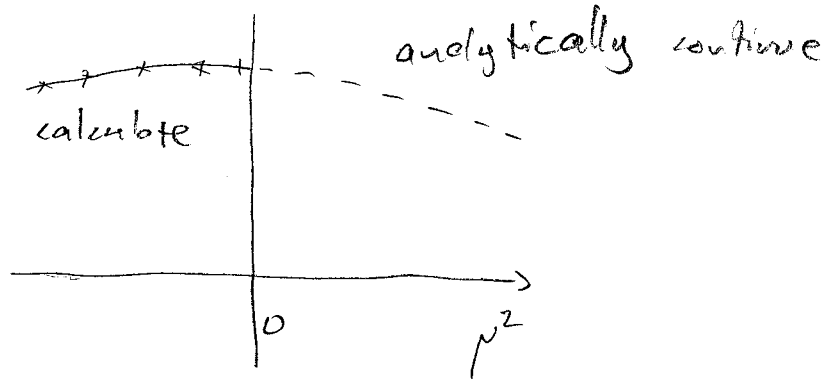
b) use $\mu \to i\mu$ imaginary?

Now $M^{+}(i\mu) = \gamma_5 M(i\mu) \gamma_5$

is γ_5-hermitean

$\Rightarrow \det M(i\mu) \in \mathbb{R}$. With 2 flavours, we get again $\det[(M(i\mu))^{+}(M(i\mu))]$ which is positive definite.

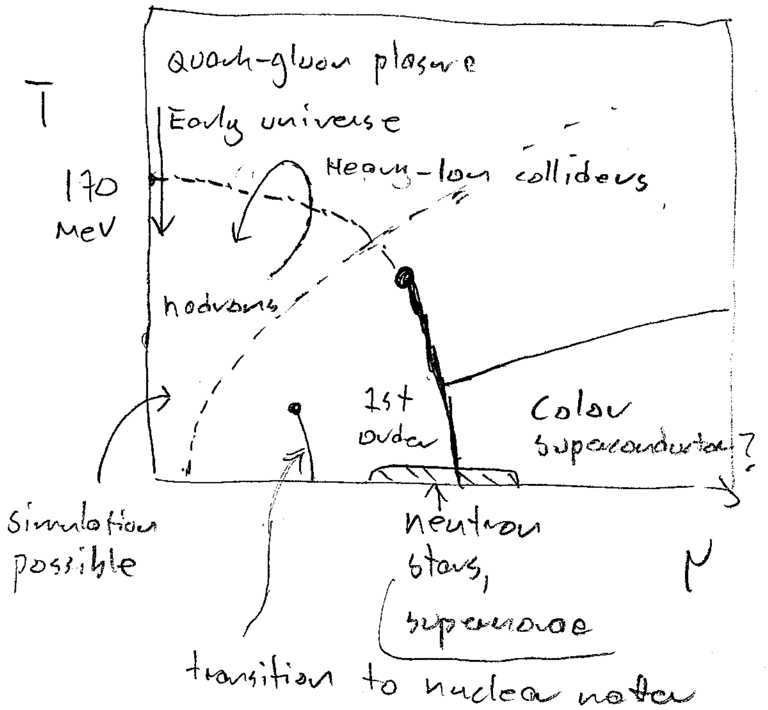
Results at $i\mu$ can be analytically continued to real μ:



Eg. the location of the cross-over T_c symmetric in $(\mu \leftrightarrow -\mu) \Rightarrow$ function of μ^2.

fit Taylor series of μ^2: $A + B\mu^2 + C\mu^4 + \ldots$

$\Rightarrow$ finite radius of convergence;
does not work at large μ



Phase diagram of QCD

- Simulations are possible only at small μ (grows as T grows)
- At very large T, μ pert. theory reliable
  - intermediate range conjectural!
- At high μ, there probably exists a color superconductor phase (cooper pairs)
  - Analogue of superconductivity in metals
  - Seen in p.t. at high μ

- Solution sorely needed!

Chiral symmetry

- Let us have a look at chiral effective theory, which gives a description of meson (and also baryon, but not as naturally) mass spectrum.

- Nothing to do w. lattice per se.

- Recall: with N_f massless fermions, we have symmetries

$$U(N_f)_V \otimes U(N_f)_A = SU(N_f)_V \otimes SU(N_f)_A \otimes U(1)_V \otimes U_1(A)$$

broken by anomaly

with V: $\psi \to e^{i\theta^a\lambda^a}\psi \, ; \, \bar\psi \to \bar\psi e^{-i\theta^a\lambda^a}$

A: $\psi \to e^{i\theta^a\lambda^a\gamma^5}\psi \, ; \, \bar\psi \to \bar\psi e^{i\theta^a\lambda^a\gamma_5}\psi$

- $m > 0$ breaks "A" symmetry explicitly.

- Often expressed with "left" and "right" symmetries:

- Define

$$P_L = \frac{1}{2}(1-\gamma_5) \quad ; \quad P_R = \frac{1}{2}(1+\gamma_5)$$

$$P_L + P_R = 1 \quad ; \quad P_L P_R = P_R P_L = 0$$

$$P_L^2 = P_L \quad ; \quad P_R^2 = P_R$$

projection operators

- Transformations can be written as

$$\begin{cases} \psi \to V\psi \\ \bar{\psi} \to \bar{\psi}\,\bar{V} \end{cases} \quad \text{with} \quad \begin{cases} V = V_L P_L + V_R P_R \\ \bar{V} = V_L^\dagger P_R + V_R^\dagger P_L \end{cases}$$ (homework

where $V_{L,R} \in U(N_f)$ (or $SU(N_f)$)

- kinetic term invariant: (here $\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{N_f} \end{pmatrix}$)

$$\bar{\psi} \not{D} \psi \to \bar{\psi}\,\bar{V}\gamma_\nu V D_\nu \psi$$

$$= \bar{\psi}\gamma_\nu (V_L^\dagger V_L + V_R^\dagger V_R) D_\nu \psi = \bar{\psi}\gamma_\nu D_\nu \psi$$

- mass term ($m \propto m\mathbb{1}$)

$$\bar{\psi} m \psi \to \bar{\psi} m (V_R^\dagger V_L P_L + V_L^\dagger V_R P_R)\psi$$

is invariant only if $V_R = V_L$ (= vector symmetry)

- Consider matrix:

$$\Phi_{fg} \equiv \bar{\psi}_g P_L \psi_f \qquad g, f = 1 .. N_f$$

or $\Phi \equiv \bar{\psi} P_L \psi$

- This transforms as $\Phi \to \bar{\psi} \bar{V} P_L V \psi$
$$= \bar{\psi} V_R^\dagger V_L \psi$$

or, in components

$$\Phi_{fg} \to \bar{\psi}_{g'} \bar{V}_{g'g} P_L V_{ff'} \psi_{f'}$$
$$= V_{L ff'} (\bar{\psi}_{g'} P_L \psi_{f'}) V^\dagger_{R g'g}$$
$$= V_{L ff'} \phi_{f'g'} V^\dagger_{g'g}$$

or
$$\underline{\Phi \to V_L \Phi V_R^\dagger .}$$

- Now $\underline{\bar{\psi}_g P_R \psi_f} = (\bar{\psi}_f P_L \psi_g)^\dagger = (\Phi_{gf})^*$
$$= \underline{\Phi^\dagger{}_{fg}}$$

- Φ, fermion bilinear, encodes the chiral symmetry

- Let us write effective action for ϕ which has the same symmetry as Φ.
  " $N_f \times N_f$ matrix

- transforms as $\phi \to V_L \phi V_R^+$, $V_{L,R} \in \begin{cases} U(N_f) \\ SU(N_f) \end{cases}$ or

- Combination $Tr((\phi^+\phi)^k)$ is invariant.

  Parity, and using only 2 derivatives, gives

$$S = \int d^4x \, Tr\left(F_2 \, \partial_\mu \phi^+ \, F_1 \, \partial_\mu \phi + G\right)$$

with $F_1 = \sum_k f_{1k} (\phi\phi^+)^k \qquad F_2 = \sum_k f_{2k} (\phi^+\phi)^k$

$$G = \sum_k g_k (\phi\phi^+)^k$$

parity demands $f_{1k} = f_{2k}$.

(under parity $\bar{\psi}_f P_L \psi_g \to \bar{\psi}_g P_R \psi_f$, and $\bar{x} \to -\bar{x}$

thus $\phi(x_0, \bar{x}) \underset{\text{Parity}}{\to} \phi^+(x_0, -\bar{x})$)

- Higher order (e.g. $(Tr())^2$) terms? neglect.

- Ground state: $\partial_\mu \phi = 0$. Minimize $Tr\, G$:

  - if λ_i are eigenvalues of $\phi^+\phi$ (Hermitean, ≥ 0)

$$\Rightarrow \quad Tr\, G = \sum_k g_k (\lambda_1^k + \ldots + \lambda_{N_f}^k)$$

- At minimum,

$$0 = \frac{\partial}{\partial \lambda_j} \operatorname{Tr} G = \sum_k g_k \, k \lambda_j^{k-1}$$

- j-independent, thus, solution is

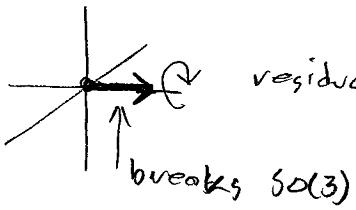
$$\lambda_1 = \dots = \lambda_{N_f} = \lambda$$

and thus $\underline{\phi^+ \phi = \lambda \mathbb{1}}$. $\lambda \geq 0$ because $\phi^+\phi$ positive.

- If now $\underline{\lambda > 0}$ (assume!), ϕ must be $\neq 0$ in the ground state. Symmetry spontaneously broken!

- What is the residual symmetry (which leaves the ground state invariant)

(e.g. $SO(3)$ rotations of vector in 3d



residual symmetry: rotations around ground state vector $= U(1)$)

- "Polar decomposition":

Def. $H = \sqrt{\phi \phi^+}$; $U = H^{-1} \phi$; now $H^+ = H$; $U^+ = U^{-1}$

$\Rightarrow \phi = H U$ $\qquad U \in U(N_f)$

- Ground state: $H = \sqrt{\lambda}\, \mathbb{1}$; $U = \mathbb{1}$.

In chiral transformations, $\phi \Rightarrow V_L U V_R^+ \Rightarrow U \to V_L U V_R^+$

If $U = \mathbb{1}$, this is invariant when $\underline{V_L = V_R}$ - vector symmetry.

- Thus, $U(N_f)_L \otimes U(N_f)_R \to U(N_f)_V$

- D.o.f's in U change direction of the ground state $\to$ Goldstone modes

- let $H = v + h$, $\quad v = \sqrt{\lambda}$

  $U = e^{i\alpha}$ $\quad \alpha \in su(N_f)$

$$\phi = (v+h)(1 + i\alpha - \tfrac{1}{2}\alpha^2 + \ldots)$$

- Approximate $F_1 = F_2 = F = F_1(v)$ from TWB

$$\Rightarrow \quad S = \int d^4x \, \mathrm{Tr}(F^2 v^2 \partial_\mu \alpha \partial_\mu \alpha + F^2 \partial_\mu h \partial_\mu h + v h^2) + \ldots$$

- There are light goldstone modes and heavier "radial" h-modes. Let us freeze heavy modes; eg. fix $H = \mathbb{1} v$:

$$\underline{S = \int d^4x \frac{f^2}{4} \mathrm{Tr}[\partial_\mu U^\dagger \partial_\mu U]} \qquad f^2 = 4F^2 v^2$$

"pseudoscalar decay constant", f_π

for $U(N_f)$, there are N_f^2 goldstones ($N_f^2 - 1$ if $U(1)$ factored out)

• Mass : in QCD,

$$S_{mass} = \int d^4x\, \bar{\psi}\, m(P_L + P_R)\psi$$

$$= \int d^4x\, \mathrm{Tr}(J^\dagger \Phi + \Phi^\dagger J) \quad , \quad J = m$$

not necessarily $\propto \mathbb{1}$.

• Thus, we obtain

$$S = \int d^4x \left\{ \frac{f^2}{4} \mathrm{Tr}\left[\partial_\mu U^\dagger \partial_\mu U\right] + v\, \mathrm{Tr}\left[m(U + U^\dagger)\right] \right\}$$

breaks symmetry!

Using $U = e^{i\alpha} \simeq 1 + i\alpha - \frac{1}{2}\alpha^2 + \ldots$

we get

$$\mathrm{Tr}\, m(U + U^\dagger) = \mathrm{Tr}\, \cancel{2m} - \mathrm{Tr}\, m\alpha^2$$

$$= - m_f \alpha_{fg} \alpha_{gf}$$

Because $\alpha^\dagger = \alpha$ (algebra), $\alpha_{gf} = \alpha_{fg}^*$

Thus, the mass term

$$v \sum_{f<g} (m_f + m_g)\, \alpha_{fg}^* \alpha_{fg} + v \sum_f m_f\, \alpha_{ff}^2$$