Expanding the kinetic term

$$\frac{f^2}{4}\mathrm{Tr}[\partial_\mu U^\dagger \partial_\mu U] \rightarrow \frac{f^2}{4}\partial_\mu \alpha_{fg}\partial_\mu \alpha_{gf} + \ldots$$

$$= \frac{f^2}{4}\Big(2\sum_{f<g}\partial_\mu \alpha^*_{fg}\partial_\mu \alpha_{fg} + \sum_f \partial_\mu \alpha_{ff}\partial_\mu \alpha_{ff}\Big) + \ldots$$

Thus, we recognize effective mass (scalar) terms

$$\begin{cases} m^2_{ff} = \frac{4v}{f^2} m_f \\ m^2_{fg} = \frac{2v}{f^2}(m_f + m_g) \quad \left(= \frac{4v}{f^2} m_f, \text{ when } f=g\right) \end{cases}$$

- If we couple our effective action to electroweak fields, it can be shown that f determines the leptonic decays of the pseudoscalar mesons. $f = f_\pi$, Pion decay constant,

$$f_\pi = 93 \text{ MeV}.$$

It is related to the (bare) chiral condensate

$$\langle\bar\psi\psi\rangle = \sum_f \langle\bar\psi_f \psi_f\rangle = \sum_f \langle \bar\psi_f (P_L + P_R)\psi_f\rangle =$$

$$= \sum_f \langle \phi_{ff} + \phi^*_{ff}\rangle = 2\sum_f \langle\phi_{ff}\rangle = 2N_f v$$

If we write effective field for pseudoscalars, $P_{fg} \equiv i(\phi^+ - \phi)_{fg} = i\bar{\psi}_f \gamma_5 \psi_g$, and defining pseudoscalar wave function renormalization through

$$Z \, \partial_\nu P^* \partial_\nu P$$

$$\Rightarrow Z^{-1} = f^2/8v^2 \Rightarrow f = \frac{\sqrt{8}\, v}{\sqrt{Z}} = \frac{\sqrt{2} \langle \bar{\psi} \psi \rangle}{N_f \sqrt{Z}}$$

Pseudoscalar masses:

$\pi^\pm$: $m^2_{\pi^+} \simeq 0.0195$ GeV^2

$K^\pm$: $m^2_{K^+} \simeq 0.244$ GeV^2

$K^0, \bar{K}^0$: $m_{K^0} \simeq 0.248$ GeV^2

π^0: $m^2_{\pi^0} \simeq 0.0182$ GeV^2

η: $m^2_\eta \simeq 0.301$ GeV^2

η': $m^2_{\eta'} \simeq 0.917$ GeV^2

Do these fit? Things work OK for non-isosinglets $(f \neq g)$, but not for $f = g$.

Take $N_f = 3$. Now we can "guess" that $\pi^+ = u\bar{d}$; $K^+ = u\bar{s}$, $K^0 = d\bar{s}$.

Thus, $$\frac{m^2_{\pi^+}}{m^2_{K^+}} \simeq \frac{B(m_u + m_d)}{B(m_u + m_s)} = \frac{m_u + m_d}{m_u + m_s} = A$$

$$\frac{m^2_{K^+}}{m^2_{K^0}} = \frac{m_u + m_s}{m_d + m_s} \simeq B$$

$$\Rightarrow \frac{\frac{m_u}{m_s} + \frac{m_d}{m_s}}{\frac{m_u}{m_s} + 1} = A \quad , \quad \frac{\frac{m_u}{m_s} + 1}{\frac{m_d}{m_s} + 1} \simeq B$$

$$\Rightarrow \frac{m_u}{m_s}(1 - A) = A - \frac{m_d}{m_s} \quad , \quad \frac{m_d}{m_s} + 1 = \frac{1}{B}\left(\frac{m_u}{m_s} + 1\right)$$

$$\Rightarrow \frac{m_u}{m_s}(1 - A) = A - \frac{1}{B}\left(\frac{m_u}{m_s} + 1\right) + 1$$

$$\Rightarrow \frac{m_u}{m_s} = \frac{A - \frac{1}{B} + 1}{1 - (A - \frac{1}{B})} \simeq \frac{1}{31} \; ;$$

$$\frac{m_d}{m_s} \simeq \frac{1}{20} \quad \text{OR} \quad \frac{m_u}{m_d} \simeq \frac{2}{3} .$$

This just gives the quark mass ratios, but does not predict anything (m_q : directly unobservable).

If we take other states,

$$m^2_{uu} = \frac{2m_u}{m_u + m_d} m^2_{\pi^+} \simeq 0.0155 \text{ GeV}^2$$

$$m^2_{dd} = \frac{2m_d}{m_u + m_d} m^2_{\pi^+} \simeq 0.0235 \text{ GeV}^2$$

$$m^2_{ss} \simeq \frac{2m_s}{m_u + m_d} m^2_{\pi^+} \simeq 0.473 \text{ GeV}^2$$

These do not fit to anything.

Model fails.

This is due to the U(1) chiral (axial) anomaly:
Our $U \in U(N_f)$, and the effective action is symmetric wrt. $U(1)_A$:

This corresponds to
$$V_L = V_R^+ = e^{i\omega} \mathbb{1} \qquad (\text{or, rather, } \det V_L \neq 1)$$

$$U \to V_L U V_R^+ \; ; \; U^+ \to V_R U^+ V_L^+$$
is invariant (if $m = 0$).

- We can break $U(1)_A$ explicitly by adding a term which is not invariant:

$$\det U \to \det V_L U V_R^+ = \det U \det V_L V_R^+$$
$$= \det U$$

if $V_L = V_R$ (vector) or $\det V_L = \det V_R = 1$

$U(N_f)_A \to SU(N_f)_A$

$U(1)_A$ broken.

Thus, we can add a term

$$\Delta S = \int d^4x \, c \, (\det U + \det U^+)$$

to our effective action

Or, often $(\ln \det U - \ln \det U^+)^2$ is used, motivated by large-N_c QCD)

Using $\det U = e^{\mathrm{Tr} \ln U}$

and $U = e^{i\alpha}$

$\Rightarrow \det U \simeq e^{i \sum \alpha_{ff}} \simeq 1 + i \sum \alpha_{ff} - \frac{1}{2} (\sum \alpha_{ff})^2 + \ldots$

or

$$\Delta S \simeq -\int d^4x \, C \left(\sum_f \alpha_{ff} \right)^2$$

Thus, m^2_{fg}, $f \neq g$ are unaffected, but the "diagonal" terms have effective action

$$S = \int d^4x \left[\frac{1}{2} \sum_f \partial_\nu \alpha_{ff} \partial_\nu \alpha_{ff} + \frac{1}{2} \sum_f m^2_{ff} \alpha^2_{ff} + \frac{\lambda}{2} \left(\sum_f \alpha_{ff} \right)^2 \right]$$

or, the effective mass matrix is

$$m^2 = \frac{4v}{f^2} \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{bmatrix} + \lambda \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

with $\lambda = -\frac{2C}{f^2}$.

If $m_u = m_d = m_s = 0$, m^2 has eigenmodes:

$$m_3^2 = 0, \quad \phi_3 = \frac{1}{\sqrt{2}} (1, -1, 0)^T$$

$$m_8^2 = 0, \quad \phi_8 = \frac{1}{\sqrt{6}} (1, 1, -2)^T$$

$$m_0^2 = 3\lambda, \quad \phi_0 = \frac{1}{\sqrt{3}} (1, 1, 1)^T$$

These correspond to

$$\phi_3 \triangleq \frac{1}{\sqrt{2}}(u\bar{u}-d\bar{d}) \quad \sim \quad \pi^0$$

$$\phi_8 \triangleq \frac{1}{\sqrt{6}}(u\bar{u}+d\bar{d}-2s\bar{s}) \quad \sim \quad \eta$$

$$\phi_0 \triangleq \frac{1}{\sqrt{3}}(u\bar{u}+d\bar{d}+s\bar{s}) \quad \sim \quad \eta'$$

Thus, even when $m_f \to 0$, η'-pseudoscalar remains massive!

Let us now add quark masses. Without diagonalizing the full matrix, we can treat m_f's as small perturbations: $m_i^2 \simeq \phi_i^T m \phi_i$

$$m_{\pi^0}^2 \simeq \frac{2v}{f^2}(m_u+m_d) \quad = m_{\pi^+}^2 \quad \simeq \quad 0.0195 \text{ GeV}^2$$

$$m_\eta^2 = \frac{2v}{f^2}\frac{1}{3}(m_u+m_d+4m_s) = \frac{2}{3}(m_{K^+}^2+m_{K^0}^2)-\frac{1}{3}m_{\pi^+}^2$$
$$\simeq 0.3245 \text{ GeV}^2$$

$$m_{\eta'}^2 = 3\lambda + \frac{2v}{f^2}\frac{2}{3}(m_u+m_d+m_s)$$
$$= 3\lambda + \frac{1}{2}(m_{\pi^0}^2 + m_\eta^2)$$
$$\Rightarrow \lambda \simeq 0.252 \text{ GeV}^2$$

$\leftarrow$ Characterizes $U(1)_A$ breaking.

π_0, η-masses are <u>predictions</u>, λ is fitted to $m_{\eta'}^2$.

- When $m_q = 0$, we chose ϕ_3, ϕ_8 in a particular way. Naturally we could have chosen *any* orthogonal combo.

Why this?

- when $m_u \simeq m_d \ll m_s$, the eigenmodes are close to these vectors.
- True diagonalisation reveals "better" eigenmodes, small difference between $m_{\pi^+}^2, m_{\pi^0}^2$.

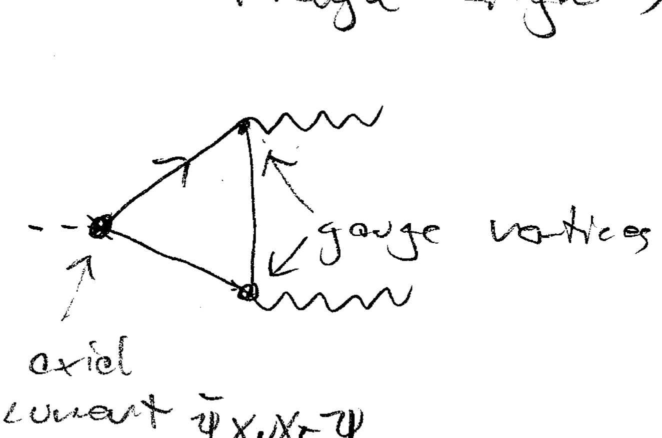
Axial anomaly and topology

- Anomaly appears in non-conservation of

$$j_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi$$

$$\partial_\mu j_\mu = A$$

- In QCD, diagrammatically the anomaly appears in triangle diagrams



In Euclidean spacetime, the anomaly is

$$\partial_\mu(\bar\psi_f \gamma_\mu \gamma_5 \psi_g) = (m_f + m_g)\bar\psi_f \gamma_5 \psi_g + \delta_{fg}\, 2q$$

not anomaly – mass breaks symmetry

anomaly

$$q = \frac{g^2}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\,\mathrm{Tr}\,F_{\mu\nu}F_{\rho\sigma}$$

q: Topological charge density

This breaks exactly $U(1)_A$; when $m=0$, r.h.s. is proportional to $\delta_{fg} = \mathbb{1}_{fg}$

$$F_{\mu\nu} = F^a_{\mu\nu}\lambda^a = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

On topologically non-trivial 4d manifold (for example torus T^4) gauge configurations can be divided into topologically distinct classes:

$$Q = \int d^4x\, q(x) \qquad \text{topological charge}$$

$$Q \in \mathbb{Z}!$$

Classical solution with $Q = \pm 1$: Instanton anti-instanton

$$S_{inst} = \frac{8\pi^2}{g^2}$$

Thus, topological charge ↔ anomaly

Atiyah-Singer index theorem:

$$Q = n_+ - n_-$$

where $n_\pm$ are the number of zero modes of $\not{D}$ with chirality $\gamma_5 = \pm 1$

Witten-Veneziano formula:

$$\lambda \simeq \frac{1}{2f_\pi^2}\chi_{top} \quad ; \quad \chi_{top} = \int d^4x \langle q(0)\, q(x)\rangle$$

$U(1)_A$ breaking mass

topological susceptibility.

$$\chi_{top} \simeq (180\ \mathrm{MeV})^4$$

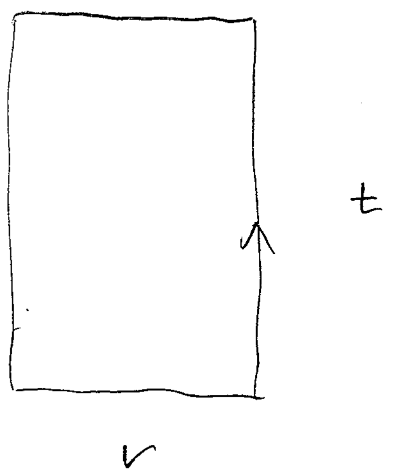
- All this is valid in continuum.

Wilson fermion action does <u>not</u> have chiral symmetry at $a>0$; thus, studying chiral phenomena is problematic (but possible).

Much easier with overlap or domain wall.

Lattice perturbation theory and Coulomb potential

We already looked at the Wilson loop at strong coupling:



potential

$$V(r) = -\lim_{t\to\infty} \frac{1}{t} \ln W(r,t)$$

or $W = e^{-tV(r)}$, t large

At strong coupling: $V(r) \sim \sigma r$, "Area law"

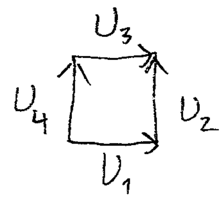
$$W \sim e^{-\sigma r t}$$

• Consider $U(1)$:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

On the lattice: $$S_L = \frac{1}{g^2} \sum_\Box \big(\overbrace{1 - \mathrm{Re}\, U_\Box}^{\cos\theta_\Box}\big)$$

$$U_\Box = U_1 U_2 U_3^+ U_4^+$$



$$U_\mu(x) = e^{igaA_\mu}$$

$$\theta_\Box = ga(A_1 + A_2 - A_3 - A_4)$$

"compact" $U(1)$; non-compact: $S = \frac{1}{g^2} \sum_\Box \frac{1}{2} \theta_\Box^2$

- EM source terms:

$$Z = \int dU \, e^{-S + \int d^4x \, J_\mu A_\mu}$$

- On the lattice $\int d^4x \, J_\mu A_\mu \to \sum_{x,\mu} J_\mu(x) \, a A_\mu(x)$

where $J_\mu(x)$ describes current along link $x;\mu$.

- Describes Wilson loop with $J_\mu(x) = ig$ on links along the loop, oriented to direction of the loop.

- <u>Gauge fixing</u>
  - In p.t. it is necessary to fix the gauge, even for the definition of the propagator.
  - $$S_{GF} = \int d^4x \, G(A)^2 = \frac{1}{2\xi} \int d^4x \left(\partial_\mu A_\mu\right)^2$$

    gauge parameter covariant gauge

    (cont. QFT courses)
  - Gives rise to <u>ghosts</u> in non-abelian gauge; do not play a role in U(1). Ignore.

$\xi = 0$: Landau gauge $\hat{=} \ \delta(G(A))$

$\xi = 1$: Feynman gauge

In continuum,

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{2\xi} (\partial_\mu A_\mu)^2$$

$$= \frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{1}{2\xi} (\partial_\mu A_\nu)^2$$

integr. by parts

$$= -\frac{1}{2} A_\mu \left(\partial_\rho \partial_\rho \delta_{\mu\nu} - \partial_\mu \partial_\nu \left(1 - \frac{1}{\xi}\right)\right) A_\nu$$

Thus, propagator is

$$D^{-1}_{(x)} = -\partial_\rho \partial_\rho \delta_{\mu\nu} + \partial_\mu \partial_\nu \left(1 - \frac{1}{\xi}\right) \to D^{-1}(p) = p^2 \delta_{\mu\nu} - p_\mu p_\nu \left(1 - \frac{1}{\xi}\right)$$

$\Rightarrow$

$$D(p) = \frac{1}{p^2} \left(\delta_{\mu\nu} - (1 - \xi) \frac{p_\mu p_\nu}{p^2} \right)$$

- On the lattice: $p_\mu \to \hat{p}_\mu \equiv \frac{2}{a} \sin \frac{p_\mu a}{2}$

In numerical work, often use

$$S_{GF} = \frac{1}{\xi} \sum_{x,\mu} \Big(\Delta_\mu \underbrace{\mathrm{Im}\, U_{x,\mu}}_{\sim A_\mu} \Big)^2$$

and Landau gauge ($\xi \to 0$) is obtained by minimizing the functional.

- Gribov copies: gauge which satisfies the gf-condition not necessarily unique; there can be other configurations "far" from it which also satisfy the gauge.