Staggered fermions (Kogut-Susskind)

- Staggered fermions get the doublers from $16 \to 4$, at the cost of breaking most of the chiral symmetry
- Change $\psi \to \chi$ so that

$$\begin{cases} \psi_x = T(x)\,\chi(x) \\ \bar{\psi}_x = \bar{\chi}(x)\,T^{+}(x) \end{cases}$$

integer coordinates

where $T(x) = \gamma_0^{x_0}\,\gamma_1^{x_1}\,\gamma_2^{x_2}\,\gamma_3^{x_3}$

- Now

$$T_x^{+}\,\gamma_\mu\,T_{x+\hat{\mu}} = (-1)^{x_0+x_1+\dots x_{\mu-1}} \equiv \eta_\mu(x)$$

contains 1 γ_μ more than T_x^+

$$T_x^{+}\,T_x = 1$$

- Thus, naive action becomes

$$S = \sum_{x,\mu} a^4\,\bar{\chi}_x\,\eta_\mu(x)\left[U_\mu(x)\,\chi_{x+\hat{\mu}} - U_\mu^{+}(x-\hat{\mu})\,\chi_{x-\hat{\mu}}\right]\frac{1}{2a}$$
$$+\, m \sum_x a^4\,\bar{\chi}_x\,\chi_x$$

γ-matrices vanished! (or, all became $\mathbb{1}$)

Therefore, all components are equivalent!

- Let us discard 3 of them;

1 remains:

$$S_{stag.} = \sum a^4 \bar{\chi}_x (\eta_\mu \Delta_\mu + m) \chi_x$$

$$\left(\Delta_\mu \chi_x = \frac{1}{2a}(\chi_{x+\hat{\mu}} - \chi_{x-\hat{\mu}})\right)$$

- This still has a "pole" at each corner of the hypercube: 16 scalars → 4 4-comp. Dirac spinors.

⇒ doublers 16 → 4.

- In modern parlance, these are called "tastes".

- Remnant symmetry: $U(1)_V \otimes U(1)^{m=0}$

special staggered symmetry

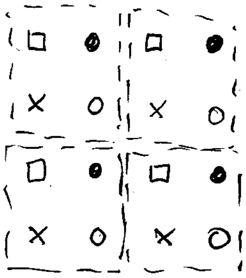
$$\chi \to e^{i\Gamma_5 \theta} \chi$$
$$\bar{\chi} \to \bar{\chi} e^{i\Gamma_5 \theta}$$

where

$$\Gamma_5(x) = \begin{cases} +1, & \sum x_\mu = \text{even} \\ -1, & \sum x_\mu = \text{odd} \end{cases}$$

This symmetry protects mass from additive renormalization.

- In principle, staggered fermions "live" on 2^4 hypercubes



deriviate couples only "same" sites

Make a block transformation:

$$\chi^{\alpha} \longrightarrow \psi^{\alpha}_{a}$$

α ← Dirac 1..4

a ← Flavour, "taste", 1..4

α: index within cube, $\alpha = 1..16$

(see Rothe)

$b = 2a \Rightarrow$

Brillouin zone cut in ½ and ψ's do not have doublers

$(ap)_{New} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

- When there are no gauge fields, action can be transformed into

$$S_{KS} = b^4 \sum_x \bar{\psi} \left[(\gamma_\mu \otimes \mathbb{1}) \Delta_\mu + \frac{b}{2} (\gamma_5 \otimes \gamma_\mu^* \gamma_5) \Delta_\mu^2 + 2m (\mathbb{1} \otimes \mathbb{1}) \right] \psi$$

spinor matrix — flavor

flavor non-diagonal!

- In detail: $\psi^{\alpha}_{a} = \frac{1}{8} \sum_y' T_y^{\alpha a} \chi_y$; $T_y = \gamma_0^{y_0} \gamma_1^{y_1} \gamma_2^{y_2} \gamma_3^{y_3}$

where $y_\mu = 0, 1$ goes over the 2^4-hypercube coordinates.

The inverse transformation is

$$\chi_y = 2\mathrm{Tr}[T_y^\dagger \psi]$$

Here it is understood that x is the hypercube coordinate, that is, $\chi_{2x+y} = 2\mathrm{Tr}[T_y^\dagger \psi_x]$.

- For interacting theory (w. gauge fields) the block transformation becomes more complicated; or rather, the $O(a)$ -term receives gauge field contributions.

However, $\gamma_5 \otimes \gamma_5$ -symmetry remains (if $m=0$)

- Error w. staggered is actually $O(a^2)$!

- "Rooting" : staggered fermions in the continuum have 4 flavours. What if we want less ?

$$\int DU\, D\bar\psi D\psi\, e^{-S_G - \bar\psi M \psi} = \int DU \det M\, e^{-S_G}$$

$\det M$ ← fermion determinant

with 2 fermions (degenerate)

$$\int DU\, D\bar\psi\, D\psi\, e^{-S_G - \bar\psi_1 M \psi_1 - \bar\psi_2 M \psi_2} = \int DU (\det M)^2 e^{-S_G}$$

Thus, if staggered M describes 4 flavors, perhaps

$$\int DU\,(\det M)^{1/4}\, e^{-S_G}$$

describes 1? Rooting trick.

- Now, it is known that rooting is not exact at finite a. Continuum limit? Not known; probably not quite exact. However, many calculations use this procedure.

Exact chiral symmetry on the lattice

- Let us have a short discussion of modern approaches to exactly chiral doubler-free fermions. Price to pay: non-locality, expensive!

- Let us have $\mathcal{L} = \bar\psi D \psi$, where D is our new lattice Dirac operator (massless)

- Propose "chiral" symmetry

$$\delta\psi = i\theta\gamma_5\left(1-\frac{a}{2r_0}D\right)\psi$$

$$\delta\bar\psi = \bar\psi\left(1-\frac{a}{2r_0}D\right)\gamma_5\cdot i\theta$$

$a\to 0$ these reduce to std. chiral transformations

If now $\delta \mathcal{L} = 0 \Rightarrow$

$$\{\gamma_5, D\} = \frac{a}{r_0} D \gamma_5 D$$

Ginsparg-Wilson relation

$a \to 0$: std. chiral D

This implies

$$\{\gamma_5, D^{-1}\} = \frac{a}{r_0}$$

and, using $\gamma_5 D \gamma_5 = D^+$ (all D's obey this, incl. Wilson)

$$D + D^+ = \frac{a}{r_0} D^+ D$$

D has "good enough" chiral symmetry; same number of symmetries as continuum $\not{D}$ and $D \Rightarrow \not{D}_{cont}$ as $a \to 0$. Can we find D obeying G-W relation and does not have doublers?

YES: overlap fermions (Neuberger 1998)

$$D = \frac{r_0}{a}\left[1 + \frac{d}{\sqrt{d^+ d}}\right]$$

where $d = D_W - \frac{r_0}{a} = \gamma_\mu \Delta_\mu - \frac{a}{2}\Delta_\mu^2 - \frac{r_0}{a}$

Wilson $\not{D}$ with large negative mass

Because Wilson fermions do not have doublers, neither have overlap.

Defining hermitean operator $h(m) = \gamma_5 (D_W + m)$ overlap is often written as

$$D = \frac{v_0}{a}\left[1 + \gamma_5\, \epsilon\left[h\left(-\frac{v_0}{a}\right)\right]\right]$$

where $\epsilon(h) = \frac{h}{\sqrt{h^+ h}}$ is a matrix "step function".

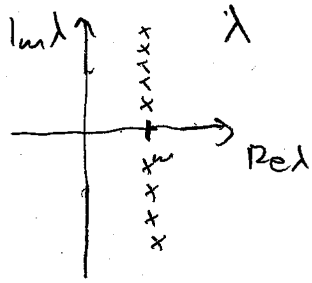
Mass term is added separately: $\mathcal{L}_F = \bar{\psi}(D + m)\psi$

Ex.: show that overlap obeys the full chiral symmetry when $m = 0$, and that the formal continuum limit $a \to 0$ is ok.

The parameter v_0 is chosen so that "$|D_W| \ll \frac{v_0}{a}$"; i.e. eigenvalues of D_W are small wrt. $\frac{v_0}{a}$.
(Note: this is true only approximately; lot of technical points)

Overlap Dirac operator is very non-local:
if we e.g. expand h around $-\frac{m}{a}$ we
get infinitely many P_W's. Need to truncate $\Rightarrow$
χ symmetry not exact.

Eigenvalues:



- Continuum $\mathcal{L} = \bar\psi\,(\overbrace{\not{D} + m}^{M})\,\psi$

antihermitean

- Wilson (or Clover):

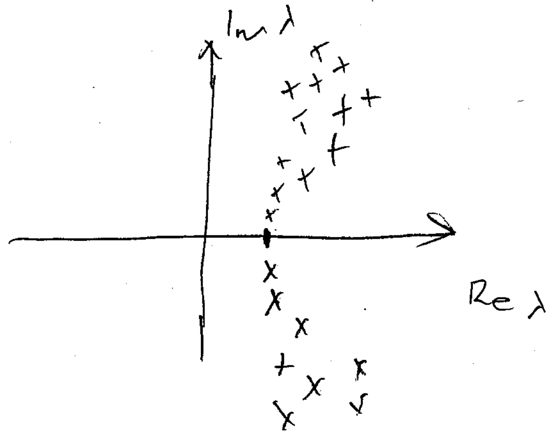
$$M = \gamma_\mu \Delta_\mu - \frac{a}{2}\Delta_\mu^2 + m$$

antihermitean　　hermitean

$\Rightarrow$ λ's all over plane

At small enough a,
smallest λ's:

"pinch" at m



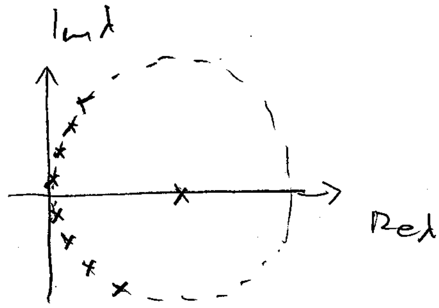
Overlap:

if $D\phi = \lambda\phi$, with $\lambda = x + iy$,

$$D^\dagger + D = \frac{a}{v_0} D^\dagger D \Rightarrow 2x = \frac{a}{v_0}(x^2 + y^2)$$

$$\Rightarrow \left(x - \frac{v_0}{a}\right)^2 + y^2 = \left(\frac{v_0}{a}\right)^2$$

E.V.'s are on a circle :
of radius v_0/a



m shifts to positive $\mathrm{Re}\,\lambda$ direction → resembles continuum.