Fermionic observables

- Introduce fermion sources, $\phi, \bar{\phi}$: (Grassmann)

$$Z[\phi,\bar{\phi}] = \int DU\, D\psi D\bar{\psi}\, e^{-S_G - \bar{\psi}M\psi + \bar{\phi}\psi + \bar{\psi}\phi}$$

(p. 122, 123):

$$= \int DU \det M\, e^{-S_G}\, e^{-\bar{\phi}M^{-1}\phi}$$

- Thus, a correlation function

$$\underbrace{\langle \psi_x \bar{\psi}_y \rangle} \quad \text{(matrix)}$$

$$= \frac{1}{Z}\int d\, \psi_x \bar{\psi}_y e^{-S} = \frac{\delta}{\delta\bar{\phi}_x}\frac{\delta}{\delta\phi_y} \ln Z[\phi,\bar{\phi}] \Big|_{\phi,\bar{\phi}=0}$$

$$= \langle M^{-1}{}_{xy} \rangle$$

$\langle\cdot\rangle$ because need to average over U's.

$$= \overbrace{S_{xy}}, \text{ fermion prop.}$$

- Now, take e.g. field combo which describes a pion (pseudoscalar): π^+_x: $\bar{\psi}^d_x \gamma_5 \psi^u_x$

d: down
u: up,
2 diff. flavours

- Now pion-pion correlation function

$$\langle \pi^{+\dagger}_x \pi^+_y \rangle = -\langle \bar{\psi}^u_x \gamma_5 \psi^d_x\, \bar{\psi}^d_y \gamma_5 \psi^u_y \rangle$$

$$= -\langle \mathrm{Tr}(\bar{\psi}^u_x \gamma_5 \psi^d_x \bar{\psi}^d_y \gamma_5 \psi^u_y) \rangle$$

$$= +\langle \mathrm{Tr}(\gamma_5 \underbrace{\psi^d_x \bar{\psi}^d_y} \gamma_5 \underbrace{\psi^u_y \bar{\psi}^u_x}) \rangle$$

$$= +\langle \mathrm{Tr}(\gamma_5 M^{d\,-1}_{xy} \gamma_5 M^{u\,-1}_{yx}) \rangle$$

Here $M^{-1}_{xy} = \gamma_5 M^{-1*}_{yx} \gamma_5$

Thus, correlation functions can be constructed from quark propagators.

- At long distance: $\langle \pi^\dagger_x \pi_y \rangle \sim e^{-m_\pi (x-y)}$

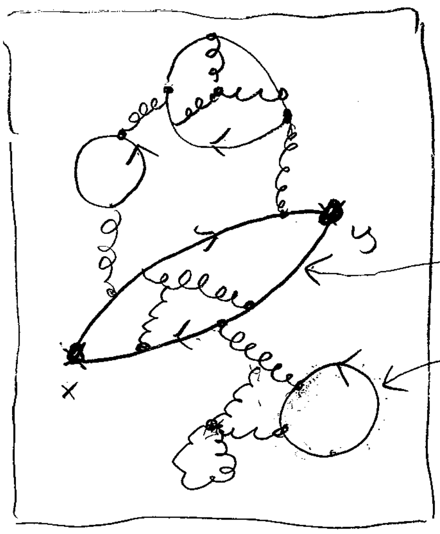
- M^{-1}_{xy} can be measured e.g. as follows:
  - set $y=0$, for concreteness.
  - Let $\phi = 0$, except $\phi^{a,\alpha}_{(y=0)} = 1$ for fixed a, α (a: color, α: Dirac)
  - Calculate $\chi^{a',\alpha'}_x = M^{-1}\phi$ using e.g. Conjugate gradient

    The elements $\chi^{a',\alpha'}_x \equiv M^{-1}{}^{a'a,\alpha'\alpha}_{x0}$ are the elements of M^{-1}. Repeat for other a, α, and construct propagators.

- Using different γ-matrices gives different mesons.
- Baryons: 3 quark sources

- Note that when propagator is calculated, it cares only about the gauge field background – not the fermion fields used in generating the fields.

- Quenched simulations : use only gauge action in generating the configurations
  ⟹ Much cheaper

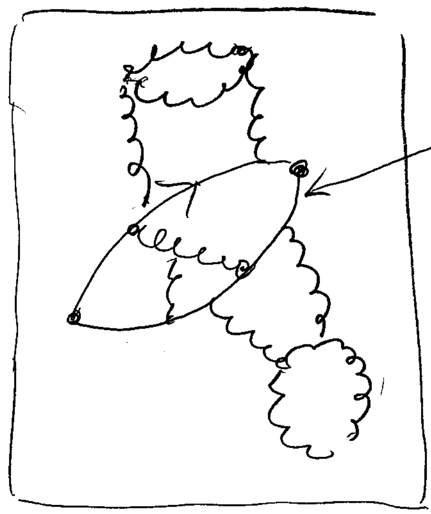


"Valence" quarks : used in propagator

"sea" quark loops

In full QCD, $m_{sea} = m_{valence}$

"Full QCD"



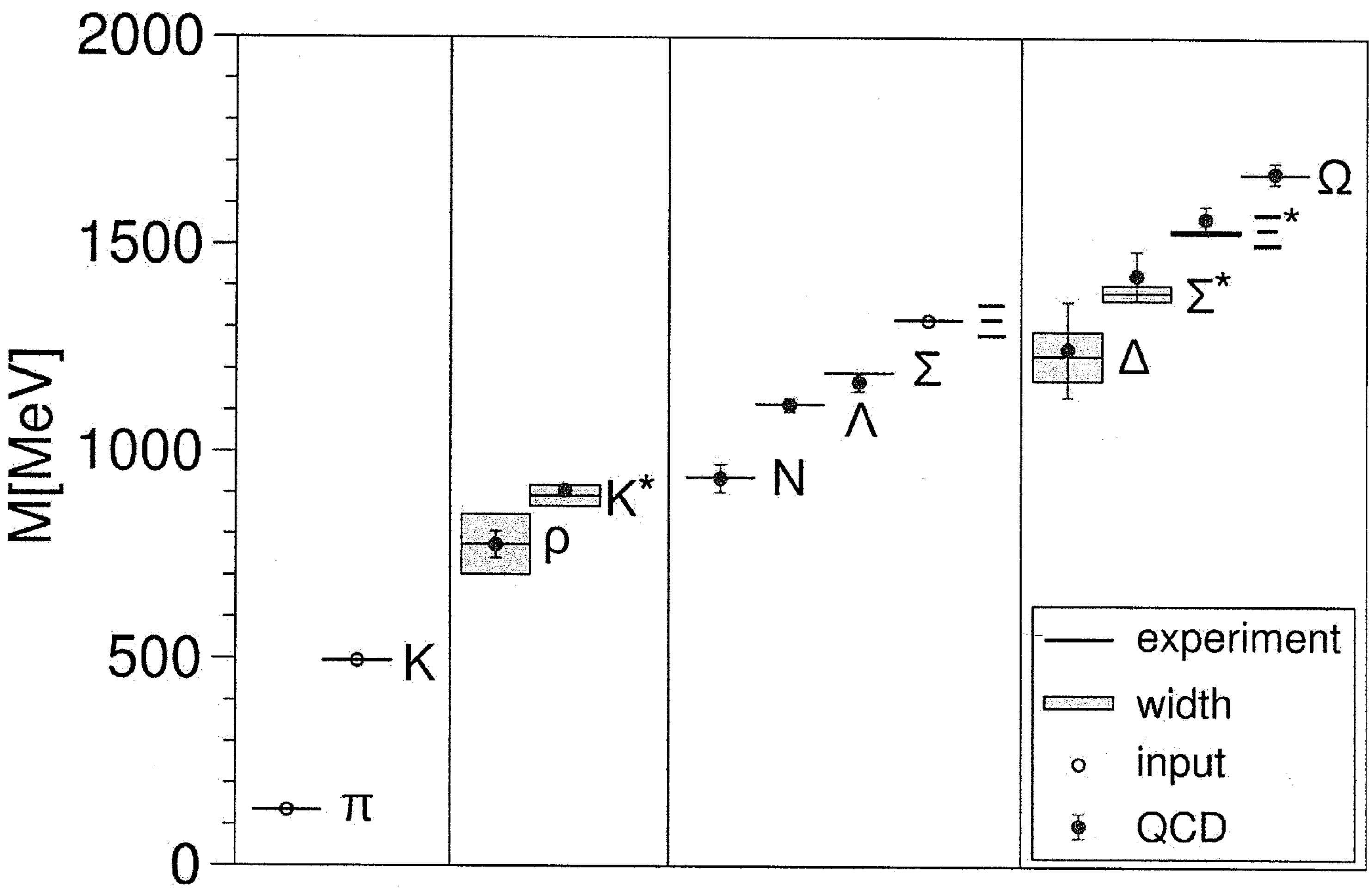
Valence

no sea quark loops!

Quenched approximation was used a lot in history, not much currently.

(not real QCD or field theory)

"Quenched QCD"



Full QCD spectroscopy:

m_π, m_K, m_Σ are used to set

$m_{u,d}$, m_s, m_c

Finite temperature QCD

- We obtained $_{\tau}\langle b|\, e^{-\tau\hat{H}}\,|a\rangle = \int D\phi\, e^{-S_E(\phi)}$

  where $S_E = \int_0^{\tau} d\tau' \int d^3\bar{x}\; \mathcal{L}_E$

  and $\phi(\bar{x},\tau'=0) = \phi_a(\bar{x}) = \langle\phi|a\rangle$

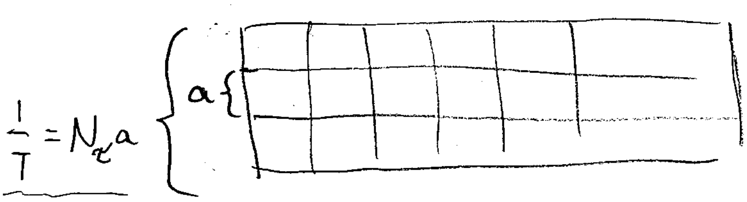
  $\phi(\bar{x},\tau'=\tau) = \phi_b(\bar{x}) = \langle\phi|b\rangle$.

- Thus, $\underline{\mathrm{Tr}\, e^{-\frac{1}{T}\hat{H}}} \equiv \sum_a \langle a|e^{-\frac{1}{T}\hat{H}}|a\rangle = \underline{\int D\phi\, e^{-S_E(\phi)}}$

  with $S_E = \int_0^{1/T} d\tau' \int d^3x\; \mathcal{L}_E(\phi)$

  with periodic boundary : $\phi(1/T) = \phi(0)$.

- Thus, finite temperature QFT can be calculated on a box (lattice) with <u>finite</u> $= \frac{1}{T}$ and <u>periodic</u> imaginary time:



T increases
$\Rightarrow$ system becomes "flatter".

- This formalism is used both in continuum <u>and</u> on the lattice.

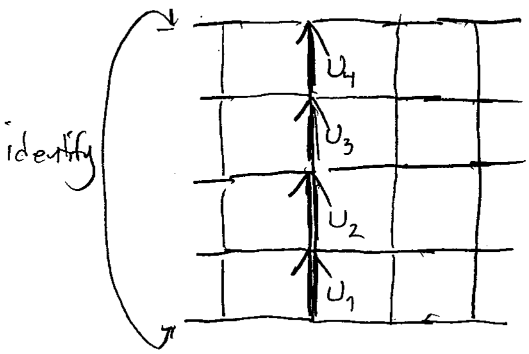
- For fermions: the b.c. in τ-direction becomes antiperiodic (not shown here):

$$\psi\left(\tfrac{1}{T}\right) = -\psi(0) \qquad \left(\text{or, rather, } \psi\left(\tau+\tfrac{1}{T}\right) = -\psi(\tau).\right)$$

- QCD (or theory w. fermions and gauge)

$$Z = \int DU\, D\psi D\bar{\psi}\, e^{-\int_0^{1/T} d\tau \int d^3x \left(\mathcal{L}_G + \bar{\psi} M \psi\right)}$$

- Gauge field at finite T:



Polyakov line/loop:

$$L(\bar{x}) = \prod_{\tau=0}^{N_\tau - 1} U_0(\tau, \bar{x})$$

L is a closed loop,

thus, $\mathrm{Tr}\, L(x)$ is gauge invariant.

- Take $SU(3)$, for concreteness.

- Center of SU(3):

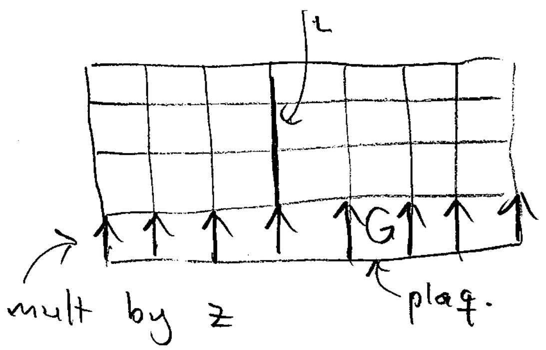
$$\{ Z \in SU(3) \mid [Z, U] = 0 \text{ for all } U \in SU(3) \}$$

- Must be proportional to $\mathbb{1}$. For SU(3),

$$Z = \mathbb{1}\, e^{in2\pi/3}, \quad n = 0, 1, 2$$

$$= \mathbb{1},\ \mathbb{1}\, e^{i2\pi/3},\ \mathbb{1}\, e^{i4\pi/3}$$

- For SU(N), center $= \{ \mathbb{1}\, e^{in2\pi/N}, \ n = 0..(N-1) \}$



Center symmetry:
Take fixed τ, e.g. $\tau = 0$. Multiply all $U_0(\tau=0)$ by $Z \in$ Center

$$U_0(0) \to Z\, U_0(0)$$

- Plaquettes do not change (or any closed loops which do not wrap over $\frac{1}{T}$)

$\Rightarrow$ S_{Gauge} invariant. Symmetry of action

- Polyakov line $\text{Tr}\, L \to e^{in2\pi/3}\, \text{Tr}\, L$

changes (if $n \neq 0$)

- Spontaneous symmetry breaking at high T:

Tr L order parameter

$\langle \text{Tr}\, L \rangle \simeq 0$ at low T

$\langle \text{Tr}\, L \rangle = \text{const.}\, e^{in2\pi/3}$ for some n at high T

- Deconfinement :

Interpretation of the Polyakov line :

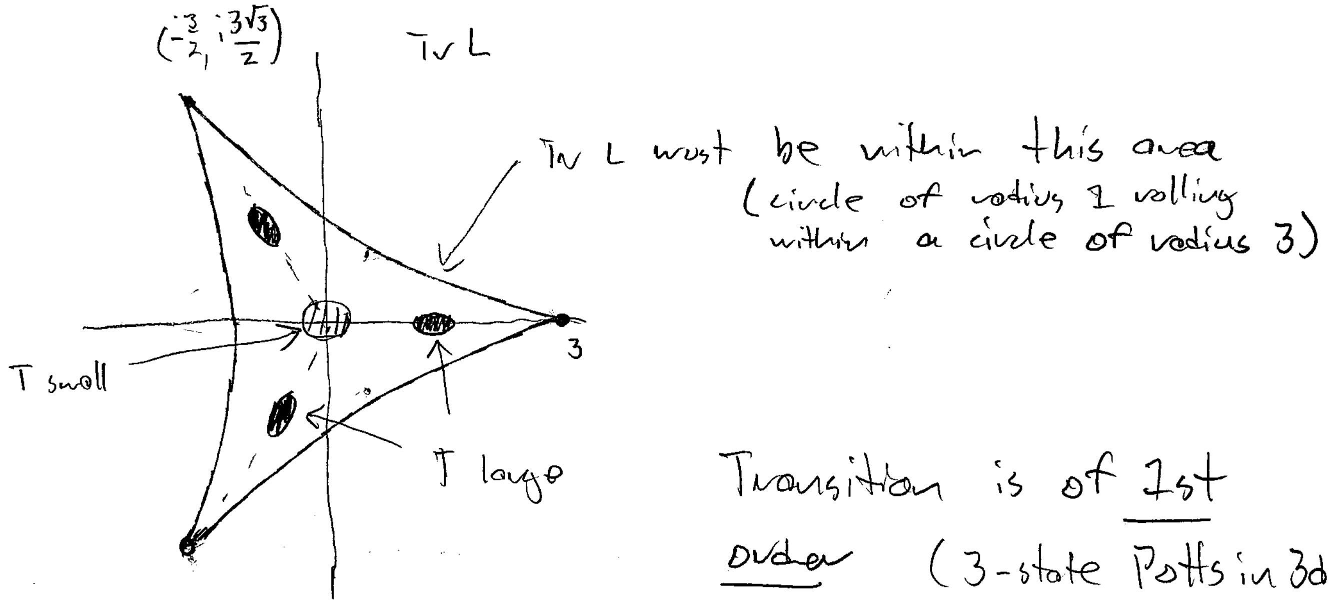
$\frac{1}{V} \langle | \sum_x \mathrm{Tr} L | \rangle \propto e^{-f/T}$

where f is the "free energy" of a static test charge (color)

- Can be done more rigorously using correlators

- $\langle \mathrm{Tr} L \rangle = 0 \quad \Rightarrow f = \infty$, no free charges
  $\Rightarrow$ confinement (Low T)

- $\langle \mathrm{Tr} L \rangle \neq 0 \quad \Rightarrow f$ finite, free charges OK
  $\Rightarrow$ deconfinement (high T)



Transition is of 1st order (3-state Potts in 3d)

- Generalizes to SU(N). 1st order for $N \geq 3$,
  2nd for SU(2) (3d ising)

- U(1) ? No transition

- Finite T with gauge fields + matter (quarks)

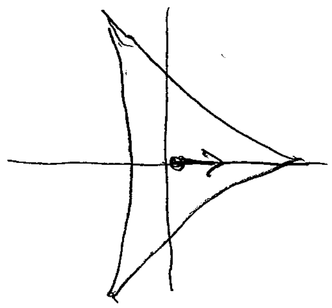
- $\bar{\psi} M \psi$ not invariant under $U \to zU$!

  (in general: $\bar{\psi}_{x,t} U_0 \psi_{x,t+1}$ is <u>not invariant</u>; not even when ψ's are redefined)

- Center symmetry <u>not exact</u>
- The heavier the quarks, the <u>less</u> they affect the gauge fields

- Normally, matter prefers $U_0 \to \mathbb{I}$
  $\Rightarrow \mathrm{Tr}\, L \to \alpha \mathbb{1}$.

$\Rightarrow$



fermions act as an "external field" to positive real axis direction for Tr L. Strength $\propto \frac{1}{m}$

- If quarks light enough, 1st order transition <u>vanishes</u>

- Chiral transition

  - Consider 2 light quarks (u,d)
  - When $m_{u,d} \to 0$, we have symmetries $\underline{SU(2)_V \otimes SU(2)_A}$ (neglect $U(1)_{V,A}$)

$$\psi \to e^{i\sigma_a \theta_a}\psi \;;\; \psi \to e^{i\sigma_a\theta_a\gamma_5}\psi$$

Pauli matrix, mixing of flavour.

- $n.\ m \neq 0$, this is explicitly broken to $SU(2)_V$.

- HOWEVER: even at $m_{u,d} = 0$, the symmetry is broken *spontaneously*:

lot of hadron + meson phenomenology!

chiral condensate $\langle \bar\psi \psi \rangle \neq 0$

(vacuum exp. value) $= \langle \bar\psi_u \psi_u + \bar\psi_d \psi_d \rangle$

  - (actually, when $m = 0$ any direction in $SU(2) \otimes SU(2)$-space is equally likely. However, mass $m > 0 \to \langle\bar\psi\psi\rangle$-direction; and because in reality m is not quite $=0$, $\langle\bar\psi\psi\rangle$ is chosen)

- $\langle \bar\psi_x \psi_x \rangle = \langle M^{-1}_{xx} \rangle$, measureable (in principle)

- Turns out that even at $m=0$:

  at low T: $\langle\bar{\psi}\psi\rangle \neq 0$

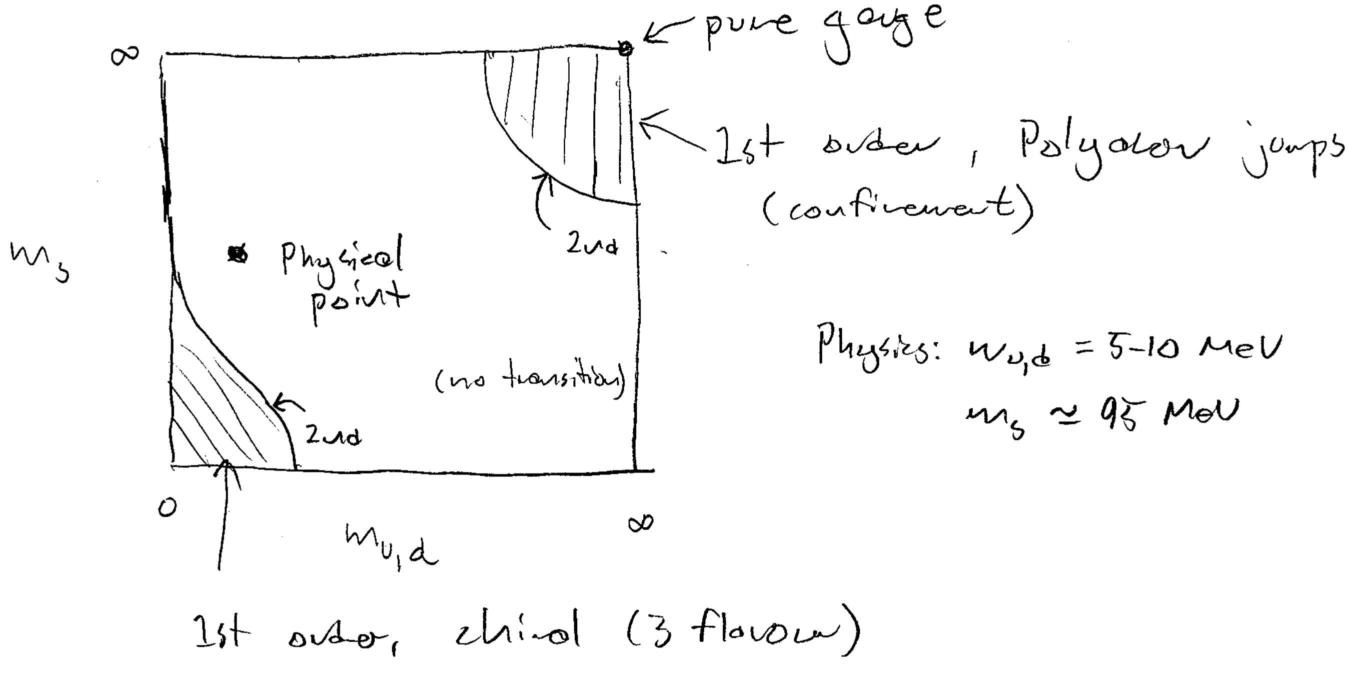
  at high T: $\langle\bar{\psi}\psi\rangle = 0$

<u>chiral</u> <u>symmetry</u> <u>restoration</u> at high T!

- for 2 light quarks ($m=0$), transition is of <u>2nd order</u>

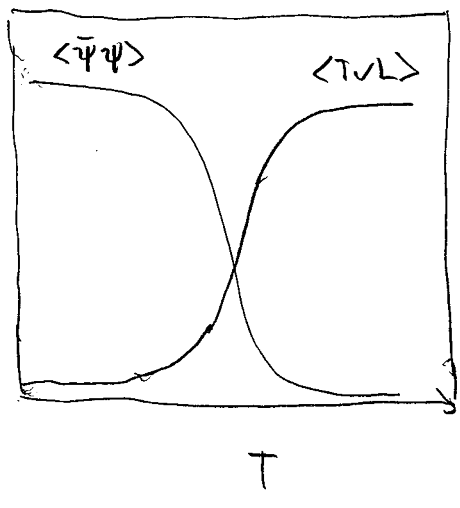
  >2 $m=0$ quarks, <u>1st order</u>

- $m \neq 0$ breaks the symmetry.



Physics: $m_{u,d} = 5$–10 MeV

$m_s \simeq 95$ MeV



Transition is continuous <u>cross-over</u> (no real transition)

$T_c \simeq 170$ MeV (for qcd)

- There is no real high-T phase transition in real QCD – cross-over

- Low T: hadron gas, confinement

  High T: plasma of quarks and gluons

Chemical potential

- So far, we have been discussing zero chemical potential – "vacuum" at $T=0$, Finite T plasma with vanishing baryon number $\langle n_B \rangle = 0$.

- Baryon/quark chemical potential needed for finite density matter

- Baryon number is conserved:
  - corresponds to $U(1)_V$ symmetry,

    $\psi \to e^{i\theta}\psi; \quad \bar{\psi} \to \bar{\psi} e^{-i\theta}$

  - Noether current $j_\mu = \bar{\psi}\gamma_\mu\psi$

$$\left(j_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \psi \right)$$

- Thus, quark number density $j_0 = \bar{\psi}\gamma_0\psi$,
  and total number $N_q = \int d^3x\, \bar{\psi}\gamma_0\psi$
- N is conserved; $\partial_0 N = 0$
- Note: N_q still fluctuates configuration by configuration, and $\langle N_q \rangle = 0$ if no chemical potential.
- Chemical potential couples to N_q:

$$\underset{\substack{\uparrow \\ \text{grand can.}}}{Z_{GC}} = \underline{\mathrm{Tr}\, e^{-\frac{1}{T}(H-\mu N)}}$$

$$= \int DU\, D\psi\, D\bar{\psi}\; e^{-S_E + \int d^4x\, \bar{\psi}\mu\gamma_0\psi}$$

- Thus, effectively $igA_0 \to \underset{\substack{\uparrow \\ \in \text{ algebra of } SU(N)}}{igA_0} + \mu$ ← $\propto \mathbb{1}$, $\notin$ algebra!
- This can be taken into account by modifying temporal links:

$$\left\{\begin{array}{l} U_0(x) \to e^{a\mu}\, U_0(x) \\ U_0^+(x) \to e^{-a\mu}\, U_0^+(x) \end{array}\right. \qquad \notin SU(N)!$$

- This works with all lattice fermion actions.

- Closed gauge loops invariant (Gauge action)

- Naive derivative:

$$\bar{\psi}\gamma_0\Delta_0\psi \to \frac{1}{2a}\bar{\psi}_\tau\left(e^{a\mu}U_0(\tau)\psi_{\tau+1} - e^{-a\mu}U_0^\dagger(\tau-1)\psi_{\tau-1}\right)$$

intuitively: $\mu>0$ favours quarks going forward in time (disfavours backwards)

- However: M is $\underline{\text{not}}$ γ_5-hermitean:

$$\gamma_5 M \gamma_5 \neq M^\dagger$$

Already evident from $\gamma_5(\mu\gamma_0)\gamma_5 = -\mu\gamma_0 \neq (\mu\gamma_0)^\dagger$

$$\Rightarrow \underline{\int D\psi D\bar{\psi}\, e^{-\bar{\psi}M\psi} = \det M \notin \mathbb{R}}$$

$\Rightarrow$ Wild fluctuations configuration by configuration: