7. Low and high T expansions in Ising model

In addition to the Monte Carlo methods, we can generate results analytically from lattice models. The most important expansions are the *low- and high-temperature* expansions:

- *Low temperature expansion:*

  - Expansion of $Z = \int [d\phi] e^{-E/T}$ around $T = 0$

  - Perturbations around a $S = S_{\rm min}$ state

  - In field theory this corresponds to the *weak coupling expansion*. For continuously varying fields, this gives the standard perturbation theory (in continuum or on the lattice)

- *High temperature expansion:*

  $e^{-E/T} \sim 1 - E/T + \frac{1}{2}(E/T)^2 + \ldots$

  - Expansion around a “random” state

  - *Strong coupling expansion*

- *Hopping parameter expansion*
- No direct continuum counterpart

- *Mean field approximation*
- *Exact results: dualities etc.*

Ising model: low + high temperature expansions

Ising model: every lattice point has a spin $s_i = \pm 1$

$$Z = \sum_{s_i} e^{-\beta H} , \qquad H = -\frac{1}{2} \sum_{<ij>} s_i s_j$$

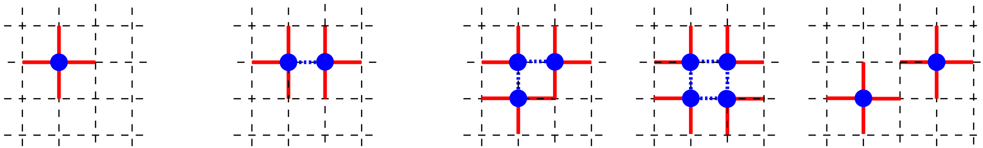
$< ij >$: nearest neighbour pairs, $\beta = 1/T$
The model has a symmetry breaking phase transition at a Curie point β_c (in 2D: $\beta_c = \log(1 + \sqrt{2}) \approx 0.8814$).
If $\beta \le \beta_c$ ($T \ge T_c$), $\langle s \rangle = 0$, whereas if $\beta > \beta_c$, $\langle s \rangle \neq 0$.

7.1. Low temperature expansion for 2d Ising

- 2d Ising model at $\beta \gg 1$. For simplicity, redefine $H \to H' = \sum_{<ij>} [1 - \delta(s_i, s_j)]$ so that a completely ordered system (all $s_i = +1$ or -1) has $H = 0$.

- Assume that $\langle s \rangle > 0$ (for example, boundaries fixed to $s = 1$)
- Classify configurations by the number of *frustrated* bonds $n_f = 0, 4, 6 \ldots$
- Partition function

$$
\begin{aligned}
Z &= \sum_{s_i} e^{-\beta H} = \sum_{s_i} \prod_{<ij>} e^{-\beta(1-\delta(s_i,s_j))} = \sum_{s_i} e^{-\beta n_f[s]} \\
&= 1 + V e^{-4\beta} + 2V e^{-6\beta} + (6V + V + V(V-5))e^{-8\beta} + O(e^{-10\beta})
\end{aligned}
$$



- Likewise, expectation value $\langle s \rangle = \langle s_x \rangle$ (using translation invariance:

$$\begin{aligned}
\langle s_x \rangle &= \frac{1}{Z} \sum_{s_i} s_x e^{-\beta H} \\
&= \frac{1 + (V-2)e^{-4\beta} + (2V-8)e^{-6\beta} + O(e^{-8\beta})}{1 + V e^{-4\beta} + 2V e^{-6\beta} + O(e^{-8\beta})} \\
&= 1 - 2e^{-4\beta} - 8e^{-6\beta} + O(e^{-8\beta})
\end{aligned}$$

- 2d Ising: expansion to order $e^{-76\beta}$ [I. G. Enting, A, J. Guttmann, I. Jensen, J.Phys.A27 (1994)]

  3d Ising: expansion to $e^{-26\beta}$ [I. G. Enting, A, J. Guttmann, J.Phys.A26 (1993)]

- Note: expansion of $F = -\log Z$ has only connected graphs and is $\propto V$!

- Does not work for continuous d.o.f's

From [I. G. Enting, A, J. Guttmann, I. Jensen, J.Phys. A27 (1994) 6987-7006]

Table II: New low-temperature series for the spin-$\frac{1}{2}$ 2-dimensional Ising magnetisation ($M(u) = \sum_n m_n u^n$), susceptibility ($\chi(u) = \sum_n x_n u^n$), and specific heat ($C_v(u) = \sum_n c_n u^n$). All terms with odd n are zero.

n	m_n	x_n	c_n
0	1	0	0
2	0	0	0
4	-2	1	16
6	-8	8	72
8	-34	60	288
10	-152	416	1200
12	-714	2791	5376
14	-3472	18296	25480
16	-17318	118016	125504
18	-88048	752008	634608
20	-454378	4746341	3269680
22	-2373048	29727472	17086168
24	-12515634	185016612	90282240
26	-66551016	1145415208	481347152
28	-356345666	7059265827	2585485504
30	-1919453984	43338407712	13974825960
32	-10392792766	265168691392	75941188736

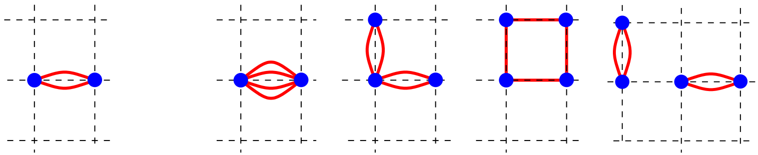
n	m_n	x_n	c_n
34	-56527200992	1617656173824	414593263952
36	-308691183938	9842665771649	2272626444528
38	-1691769619240	59748291677832	12502223573304
40	-9301374102034	361933688520940	68996534259040
42	-51286672777080	2188328005246304	381858968527680
44	-283527726282794	13208464812265559	2118806030647328
46	-1571151822119216	79600379336505560	11783826597027256
48	-8725364469143718	479025509574159232	65674579024955904
50	-48552769461088336	2878946431929191656	366728645195006000
52	-270670485377401738	17281629934637476365	2051443799934043632
54	-1511484024051198680	103621922312364296112	11494250259278105304
56	-8453722260102884930	620682823263814178484	64499139095733378176
58	-47350642314439048648	3714244852389988540072	362436080938852037648
60	-265579129813183372802	22206617664989885664363	2039249170926323834880
62	-1491465339550559632448	132657236460768679560864	11487673072269872540904
64	-8385872784303807639294	791843294876287279547520	64786142191741932873984
66	-47202746620874986470336	4723112509660327575046688	365754067103461706996304
68	-265975151780412455885826	28152514246598001579534217	2066925549185792626090544
70	-1500179080790296495333960	167696255471026758161692328	11691314122170272566638200
72	-8469330846027919131108866	998303936498277539688401212	66188283453887221177721568
74	-47856040705247407564621400	5939502715888619728011515904	375021938737150106426702208
76	-270636033194089067428986890	35318214476286590871820680287	2126523853550658555941372768

7.2. High temperature expansion for 2d Ising

[A.J.Guttmann, in *Phase transitions and critical phenomena*, Vol. 13, eds. Domb and Lebowitz (Academic Press, 1989)]

- Now most convenient to use $H = -\sum_{<ij>} s_i s_j$
- Partition function at $\beta \ll 1$: expand in β, only terms which have s_i^{2n} survive!

$$
\begin{aligned}
Z &= \sum_{s_i} \prod_{<ij>} e^{\beta s_i s_j} \\
&= \sum_{s_i} \prod_{<ij>} \left(1 + \beta s_i s_j + \frac{1}{2!}\beta^2 (s_i s_j)^2 + \frac{1}{3!}\beta^3 (s_i s_j)^3 + \frac{1}{4!}\beta^4 (s_i s_j)^4 + \ldots\right) \\
&= 2^V \left[1 + \beta^2 \frac{2V}{2} + \beta^4 \left(\frac{2V}{4!} + \frac{6V}{2^2} + V + \frac{1}{2}\frac{2V(2V-7)}{2^2}\right) + O(\beta^6)\right]
\end{aligned}
$$



- $\mapsto$ Partition function as a sum of closed graphs:

$$Z = 2^V \sum_G \beta^{L(G)} \prod_{<ij>} \frac{1}{m_{ij}(G)\,!}$$

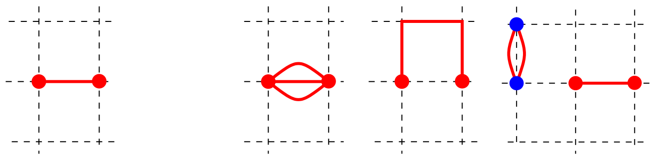
where $L(G)$ is the number of the links in the graph G (including links with n hops n times), and $m_{ij}(G)$ is the number of hops over link $< ij >$.

- Expectation values for spin operators: the operators we measure are products (and sums of products) of spins.

– $\langle \prod^N s_i \rangle = 0$, if N odd

– for N even, construct graphs which connect the "sources".

For example, a nn-pair $\langle s_a s_b \rangle$ has the following graphs up to 3 hops:



This gives (taking into account the combinatorics)

$$
\begin{aligned}
\langle s_a s_b \rangle &= \frac{1}{Z}\sum_{s_i} s_a s_b e^{-\beta H} \\
&= \frac{1}{Z}\sum_{s_i} s_a s_b \left(\beta s_a s_b + \beta^3 \left[\frac{1}{3!}(s_a s_b)^3 + \sum_{cd \in \sqcup} (s_a s_c)(s_c s_d)(s_d s_b) \right.\right. \\
&\quad \left.\left. + \frac{1}{2} \sum_{\langle ij \rangle \neq \langle ab \rangle} (s_i s_j)^2 s_a s_b \right] + O(\beta^5) \right) \\
&= \frac{\beta + \beta^3(\frac{1}{3!} + 2 + \frac{V-1}{2}) + O(\beta^5)}{1 + \beta^2 V + O(\beta^4)} = \beta + \beta^3 \frac{5}{3} + O(\beta^5)
\end{aligned}
$$

Here the last line could have been written directly by inspecting the graphs. Each link gives a factor of β, and the "multiplicity" gives a factor $1/m!$.

- Note: again $F = -\log Z$ contains only connected graphs, and is $\propto V$.

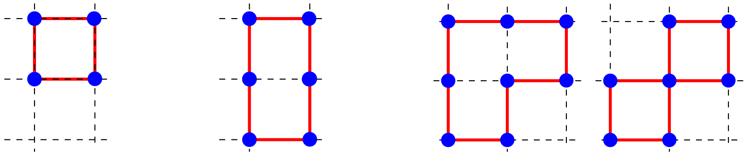
7.3. High temperature expansion using "character expansion"

- A more efficient way to perform the high-temperature expansion is to use the "character expansion":

$$\exp[\beta s_i s_j] = a(1 + b s_i s_j)$$

where $a = \cosh\beta$ and $b = \tanh\beta$; $O(b) = O(\beta)$. Now

$$\begin{aligned} Z &= \sum_{s_i} \prod_{<ij>} e^{\beta s_i s_j} = a^{2V} \sum_{s_i} \prod_{<ij>} (1 + b s_i s_j) \\ &= a^{2V} 2^V \left[1 + b^4 V + b^6 2V + b^8 \left(6V + \frac{1}{2} V(V-5) \right) + O(b^{10}) \right] \end{aligned}$$



- Only single-link graphs here! Much easier to enumerate.

- If we want the expansion in β, we have to expand a and b.
- Note: β^2 -term comes from the “vacuum”;

$$a^{2V} 2^V \times 1 = 2^V (1 + \beta^2/2 + \ldots)^{2V} = 2^V (1 + \beta^2 V + O(\beta^4))$$

- Partition function as a sum of closed graphs is simply

$$Z = a^{2V} 2^V \sum_G b^{L(G)}$$

where $L(G)$ is the number of the links in the graph G.

- And nn-pair expectation value comes as before, from the expansion where the source points are connected by links:

$$\begin{aligned} \langle s_a s_b \rangle &= \frac{1}{Z} \sum_{s_i} s_a s_b e^{-\beta H} \\ &= \frac{1}{Z} a^{2V} \sum_{s_i} s_a s_b \left(b s_a s_b + b^3 \sum_{cd \in \sqcup} (s_a s_c)(s_c s_d)(s_d s_b) + O(b^5) \right) \\ &= b + 2b^3 + O(b^4) = \beta + \beta^3 \frac{5}{3} + O(\beta^4). \end{aligned}$$

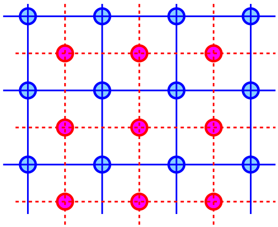
Much simpler graphs to work with than before!

- Generalizes to continuous spins: *hopping parameter expansion*

7.4. Duality in 2D Ising model

- Duality relations are (usually) exact relations which map a lattice system to another. Generically, they map low-temperature (weak coupling) $\leftrightarrow$ high-temperature (strong coupling).

- *Dual lattice* is a lattice which lives at the center of the original lattice hypercubes. In 2 dimensions:



- 2D Ising model is *self-dual*, i.e. the dual model is 2d ising too, but with different coupling.

- Start from the graph expansion from previous section ($a = \cosh\beta, b = \tanh\beta$):

$$
\begin{aligned}
Z &= a^{2V} \sum_{s_i} \prod_{<ij>} (1 + b s_i s_j) \\
&= a^{2V} 2^V \sum_G b^{L(G)} = a^{2V} 2^V \sum_G \prod_i b^{n_i(G)/2}
\end{aligned}
$$

where $n_i(G)$ is the number of links in closed graph G connecting to point i. Naturally, it has values $n_i = 0, 2, 4$.

- n_i lives on site i on the original lattice. Now comes the crucial point: we can map any graph to dual variables $\sigma_\alpha = \pm 1$, living on the dual lattice, so that $\sigma_\alpha \sigma_{\alpha'} = -1$, if link which crosses the dual link (α, α') belongs to the graph G; +1 otherwise.

- In other words, the original graphs divide the dual lattice in domains; neighbouring domains have different σ.

- Thus, each graph $\leftrightarrow$ 2 σ-configurations (all $\sigma_\alpha \rightarrow -\sigma_\alpha$ symmetry).

- If σ_i, $i = 1, 2, 3, 4$ surround point i, then we can identify (normalizing)

$$n_i = -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_4 + \sigma_4\sigma_1)/2 + 2.$$

Substituting this into partition function, we obtain (note: each dual link appears twice!)

$$\begin{aligned} Z &= a^{2V} 2^V 2 \sum_\sigma b^{\frac{1}{2}\sum_{<\alpha\gamma>} 1 - \sigma_\alpha \sigma_\gamma} \\ &\propto \sum_\sigma e^{\beta' \sum_{<\alpha\gamma>} \sigma_\alpha \sigma_\gamma} \end{aligned}$$

where β' is defined through

$$b^{1/2} = \tanh^{1/2}\beta = e^{-\beta'} \qquad \Rightarrow \qquad \beta' = -\frac{1}{2}\ln\tanh\beta.$$

Thus, 2 Ising models with β and β' are exactly dual – equivalent! – to each other. Note: if

$$\beta \rightarrow \begin{cases} 0 \\ \infty \end{cases} \qquad \Rightarrow \qquad \beta' \rightarrow \begin{cases} \infty \\ 0 \end{cases}.$$

Thus, the hot phase is mapped to the cold one and vice versa.

- What if $\beta' = \beta$: now $\beta = \frac{1}{2}\ln(1+\sqrt{2})$, i.e. the critical point of the Ising model!
- Only in 2d the dual of a lattice spin model is a spin model. Very few models are self-dual (Ising and *Potts* models).
- In 3D, the dual of a spin model is a *gauge theory*. For example, the dual of a 3D Ising model is a 3D Ising gauge theory. Very useful relation! We do not know efficient (cluster) algorithms for Ising gauge theory, but do for the Ising model.

- In 4D, the dual of a gauge theory is another gauge theory.

Hopping parameter expansion

- Scalar theory in d-dimensions:

$$S = \int d^d x \; \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda_0 \phi^4$$

- Conventional (not perhaps the best) lattice discretization: define (note: in d-dim. $[\lambda_0]$ = GeV^{4-d})

- $\varphi = \sqrt{\kappa} a^{d-2}/2\phi$

- $\lambda = \dfrac{1}{4!} \lambda_0 a^{4-d} \kappa^2$

- *Hopping parameter* κ is fixed through $(d + \frac{1}{2}(ma)^2)\kappa + 2\lambda = 1$

$$S_{\text{latt}} = \sum_x \left[-\kappa \sum_\mu \varphi_x \varphi_{x+\mu} + \varphi_x^2 + \lambda(\varphi_x^2 - 1)^2 \right] = \sum_x \left[-\kappa \sum_\mu \varphi_x \varphi_{x+\mu} + u(\varphi_x) \right]$$

All quantities are dimensionless.

- $g \to \infty$: Ising model
- *Naive* continuum limit: $\kappa = \frac{1}{d} - \frac{2}{d}\lambda$.

- "High-temperature expansion": expand around $\kappa = 0$:

$$Z = \int [\prod_x d\varphi_x e^{-u(\varphi_x)}] \prod_{<xy>} e^{\kappa \varphi_x \varphi_y}$$

- Exactly as for the Ising model, we can write the last part as a sum over sets of links, *graphs* G:

$$\begin{aligned}
\prod_{<xy>} e^{\kappa\varphi_x\varphi_y} &= \prod_{<xy>} \sum_i \frac{1}{i!} \kappa^i \varphi_x^i \varphi_y^i \\
&= \sum_G \kappa^{L(G)} \prod_{<xy>\in G} \frac{1}{m_{xy}(G)!} (\varphi_x\varphi_y)^{m_{xy}(G)} \\
&= \sum_G \kappa^{L(G)} c(G) \prod_x \varphi_x^{N_G(x)}
\end{aligned}$$

- $m_{xy}(G)$: the number of times link $< ij >$ is included in G
- $c(G) \equiv \prod_{<xy>} \frac{1}{m_{xy}(G)!}$
- $N_G(x)$: # of links going to point x
- Defining

$$Z_1 = \int d\varphi e^{-u(\varphi)}, \qquad \gamma_k = \langle \varphi^k \rangle_1 = \frac{1}{Z_1} \int d\varphi \; \varphi^k e^{-u(\varphi)}$$

  we get

$$Z = Z_1^V \sum_G \kappa^{L(G)} c(G) \prod_{x \in G} \gamma_{N_G(x)}$$

- Since $\gamma_k = 0$ for odd k, we get exactly the same closed graphs as for the Ising model.

$$\begin{aligned} \frac{Z}{Z_1^V} &= 1 + \kappa^2 V d \frac{1}{2} \gamma_2^2 + \kappa^4 \left[V d \frac{1}{4!} \gamma_4^2 + V d (2d-1) \frac{1}{(2!)^2} \gamma_4 \gamma_2^2 \right. \\ &+ \left. \tfrac{1}{2} V d (d-1) \gamma_2^4 + \tfrac{1}{2} V d (V d - 4d + 1) \frac{1}{(2!)^2} \gamma_4^2 \right] + O(\kappa^6) \end{aligned}$$

- “Feynman rules” for Z:
  1. Draw graphs of length L
  2. Each link: $1/m!$
  3. Each point: γ_N
- Again, free energy $F = -\log Z$ contains only connected graphs.
- Various quantities calculated up to 14th order [M.Lüscher, P.Weisz, Nucl.Phys.B 295 (1988)]