Some character expansions and group integrals

First, let us look at $U(1)$:

$$e^{\beta \mathrm{Re}\, e^{i\theta}} = e^{\beta\cos\theta} = \sum_n c_n e^{in\theta}$$

$$\Rightarrow \quad c_n = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, e^{-in\theta}\, e^{\beta\cos\theta}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta \left[\cos(n\theta) - i\,\underbrace{\sin(n\theta)}_{\to 0}\right] e^{\beta\cos\theta}$$

(symmetry, e.g. switch $\int_{-\pi}^{\pi}$)

$$= \frac{1}{\pi}\int_0^{\pi} d\theta\, \cos(n\theta)\, e^{\beta\cos\theta} = I_n(\beta).$$

as we had before.

SU(2):

is real for SU(2)

$$\text{Now we have} \quad e^{\beta\frac{1}{2}\mathrm{Re}\,\mathrm{Tr}\, U} = \sum_R{}' c_R \chi_R(U)$$

$$= c_1 + c_2 \mathrm{Tr}(U) + \ldots$$

trivial rep. fundamental rep.

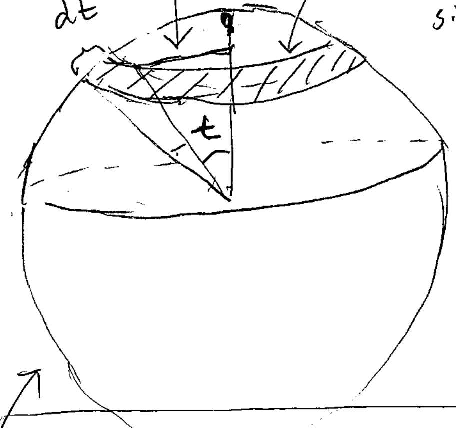
Let us use $\int dU\, \chi_R^*(U)\, \chi_{R'}(U) = \delta_{RR'}$

$$\Rightarrow \quad c_1 = \int DU\, e^{\beta\frac{1}{2}\mathrm{Tr}\, U} \cdot \underbrace{1}_{\chi_1^*}$$

How to parametrize $\mathcal{D}U$?

Recall $U = \cos(t)\,\mathbb{1} + \sin(t)\,\bar{\sigma}\cdot\bar{n}$

↑ 3d unit vector



Thus, if integrand has no $\bar{n}$-dependence,

$$dU = dt \cdot 4\pi \sin^2 t$$

Thus, $$c_1 = \int_0^{\pi} dt \cdot 4\pi \sin^2 t \, e^{\beta \cos t} \qquad (\mathrm{Tr}\,U = 2\cos t)$$

$$= -\Big|_0^{\pi} 4\pi \sin t \,\frac{1}{\beta} e^{\beta\cos t} + \int_0^{\pi} dt \cdot 4\pi \frac{1}{\beta} \cos t \, e^{\beta \cos t}$$

$$= \frac{4\pi^2}{\beta} I_1(\beta)$$

And, $$c_2 = \int \mathcal{D}U \, e^{\beta \frac{1}{2}\mathrm{Tr}U}\,\mathrm{Tr}\,U = 4\pi \int_0^{\pi} dt \, \underbrace{\sin^2 t \cos t}_{\frac{1}{2}\sin 2t \sin t} \, e^{\beta \cos t}$$

$$= -4\pi \Big|_0^{\pi} \frac{1}{2}\sin 2t \,\frac{1}{\beta} e^{\beta\cos t} +$$

$$+ \frac{4\pi}{\beta}\int_0^{\pi} dt \cos 2t \, e^{\beta\cos t} = \frac{4\pi^2}{\beta} I_2(\beta)$$

Thus, $$e^{\beta\frac{1}{2}\mathrm{Tr}U} = c_1\left(1 + f_1 \mathrm{Tr}\,U + \text{(higher order)}\right)$$

where $$f_1 = \frac{I_2(\beta)}{I_1(\beta)}.$$