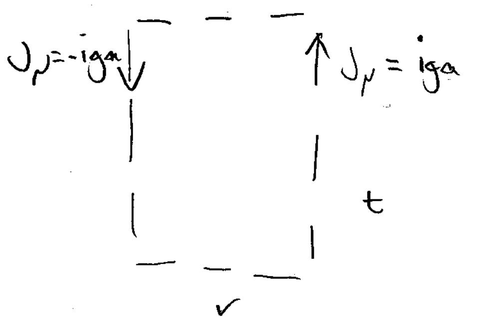


let us write $J = ia$ along the Wilson loop, otherwise zero:

$$Z[J] = \int dU \, e^{-S + g\sum_{x,\mu} J_\mu A_\mu}$$

$$= \int dU \, W \, e^{-S}$$

with $J = ia$

$$e^{gJ_\mu A_\mu} = e^{igaA_\mu} = U_\mu$$

- Now $V(\bar{r}) = -\lim_{t\to\infty} \frac{1}{t} \ln W(r,t) = -\lim_{t\to\infty} \frac{1}{t} \ln \frac{Z[J]}{Z[0]}$

- Fix to Feynman gauge: $\xi = 1$,

$$D_{\mu\nu}(p) = \frac{\delta_{\mu\nu}}{\hat{p}^2} = \delta_{\mu\nu} \frac{a^2}{\sum_\mu 4\sin^2\frac{p_\mu a}{2}}$$

- In x-space:

$$D_{\mu\nu}(x-y) = \delta_{\mu\nu} \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4} \frac{a^2}{\sum_\mu 4\sin^2\frac{p_\mu a}{2}} e^{ip(x-y)}$$

$$\Rightarrow \delta_{\mu\nu} \frac{1}{4\pi^2 (x-y)^2}, \quad \frac{(x-y)^2}{a^2} \to \infty$$

(either $a \to 0$ or $x-y \to \infty$)

- Note: in compact formulation,

$$S_L = \sum_{x\mu\nu} a^4 \left(\frac{1}{4} [F_{\mu\nu}(x)]^2 - \frac{1}{48} g^2 a^2 [F_{\mu\nu}(x)]^4 + \dots \right)$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Theory is <u>not</u> free (except at $a \to 0$!)

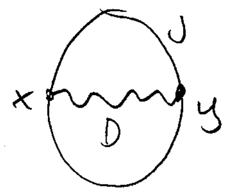
- To leading order in g^2 we obtain, as in free theory,

$$Z[J] = e^{\frac{1}{2} g^2 J_\mu D_{\mu\nu} J_\nu} Z[0]$$

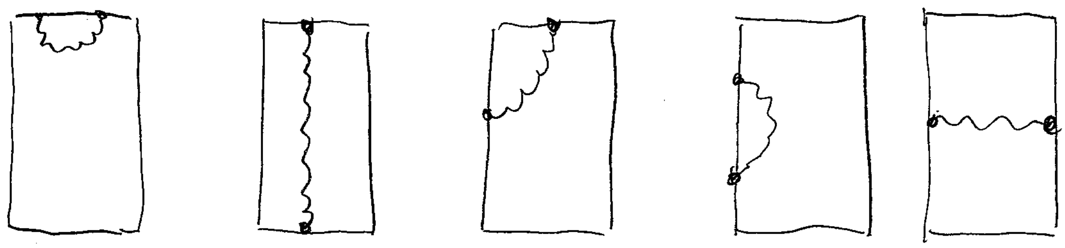
(here $J_\mu D_{\mu\nu} J_\nu \equiv \sum_{xy} J_\mu(x) D_{\mu\nu}(x-y) J_\nu(y)$)

- $$V(r) = -\lim_{t\to\infty} \frac{1}{2}\frac{g^2}{t} \sum_{xy} J_\mu(x) D_{\mu\nu}(x-y) J_\nu(y)$$

This corresponds to diagram



For Wilson loop, we get contributions



A B C D E

D is self-energy contribution:

$$\frac{1}{2} g^2 \sum_D{}' JDJ = \frac{1}{2} g^2 i^2 \sum_{x_0, y_0 = -t/2}^{t/2}{}' a^2 D_{00}(\bar{0}, x_0 - y_0)$$

t large; depends only on $x_0 - y_0$:

$$\simeq -\frac{1}{2} g^2 \frac{t}{a} \sum_{y=-\infty}^{\infty} a^2 \int_{-\pi/a}^{\pi/a} \frac{d^4 p}{(2\pi)^4} e^{ip_0 y} \frac{1}{\hat{p}^2}$$

# of links to t-dir.

$$= -\frac{1}{2} g^2 t \int_{-\pi/a}^{\pi/a} \frac{d^3\bar{p}}{(2\pi)^3} \frac{a^2}{\sum_i 4\sin^2\frac{p_i a}{2}}$$

$$= -\frac{1}{2} g^2 t \, v(0)$$

where $v(\bar{x}) = \int_{-\pi/a}^{\pi/a} \frac{d^3\bar{p}}{(2\pi)^3} \frac{1}{\hat{p}_i^2} e^{i\bar{p}\cdot\bar{x}}$

is the lattice regulated Coulomb potential.

$v(\bar{x}) \to \frac{1}{4\pi|\bar{x}|}, \quad a \to 0.$

Now $\underline{v(\bar{0}) \simeq 0.253\ldots \frac{1}{a}}$ can be summed numerically.

$\underline{D}$ does not give r-dep. contribution to $V(\bar{r})$, just a constant.

$\underline{E}$: $\frac{1}{2} g^2 \sum_E JDJ \simeq \frac{1}{2} g^2 \, i^2 \sum_{x_0, y_0 = -t/2}^{t/2} a^2 D_{00}(r, x_0 - y_0)$

$$\simeq \frac{1}{2} g^2 \int_{-t/2}^{t/2} dx_0\, dy_0 \frac{1}{4\pi^2 (r^2 + (x_0 - y_0)^2)} \quad \text{if } \underline{\underline{r \gg a}}$$

$$\simeq \frac{1}{2} g^2 t \int_{-\infty}^{\infty} dy_0 \frac{1}{4\pi^2 (r^2 + y_0^2)}$$

$$= \frac{1}{2} g^2 t \frac{1}{4\pi^2 r} \underbrace{\int_{-\infty}^{\infty} dx \frac{1}{1+x^2}}_{\arctan x \,\big|_{-\infty}^{\infty} = \pi} = \frac{1}{2} g^2 t \frac{1}{4\pi} \frac{1}{r}.$$

$\underline{B}$ must hence be $g^2 r \frac{1}{4\pi}\frac{1}{t}$; does not grow with t

and $\lim \frac{1}{t} B \to 0$.

$\underline{A} \simeq -\frac{1}{2} g^2 r\, v(0) \to$ does not grow with t

$\underline{C} \propto D_{0i} = 0$

Thus, diagrams D and E contribute as $t \to \infty$. For both of these we get contribution when $x \Leftrightarrow y$, thus, full

$$V(r) = -\lim_{t\to\infty} \frac{1}{t}(2D + 2E) = g^2\left(v(0) - v(r)\right)$$

Coulomb potential on the lattice!

- Compact $U(1)$ is interacting because of $g^2 a^2$ + higher -terms on page 190 in S
- expanding e^{-S}, these lead to interactions of form  n legs $\sim (ga)^{n-2}$

- These are of higher order in a, however, that can be compensated by $\frac{1}{a}$ - divergences in loops! Must be cheked.

Eg. diagram

leads to 'vacuum polarization tensor'

$$\Pi_{\mu\nu} = -\tfrac{1}{4} g^2 (\delta_{\mu\nu} p^2 - p_\mu p_\nu) + O(a^2)$$

and

$$D^{-1}_{\mu\nu} \to D^{-1}_{\mu\nu} = p^2 \delta_{\mu\nu} + \Pi_{\mu\nu} + O(a^2)$$

$$D_{\mu\nu} \to D_{\mu\nu} = Z(g^2)\, \delta_{\mu\nu} \frac{1}{p^2} + \# p_\mu p_\nu + O(a^2)$$

$$Z(g^2) = \left(1 - \tfrac{1}{4} g^2 + O(g^4)\right)^{-1}$$

However, turns out that terms $\propto p_\mu p_\nu$ do not contribute to Wilson loop, and more detailed analysis shows that there are no other weak coupling effects.

Thus, at weak coupling we obtain

$$V(r) = -g^2 Z(g^2) \frac{1}{4\pi r} + \text{const} + O(a^2)$$

In compact $U(1)$. $\alpha = \frac{e^2}{4\pi} = \frac{g^2 Z(g)}{4\pi}$

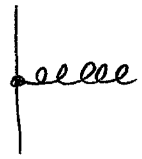
Coulomb!

In non-compact $U(1)$ there are no $(ga)^n$ - contribution and $Z(g^2) = 1$ $(g^2 = e^2)$. Free theory.

SU(N) Gauge theory and Wilson loops

Modifications:

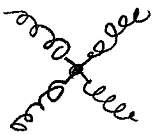
a) Color summation $\Rightarrow C_2$ to Wilson line-gluon vertex

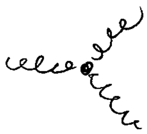


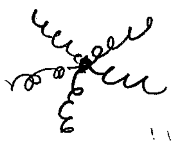
C_2 : quadratic Casimir (in the representation of the line)

If we consider fundamental rep. (e.g. quarks) line, $C_2 = \frac{N^2-1}{2N} = \frac{8}{6}$ for $SU(3)$

b) Gluon-gluon vertices :

 $\sim g^2 + O(g^2 a^2)$

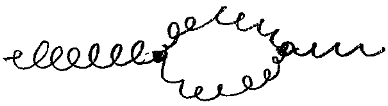
 $\sim gp + O(g a^2)$ $\quad$ $O(a^2)$ quite messy!

 $\sim O(g^{n-2} a^2)$

Outcome:

$$F(r) = \frac{\partial V(r)}{\partial r} = \frac{1}{4\pi r^2} C_2 \left[g^2 + \frac{11N}{48\pi^2} g^4 \left(\ln \frac{r^2}{a^2} + c_0 \right) + O(g^6) \right]$$

Coulomb with corrections. Log comes from

 diverges as $a \to 0$.

Divergence needs to be renormalized.

Simple prescription: define g_R through

$$F(r) = \frac{C_2 g_R^2}{4\pi d^2} \qquad d: \text{ fixed reference scale}$$

$$\Rightarrow \quad g_R^2 = g^2 + g^4 \frac{11N}{48\pi^2}\left[\ln\frac{d^2}{a^2} + C_0\right] + \dots$$

$$\text{or} \quad g^2 = g_R^2 - g_R^4 \frac{11N}{48\pi^2}\left[\ln\frac{d^2}{a^2} + C_0\right] + \dots$$

Thus, if g_R^2 is fixed, g^2 must depend on a.

Now

$$F(r) = \frac{1}{4\pi r^2} C_2 \left[g_R^2 + \frac{11N}{48\pi^2} g_R^4 \ln\frac{r^2}{d^2} + O(g_R^6)\right]$$

No a-dependence remains (and the lattice constant C_0 vanishes)

Log still blows up when $r \to 0$ or $r \to \infty$,
and expansion is not reliable there.
Only near $r \sim d$.

Renormalization group and asymptotic freedom

- Let us fix $\nu = d$ (for simplicity)

Now define $\beta_R(g_R) \equiv -\nu \frac{\partial}{\partial \nu} g_R$

$$\beta(g) \equiv -a \frac{\partial}{\partial a} g$$

(g_R depends only on ν and g; g depends on a and g_R ...)

$$\beta_R(g_R) = -\frac{11N}{48\pi^2} g_R^3 + \dots = \beta_1 g_R^3 + \underset{\text{2-loop comp.}}{\beta_2} g_R^5 + \dots$$

$$\beta(g) = -\frac{11N}{48\pi^2} g^3 + \dots = \underbrace{\beta_1 g^3 + \beta_2 g^5}_{\text{2 first terms are equal!}} + \dots$$

The equality is seen from

$$g_R = g + Ag^3 + Bg^5 + \dots$$
$$g = g_R - Ag_R^3 + \dots$$

($g = g_R$ at tree level!)

If we change $a \to \lambda a$; $\nu \to \lambda \nu$, and use $g_R = g_R(g, \frac{\nu}{a})$

$$\beta_R(g_R) = \frac{\partial g_R}{\partial \lambda} = \frac{\partial g_R}{\partial g}\frac{\partial g}{\partial \lambda} = \frac{\partial g_R}{\partial g}\beta(g)$$

$$= (1 + 3Ag^2 + 5Bg^4 + \dots)(\beta_1 g^3 + \beta_2 g^5 + \dots)$$

$$= (1 + 3Ag_R^2 + O(g_R^4))(\beta_1 g_R^3 - \beta_1 3A g_R^5 + \beta_2 g_R^5 + O(g_R^7))$$

$$= \beta_1 g_R^3 + \beta_2 g_R^5 + O(g_R^7)$$

This holds for any scheme; thus, we can use continuum dimensional regularization and $\beta_1 = \frac{11N}{48\pi^2}$; $\beta_2 = \frac{102}{121}\beta_1$

Asymptotic freedom: $\beta_R(g_R) < 0$ when g_R small
$\Rightarrow$ $g_R \to 0$ as $r \to 0$

Integrate RG eqn

$$r\frac{\partial g_R}{\partial r} = -\beta_R(g_R) \Rightarrow -\ln r = \int^{g_R} \frac{dx}{\beta_R(x)}$$

$$= \int^{g_R} dx \left[\frac{-1}{\beta_1 x^3} + \frac{\beta_2}{\beta_1^2 x} + O(x)\right]$$

$$= \frac{1}{2\beta_1 g_R^2} + \frac{\beta_2}{\beta_1^2}\ln g_R + C + O(g_R^2)$$

Shuffling constants, we get

$$-\ln(r^2\Lambda_V^2) = \frac{1}{\beta_1 g_R^2} + \frac{\beta_2}{\beta_1^2}\ln(\beta_1 g_R^2) + O(g_R^2)$$

integration const,

Λ in V-scheme

convention

Inverting this we get $\quad s \equiv -\ln(r^2\Lambda_V^2)$

$$\beta_1 g_R^2 = \frac{1}{s} - \frac{\beta_2}{\beta_1^2}\frac{1}{s^2}\ln s + \underbrace{O(s^{-3}\ln s)}_{O(g_R^{-6})}$$

This now gives $g_R^2(r)$, and now we see that $g_R^2(r\to 0) = 0$ without problems (compare w. page 196)

Inserting to $F(r) = \dfrac{C_2 g_R^2}{4\pi r^2}$ $\quad (d=r)$

$$\underline{F(r) = \frac{C_2}{4\pi r^2}\frac{\beta_1^{-1}}{s + \frac{\beta_2}{\beta_1^2}\frac{1}{s}\ln s}}$$

at very short distances $\quad F_V = \dfrac{C_2}{4\pi r^2}\dfrac{\beta_1^{-1}}{\ln\frac{1}{r^2\Lambda_V^2}}$

What was the difference with p. 96? There d was fixed reference scale; here we let $d=r$ to change with r: "RG improved" potential.

For bare coupling we obtain

$$\underline{-\ln(a^2\Lambda_L^2) = \frac{1}{\beta_1 g^2} + \frac{\beta_2}{\beta_1^2}\ln(\beta_1 g^2) + O(g^2)} \qquad \Lambda_L: \text{lattice } \Lambda$$

$$\Rightarrow \beta_1 g^2 \simeq \frac{-1}{\ln a^2\Lambda_L^2} \to 0 \quad , \text{as } a\to 0$$

Often this is written as

$$a^2 \Lambda_L^2 = (\beta_1 g^2)^{-\beta_2/\beta_1^2} e^{-1/\beta_1 g^2} \left(1 + O(g^2)\right)$$

Using $g = g(g_R^2, \frac{v}{a})$ (page 196)

and letting $a, v \to 0$ a/v fixed we get

$$\Lambda_L^2 = \Lambda_V^2 e^{-c_0}$$

; c_0 depends on details of regularization

Ratios of Λ's can be computed in different schemes

$$\frac{\Lambda_{\overline{MS}}}{\Lambda_L} = e^{\left(\frac{1}{16N} - 0.0849\ldots N\right)\frac{1}{\beta_1}} \simeq 28.81 \qquad SU(3)$$

$$\frac{\Lambda_V}{\Lambda_L} \simeq 30.19 \qquad SU(3)$$