Lattice fermions

- Free fermions

$$S = \int d^4x \, \bar{\psi} (i\partial\!\!\!/ - m) \psi \qquad \partial\!\!\!/ = \partial_\mu \gamma^\mu$$

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}$$

- In Euclidean space:

$$\underbrace{S_E = \int d^4x \, \bar{\psi} (\partial\!\!\!/ + m) \psi}$$

where $\partial\!\!\!/ = \partial_\mu \gamma^E_\mu$, $\{\gamma^E_\mu, \gamma^E_\nu\} = 2\delta_{\mu\nu}$

(e.g. $\gamma_0^E = \gamma_0^M$; $\gamma_i^E = -i\gamma_i^M$)

Common choice for γ_μ:

$$\gamma_i = \begin{pmatrix} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix} ; \quad \gamma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

$$\gamma_5 = -\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad ; \ \gamma_\mu^\dagger = \gamma_\mu$$

$$\{\gamma_5, \gamma_\mu\} = 0$$

Generating function:

$$Z[\eta, \bar{\eta}] = \int [d\psi][d\bar{\psi}] \, e^{-\int d^4x (\bar{\psi}(i\partial\!\!\!/ + m)\psi - \bar{\eta}\psi - \bar{\psi}\eta)}$$

here $\psi, \bar{\psi}$; $\eta, \bar{\eta}$ are anticommuting Grassmann numbers.

Grassmann numbers χ, η obey

$$\chi^2 = 0 , \quad \eta^2 = 0 , \quad \chi\eta = -\eta\chi \Leftrightarrow \{\chi, \eta\} = 0$$

Define derivative operator $\frac{d}{d\chi}$:

$$\frac{d}{d\chi}\chi = 1 \;;\quad \frac{d}{d\chi}\underset{\uparrow}{a} = 0 ; \quad \frac{d}{d\chi}\eta = 0$$

c-number

(rather, should write $\left\{\frac{d}{d\chi}, \chi\right\} = 1$; $\left\{\frac{d}{d\chi}, \eta\right\} = 0$)

And integral

$$\int d\chi \, \chi = 1 ; \qquad \int d\chi \, a = 0,$$

$$\int d\chi \, \eta = 0$$

$$\frac{d}{d\chi}\frac{d}{d\eta} = -\frac{d}{d\eta}\frac{d}{d\chi} \;; \quad \int d\chi d\eta = -\int d\eta d\chi$$

From the properties of Grassmann numbers follows

$$e^{\chi} = 1 + \chi$$

$$e^{\bar\eta_i \chi_i} = e^{\bar\eta_1 \chi_1 + \bar\eta_2 \chi_2} = 1 + \bar\eta_i \chi_i + \tfrac{1}{2}(\bar\eta_i \chi_i)^2$$

$$= 1 + \bar\eta_i \chi_i + \tfrac{1}{2}(\bar\eta_1 \chi_1 \bar\eta_2 \chi_2 + \bar\eta_2 \chi_2 \bar\eta_1 \chi_1)$$

or, in general,

$$\int d\chi d\bar\eta \, e^{\bar\eta_i M_{ij} \chi_j}$$

$\bar\eta, \chi$ n-comp. Grassmann vectors

M $n \times n$ matrix

= (need to exp. to order n, in order to obtain product $\bar\eta_1 \bar\eta_2 \cdots \bar\eta_n$ and $\chi_1 \cdots \chi_n$)

$$= \frac{1}{n!} \epsilon_{ijk\dots} \epsilon_{abc\dots} M_{ia} M_{jb} \dots = \epsilon_{ijk\dots} M_{i1} M_{j2} M_{k3} \dots$$

$$= \underline{\det M}$$

Thus,

$$Z[\eta,\bar{\eta}] = \int [d\psi][d\bar{\psi}]\, e^{-\int d^4x (\bar{\psi}-\bar{\eta}M^{-1}) M (\psi - M^{-1}\eta) + \int d^4x\, \bar{\eta} M^{-1}\eta}$$

can change variables; $\psi \to \psi - M^{-1}\eta$; jacobian $=1$

$$= \underbrace{\det M\, e^{\int d^4x\, \bar{\eta} M^{-1} \eta}}$$

Here $\quad M = \not{\partial} + m$

$$M^{-1} = \frac{1}{\not{\partial}+m} = \int \frac{d^4p}{(2\pi)^4} \frac{1}{i\not{p}+m} e^{ip\cdot(x-y)}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{m-i\not{p}}{p^2+m^2} e^{-ip\cdot(x-y)}$$

because $\quad \not{p}^2 = p_\mu p_\nu \gamma_\mu \gamma_\nu = \frac{1}{2} p_\mu p_\nu \{\gamma_\mu, \gamma_\nu\} = p_\mu p_\mu$

Thus, the fermion propagator in p-space is

$$\boxed{S_F(p) = \frac{m-i\not{p}}{p^2+m^2}} = \frac{-i\not{p}+m}{p^2+m^2}$$

Note: as usual, $\bar{\psi}$ and ψ are taken to be independent integration variables

On the lattice:

what if $\partial_\mu \psi \to \frac{1}{a}(\psi(x+\hat{\mu}) - \psi(x))$?

No! Now $\bar{\psi}\partial_\mu\psi \to \frac{1}{a}(\bar{\psi}(x)\,\psi(x+\hat{\mu}) - \bar{\psi}(x)\psi(x))$

This is not reflection invariant, leading to non-unitarity etc. serious problems.

This is caused by the single derivative in S.

Use instead symmetric derivative:

$$\partial_\mu \psi(x) \to \frac{1}{2a}(\psi(x+\hat{\mu}) - \psi(x-\hat{\mu}))$$

(this was not necessary for bosons because of 2 derivatives)

This gives us the naive lattice fermions:

$$S = \sum_x a^4 \left[\bar{\psi}_x \sum_\mu \gamma_\mu \frac{\psi_{x+\hat{\mu}} - \psi_{x-\hat{\mu}}}{2a} + m\bar{\psi}_x\psi_x \right]$$

Switch to p-space: $\psi_x = \int \frac{d^4p}{(2\pi)^4}\, \psi_p\, e^{ip\cdot x}$

$$\bar{\psi}_x = \int \frac{d^4p}{(2\pi)^4}\, \bar{\psi}_p\, e^{-ip\cdot x}$$

Thus,

$$S = \int \frac{d^4p}{(2\pi)^4} \bar{\psi}_p \left(\gamma_\mu \frac{e^{ip_\mu a} - e^{-ip_\mu a}}{2a} + m \right) \psi_p$$

$$= \int \frac{d^4p}{(2\pi)^4} \bar{\psi}_p \underbrace{\left(i\gamma_\mu \frac{1}{a} \sin(p_\mu a) + m \right)}_{S^{-1}(p)} \psi_p$$

Here we recognize the inverse propagator.

Thus,

$$S(p) = \frac{1}{i\gamma_\mu \frac{1}{a}\sin(p_\mu a) + m} = \frac{m - i\gamma_\mu \frac{1}{a}\sin(p_\mu a)}{m^2 + \left(\frac{1}{a}\sin(p_\mu a)\right)^2}$$

Clearly, as $a \to 0$, this approaches cont. prop.

NOTE: with bosons, we had $\hat{k}_\mu = \frac{2}{a} \sin \frac{k_\mu a}{2}$!

However, naive fermions are plagued w. doublers!

Doublers

Remember that Brillouin zone $-\frac{\pi}{a} < p_\nu \leq \frac{\pi}{a}$

Now, as $p_\nu a \to 0$, $i\gamma_\nu \frac{1}{a}\sin(p_\nu a) + m \to i\gamma_\nu p_\nu + m$.

However, if $p_\nu a \to (\pi, 0, 0, 0) \equiv p^A_\nu a$,

$$p_\nu a = (p^A_\nu + k_\nu) a$$

$$i\gamma_\nu \frac{1}{a}\sin(p_\nu a) = i\gamma_0 \frac{1}{a}\sin(\pi + k_0 a)$$

$$+ i\gamma_\alpha \frac{1}{a}\sin(k_\alpha a) \qquad \alpha \neq 0$$

$$\to -i\gamma_0 k_0 + i\gamma_\alpha k_\alpha$$

$$= i\gamma_\nu k'_\nu$$

where $k' = (-k_0, k_1, k_2, k_3)$

Thus, the corner of the Brillouin zone ($p_\nu \to \frac{\pi}{a}$) corresponds also to "low momentum" mode!

There are 16 corners : $p_\nu a = \underline{0 \text{ or } \pi}$; $2^4 = 16$

($p = -\pi$ is identical w. $p = +\pi$)

$\Rightarrow$ 16 doublers

Or, 1 naive fermion $\Rightarrow$ 16 continuum fermions!

More precisely:

Rotate to Minkowski time, $p_0^E = i p_0^M$

$$\gamma_i^E = -i\gamma_i^M$$

$$(p^E)^2 = p^E \cdot p^E =$$
$$= (ip_0^M)(ip^{0M}) + p_i^M p_i^M$$
$$= -p_0^{M2} + \bar{p}^{M2} = -p^{M2}$$

Thus, denominator of the prop. is

$$m^2 + \left(\tfrac{1}{a}\sin(p_\nu^E a)\right)^2 \to m^2 + \left(\tfrac{1}{a}\sin(p_i^M a)\right)^2 + \left(\tfrac{1}{a}\sin(ip_0^M a)\right)^2$$
$$= m^2 + \left(\tfrac{1}{a}\sin(p_i^M a)\right)^2 - \left(\tfrac{1}{a}\sinh(p_0^M a)\right)^2$$

Pole, when

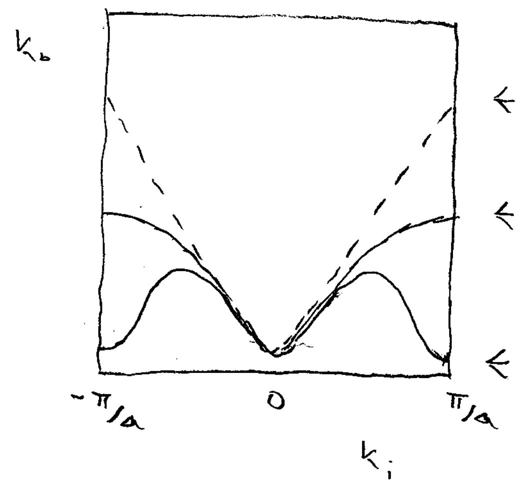
$$\sinh^2(p_0^M a) = \left((ma)^2 + \sin^2(p_i^M a)\right)$$

Thus, around $p_i a \ll 1 \Rightarrow p_0^2 = m^2 + \bar{p}^2$

but also $p_i = \frac{\pi}{a} - p_i'$; $p_i' a \ll 1 \Rightarrow p_0^2 = m^2 + \bar{p}'^2$

We get here 8 doublers. Missed the "temporal" ones, obtain from first shifting $p_0^E \to \frac{\pi}{a} + p_0^E$, and then rotating to Minkowski

Dispersion



← cont. $k_0^2 = m^2 + k_i^2$

← boson $\sinh^2 \frac{k_0 a}{2} = \sin^2 \frac{k_i a}{2} + 4(ma)^2$

← fermion $\sinh^2 k_0 a = \sin^2 k_i a + (ma)^2$

Intuitively: lattice w. spacing a

$\frac{1}{2a}(\phi(x+a) - \phi(x-a))$ $\qquad$ $ka \ll \pi$

$\simeq \partial_x \phi(x)$

$ka \simeq \pi$

$\frac{1}{2a}(\phi(x+a) - \phi(x-a))$ connects every 2nd p.t.

→ looks like very long wave mode!

Symmetries of the action

$$S = \int \bar{\psi}(\not{\partial} + m)\psi$$

- $U(1)_V$: $\psi \to e^{i\theta}\psi$; $\bar{\psi} \to \bar{\psi}e^{-i\theta}$

  is a symmetry of the action (and Z)

  Conserved quantity: fermion number

- $U(1)_A$: if $m = 0$,

  $\psi \to e^{i\theta\gamma_5}\psi$; $\bar{\psi} \to \bar{\psi}e^{i\theta\gamma_5}$

  is a symmetry of the action.

  However, it is not a symmetry of the measure $d\bar{\psi}d\psi$

  $\Rightarrow$ path integral breaks symmetry

  $\Leftrightarrow$ Axial symmetry broken by anomaly (appears at loops)

- If we have more than 1 fermion flavour, with degenerate mass m, we can write N_f spinors as a single vector

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_{N_f} \end{pmatrix} \quad ; \quad S = \int \bar{\psi}(\not{\partial} + m)\psi$$

- $SU(N_f)_V$: $\psi \to e^{i\vartheta_a\lambda_a}\psi$; $\bar{\psi} \to \bar{\psi}e^{-i\vartheta_a\lambda_a}$

  is a symmetry of action (and Z) : flavour rotation. λ_a : generator of $SU(N)$

• $SU(N_f)_A$: $\psi \to e^{i f_a \lambda_a \gamma_5} \psi$

$\bar\psi \to \bar\psi e^{i f_a \lambda_a \gamma_5}$

is a symmetry of the action if $\underline{m = 0}$.
This is not broken by anomaly; measure invariant

Total chiral symmetry is $(m=0)$

$$U(1)_V \times \underset{\substack{\uparrow \\ \text{anomalous}}}{U(1)_A} \times SU(N_f)_V \times SU(N_f)_A$$

Naive lattice fermions obey full symmetry, except $U(1)_A$ is not broken by anomaly!

Nielsen-Ninomiya theorem: (no-go)

$$\left.\begin{array}{l} \bullet \text{ local } {}^{4d}\text{ action} \\ \bullet \text{ chiral symmetry} \\ \bullet \text{ no doublers} \end{array}\right\} \text{not possible to do all!}$$

Standard methods break chiral symmetries; breaking $\propto O(a)$ $\Rightarrow$ continuum limit OK

Non-local actions (overlap; domain wall) can have chiral symmetry without doublers

A) Wilson fermions

- Add a 2nd derivative term:

$$\Delta_\mu^2 \psi(x) = \frac{1}{a^2}\left[\psi(x+\hat{\mu}) - 2\psi(x) + \psi(x-\hat{\mu})\right]$$

$$\boxed{S_W = \sum_x{}' a^4 \bar{\psi}_x \left[\gamma_\mu \Delta_\mu + m - a\frac{r}{2}\Box\right]\psi_x}$$

here $\Delta_\mu \psi = \frac{1}{2a}\left(\psi(x+\hat{\mu}) - \psi(x-\hat{\mu})\right)$

$$\Box\psi = \sum_\mu \Delta_\mu^2 \psi = \sum_\mu \frac{1}{a^2}\left(\psi(x+\hat{\mu}) - \psi(x) + \psi(x-\hat{\mu})\right)$$

- r: Wilson parameter, $\mathcal{O}(1)$
- Dimensionally, 2nd derivative term is multiplied by a → formally vanishes as $a \to 0$ ⟺ irrelevant term

Switch to momentum space:

$$S_W = \int \frac{d^4p}{(2\pi)^4} \bar{\psi}_p \left[i\gamma_\mu \frac{1}{a}\sin(p_\mu a) + m - r\frac{1}{a}\sum_\mu(\cos(p_\mu a) - 1)\right]\psi_p$$

Here we obtain the propagator

$$\frac{1}{a}S(p) = \frac{-i\gamma_\mu \sin(p_\mu a) + ma - r\sum_\mu(\cos(p_\mu a) - 1)}{\sum_\mu \sin^2(p_\mu a) + \left[ma - r\sum_\mu(\cos(p_\mu a) - 1)\right]^2}$$

when $p \to 0$,

$$\frac{1}{a} S(p) \to \frac{1}{a} \frac{-i\not{p} + m}{p^2 + m^2} \qquad \text{OK}$$

when $p \to (\pi/a, 0, 0, 0)$

$$ma - r \sum_\nu (\cos(p_\nu a) - 1) \to \underline{ma + 2r}$$

when $p \to (\pi/a, \pi/a, 0, 0)$

$$ma - r \sum_\nu (\cos(p_\nu a) - 1) \to \underline{ma + 4r}$$

and if $p \to (\pi/a, \pi/a, \pi/a, \pi/a)$

$$ma - r \sum_\nu (\cos(p_\nu a) - 1) \to \underline{ma + 8r}$$

Thus, the extra doublers get an extra mass

$$m_{\text{doubler}} = m + \frac{2r}{a} \quad \cdots \quad m + \frac{8r}{a}$$

depending on the corner.

When $a \to 0$, these become very massive and decouple.

$\Rightarrow$ only $p \sim 0$ pole remains. No doublers

However, <u>chiral symmetries are broken</u>
<u>$U(1)_A$, $SU(N)_{N_f}$</u>:

when $m \to 0$, $\bar{\psi}\, a \frac{r}{2} \Box \psi$ remains

which is <u>not</u> invariant under axial symmetry!

(no-go theorem in action!)

- Effectively, mass m gets additively renormalized: $m_{Ren} = m_0 + \delta m$
  ↖ caused by the Wilson term.

- Standard choice: $r=1$: now

$$S_W = -\sum_x a^4 \bar\psi_x \left[\sum_\mu \left(\frac{1-\gamma_\mu}{2a} \psi_{x+\hat\mu} + \frac{1+\gamma_\mu}{2a} \psi_{x-\hat\mu} \right) - M \psi_x \right]$$

here we have projection operators $\quad M \equiv m + \frac{4}{a}$

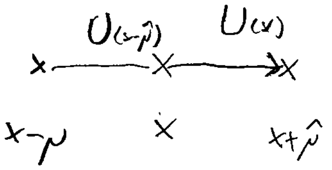
$$P_\mu^\pm = \tfrac{1}{2}(1 \pm \gamma_\mu) \; ; \quad \left(P_\mu^\pm\right)^2 = P_\mu^\pm$$

$$P_\mu^+ P_\mu^- = P_\mu^- P_\mu^+ = 0$$

$$P_\mu^+ + P_\mu^- = 1$$

- Adding gauge fields:
  (QED, QCD, EW)

  "parallel transport" $\psi(x+\hat\mu)$ to x :



$$\psi(x+\hat\mu) \rightarrow U_\mu(x)\, \psi(x+\hat\mu)$$

$$\psi(x-\hat\mu) \rightarrow U_\mu^\dagger(x-\hat\mu)\, \psi(x-\hat\mu)$$

QED, QCD

- In QED, gauge link $U_\mu = e^{ieaA_\mu}$,

  $A_\mu \in \mathbb{R}$, e : EM coupling (charge)

- In QCD, $U_\mu = e^{iga\lambda^a A^a_\mu}$

  λ^a generators of $SU(N)$; λ^a $N\times N$ Hermitean

Now $\psi = \psi^{\alpha a}$ has $N \times 4$ components

$$\underset{\text{color}}{N}\times\underset{\text{dirac}}{4}$$

Thus, now

$$S_W = -\sum_{x,\mu}' a^4 \left(\frac{1}{a}\bar\psi_x P^-_\mu U_{\mu,x}\psi_{x+\hat\mu} + \frac{1}{a}\bar\psi_x P^+_\mu U^+_{\mu,x-\hat\mu}\psi_{x-\hat\mu} \right)$$

$$+ M \sum_x a^4 \bar\psi_x \psi_x \qquad M = m + \frac{4}{a}$$

Standard rescaling : $\kappa = \frac{1}{2Ma} = \frac{1}{2ma+8}$

$\psi \to a^{-\frac{3}{2}}\sqrt{2\kappa}\,\psi$

$\bar\psi \to a^{-\frac{3}{2}}\sqrt{2\kappa}\,\bar\psi$ $\Rightarrow$

$$S_W = \sum_x' \bar\psi\psi - \kappa \sum_{x,\mu} \left[\bar\psi_x P^- U_{x,\mu}\psi_{x+\hat\mu} + \bar\psi_x P^+ U^+_{(x-\hat\mu),\mu}\psi_{x-\hat\mu} \right]$$

This is the most common form

For free fermions, $m \to 0 \Leftrightarrow \kappa \to \kappa_c = \frac{1}{8}$.

However, gauge field fluctuations reduce the magnitude of the hopping term, and $\kappa_c > \frac{1}{8}$ depending on the gauge coupling g^2 (or $\beta = \frac{2N}{g^2}$). In QCD, due to asymptotic freedom we know that as $a \to 0$, $g_0^2 \to 0$ and thus $\kappa_c \to \frac{1}{8}$.

<u>Clover action</u>:

- Wilson term was of order a; causing cut-off effects to also be of order a.

- To cancel all $O(a)$ effects, we should write <u>all</u> $O(a)$ operators and tune their couplings. These have to obey lattice symmetries (and the symmetry of the Wilson term).

$$\left.\begin{array}{l} \bar\psi\psi - O(a^{-3}) \text{, mult. by } m,\ O(a^{-4}) \\ \bar\psi \not{D} \psi - O(a^{-4}) \end{array}\right\} \text{naive action w. doublers}$$

$\bar\psi \not{D}\not{D} \psi - O(a^{-5})$ ← gives $O(a)$ terms

$\bar\psi\psi\bar\psi\psi$; $\bar\psi \not{D}\not{D}\not{D} \psi - O(a^{-6})$ - lots of terms!

At $O(a^{-5})$, we get 2 terms:

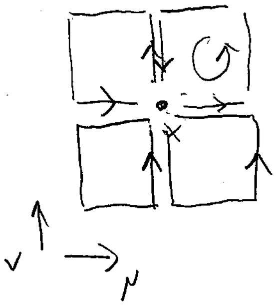
- $\bar{\psi} D^2 \psi$ – Wilson term
- $\bar{\psi} [\not{D}, \not{D}] \psi = \bar{\psi} \sigma_{\mu\nu} [D_\mu, D_\nu] \psi$

$$\propto \bar{\psi} \sigma_{\mu\nu} F_{\mu\nu} \psi$$

$$\sigma_{\mu\nu} \equiv -\frac{i}{2}[\gamma_\mu, \gamma_\nu] \quad ; \quad \sigma_{\mu\nu}^{\dagger} = \sigma_{\mu\nu}$$

Sheikholeslami-Wohlert or Clover term

- Called clover because $F_{\mu\nu}(x)$ is evaluated as a "clover" of 4 plaqs.



$$F_{\mu\nu} = \frac{1}{4} \sum_{\text{plaqs}} (U_1 U_2 U_3 U_4)_{\text{Antihermitean part}}$$

With the clover term the action becomes

$$S = S_W + \frac{c_{SW}}{2} \sum_x \bar{\psi}_x \sigma_{\mu\nu} F_{\mu\nu} \psi_x$$

Turns out that $c_{SW} = 1$ is $O(a)$ accurate at l.o. in perturbation theory. Typically $c_{SW} > 1$ at non-zero coupling.