1}{2}f_1)^4]$$



which gives

$$-a^2\sigma = -\frac{1}{RT}\log\langle W_{RT}\rangle = \log\tfrac{1}{2}f_1 + 4(\tfrac{1}{2}f_1)^4 + O([\tfrac{1}{2}f_1]^6)$$

- Order $[\frac{1}{2}f_1]^6$ gets contributions from the disconnected surfaces and Z.
- $-a^2\sigma = \log u + 4u^4 + \frac{176}{3}u^8 + \frac{10\,936}{405}u^{10} + \frac{1\,532\,044}{1\,215}u^{12} + \frac{3\,596\,102}{5\,103}u^{14}$, where $u = (\frac{1}{2}f_1)$ (see Montvay+Münster, for example)
- For U(1) this is strong coupling artefact! When $g\ 1$, there $\exists$ a phase transition. As $g \to 0$ we regain free theory.

Non-Abelian gauge fi elds

- Same, but more difficult: use *character expansion*

$$e^{1/N\beta \text{Re Tr}\, U} = A_\beta[1 + \sum_R b_R(\beta)\,\chi_R(U)]$$

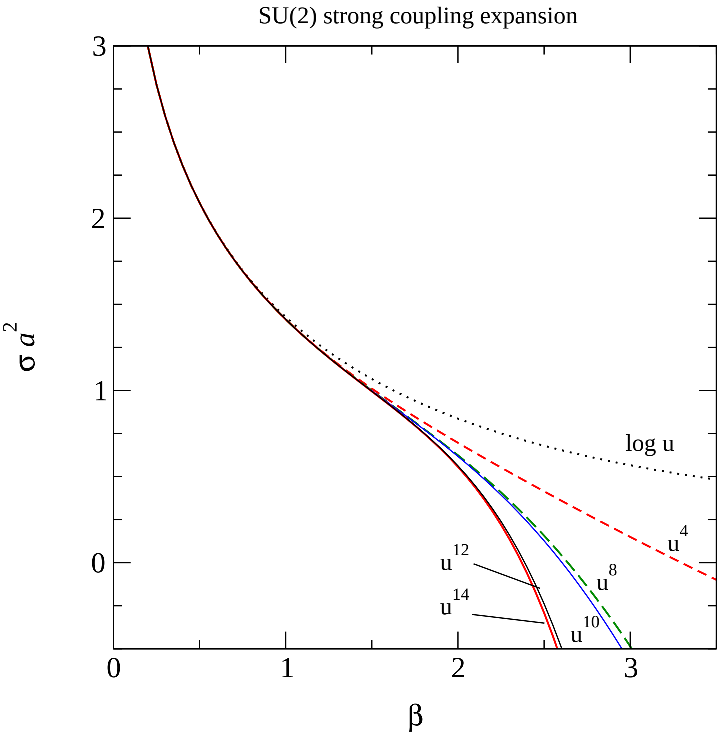
which gives "orthogonal" integration rules.

- Graphs are again surfaces – with complications!

- SU(2): $-a^2\sigma = \log u + 4u^4 + \frac{176}{3}u^8 + \ldots$,
  with $u = I_2(\beta)/I_1(\beta)$

- SU(3): $-a^2\sigma = \log u + 4u^4 + 12u^5 - 10u^6 - \ldots$
  with $u = \frac{1}{3}(x - \frac{1}{2}x^2 + \frac{2}{3}x^3 + \ldots)$, $x = \beta/6$.

- Convergence is good in *finite* range $\beta < \beta_{\max}$.
- *Mass gap:* plaquette-plaquette correlation function decays exponentially. Can be calculated using strong coupling, not in perturbation theory.
- *Roughening transition* for Wilson loops as β increases?



8.12. Realistic continuum limit

- When we change lattice spacing a, we have to adjust the *bare* coupling g in order to keep physics (at "long" distances) invariant.

- Measuring $g(a)$: choose some physical/renormalized quantity $M(a, g(a))$, (mass, string tension, Wilson loop of fixed size, vertex function ...) which can be measured/calculated. This we require to be *constant* as a is changed ("renormalization prescription")

  $\rightarrow$ *RG equation*:

$$a \frac{d}{da} M(a, g(a)) = 0 = \left(a \frac{\partial}{\partial a} - \beta_L \frac{\partial}{\partial g} \right) M$$

- β-function: $\beta_L \equiv -a \frac{\partial g}{\partial a}$

  tells how coupling g and lattice spacing a evolve.

- *Fixed point* $\beta_L(g_F) = 0$.

- If $\beta_L(g) < 0$ (> 0), g decreases (increases) as a is decreased.
- If $\partial\beta_L(g_F) < 0$, this is UV attractive fixed point and $\lim_{a\to 0} g(a) = g_F$: continuum limit
- For example: β-function at *strong coupling*, using expansion for SU(3) string tension $\sigma = -a^{-2}\log(1/(3g^2)) + O(g^{-2})$

$$\begin{aligned} 0 &= a\frac{d}{da}\sigma = -2\sigma - \beta_L \frac{2}{a^2 g} + \ldots \\ \rightarrow &\ \beta_L = -g\ \log(3g^2) + \ldots \end{aligned}$$

This is < 0, so $g \searrow$ as $a \searrow$: no continuum limit at strong coupling.

Weak coupling:

- Use perturbation theory. For QCD (for example, from renormalized 3-vertex):

$$\beta_L(g) = -b_0 g^3 - b_1 g^5 + \ldots$$

  Coefficients b_0 and b_1 are *universal*: independent of the regularization and prescription.

$$b_0 = \frac{1}{16\pi^2}\left(\frac{11N}{3} - \frac{2N_F}{3}\right)$$

$$b_1 = \frac{1}{(16\pi^2)^2}\left(\frac{34N^2}{3} - \frac{10NN_F}{3} - \frac{N_F(N^2-1)}{N}\right)$$

- Integrate: $a = \exp[-\int^g dg' \, \beta_L^{-1}(g')] \quad \rightarrow$

$$a\Lambda_L = \exp[\frac{1}{2b_0 g^2}](b_0 g^2)^{b_1/2b_0^2}[1 + O(g^2)]$$

- UV fixed point $g \rightarrow 0$ as $a \rightarrow 0$: *asymptotic freedom*. (If $b_0 > 0$!)

- Λ_L: dimensionful integration constant, Λ-parameter; sets the scale. It is regularization dependent.

- *Dimensional transmutation:* no mass scale in the original (classical) Lagrangian!

- Thus, a physical mass m should behave as

$$am = \frac{m}{\Lambda_L} \exp[\frac{1}{2b_0 g^2}](b_0 g^2)^{b_1/2b_0^2}$$

In practice, this not well satisfied in QCD; $g \sim 1$:

Asymptotic scaling does not hold.

- Scaling holds if mass ratios are $\sim$ constant:

$$\frac{m_1}{m_2}(a) = \frac{m_1}{m_2}(0) + O(a^2)$$

This has been observed to be much better satisfied than asymptotic scaling.

Numerically:

- Principle: measure a quantity – am, $a^2\sigma$ – at several g's. Then

$$\frac{(am)(g_1)}{(am)(g_2)} = \frac{a(g_1)}{a(g_2)}$$

gives a discrete sample of $a(g_1) - a(g_2) = \exp[-\int_{g_2}^{g_1} dg' 1/\beta_L(g)]$

- Matching more than one quantity simultaneously: need more than one lattice coupling.

- *Monte Carlo renormalization group (MCRG):* direct implementation of Wilson's RG in numerical simulations.

  - Write the action $S = \sum_a \kappa_a S_a$, where S_a are a set of (local) contributions to action: plaquettes, larger loops . . .

  - With couplings κ_a^A simulate on L^d lattice, and measure a set of Wilson loops.

- *block* the configurations by factor of 2: obtain $(L/2)^d$ lattice. Measure the Wilson loops.

- Repeat blocking + measuring, until lattice is too small.

- simulate directly on $(L/2)^d$ lattice, with κ^B. As before, repeat measuring Wilson loops and blocking.

- compare Wilson loops at blocking level b from simulation B to loops at level $b+1$ from A. *Tune* κ^B_a, until the measurements match (requiring a new simulation).

- repeat for $(L/4)^d$ etc.

For the plaquette term, this gives function $\Delta\beta = \beta(a) - \beta(2a) = 6/g^2 - 6/{g'}^2$

- Violation of asymptotic scalings large at practical values of β (fig.)!

- *Perfect action:* no scaling violations; evolves along RG trajectory. Requires infinitely many coupling constants.

- *Classically perfect action:* perfect action for classical configurations. Much easier to derive, and these have been used for various models. It's also 1-loop quantum perfect action (perfect in the limit $g \to 0$).

In 3 dimensions:

- Lattice coupling $\beta = \frac{2N}{g^2 a}$. Dimensions of $[g^2] = \text{GeV}$.

- Renormalization: $g_R^2 = g^2 + O(g^4 a)$ dimensionally

- Continuum limit $\beta = \frac{2N}{g_R^2 a} + O[(g_R^2 a)^0]$

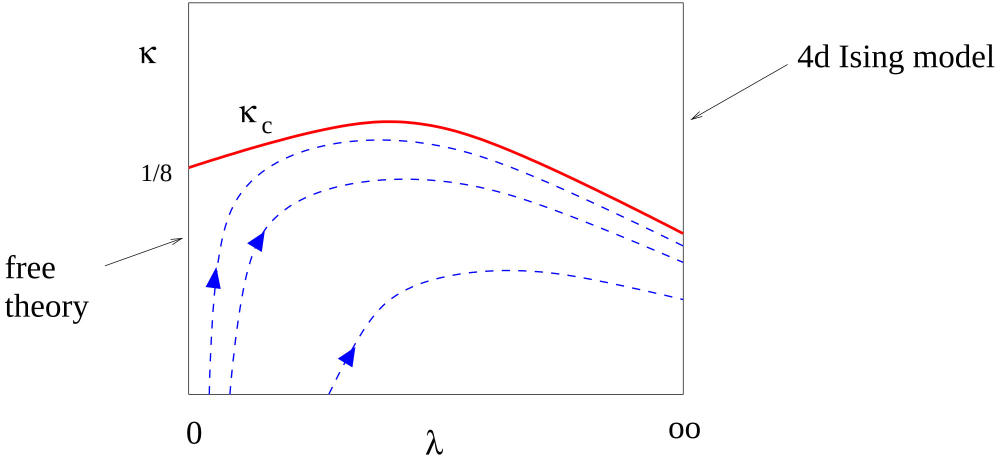
- Theory *superrenormalizable:* only finite number of divergent diagrams

8.13. Continuum limit in $\lambda\phi^4$ model

- Lattice lagrangian

$$\mathcal{L} = -\kappa \sum_{\mu} \varphi_x \varphi_{x+\mu} + \varphi_x^2 + \lambda(\varphi_x^2 - 1)^2$$

- In 4 dimensions, the RG trajectories of the bare couplings κ, λ have been solved [Lüscher, Weisz, NPB 295 (1988)] using hopping parameter expansion and perturbative RG equations:



Blue curves: RG trajectories of constant λ_R; λ_R decreases to $\nwarrow$; along κ_c we have $\lambda_R = 0$.

- Perturbatively $\beta_L(\lambda) = -a\dfrac{\partial\lambda}{\partial a} = \dfrac{3}{16\pi^2}\lambda^2 > 0$
- *No continuum limit!* RG trajectories hit the Ising limit $\lambda = \infty$ before the critical line $\kappa_c(\lambda)$ is reached; i.e. at finite a.

- Ising model: smallest possible a for any fixed λ_R.
- This is a generic feature of $\lambda\phi^4$ models in 4d

  $\longrightarrow$ *Triviality bound* for the Standard Model Higgs mass!

- In 3 dimensions: there $\exists$ continuum limit: coupling constant is dimensionful $[\lambda] = \text{GeV}$.
- $\lambda_R = \lambda + O(\lambda^2 a)$
- $m_R^2 = m_0^2 + C_1 \lambda a^{-1} + C_2 \lambda^2 + O(\lambda^3 a)$

  m_R^2 has linear ($1/a$) and ($\log a/\Lambda$) divergence. Perturbatively calculable, and can be subtracted.