Clearly, solving DY gives

$$\lambda_R = -\frac{1}{\beta_0 \ln a\Lambda} \quad , \qquad \Lambda \text{ integration constant}, \quad a\Lambda < 1$$

Now $\lambda_R \to 0$, as $a \to 0$.

Because the coupling remains weak, this is fully consistent.

$\Rightarrow$ Triviality: at long distances $\xi \gg a$, coupling $\to 0$;

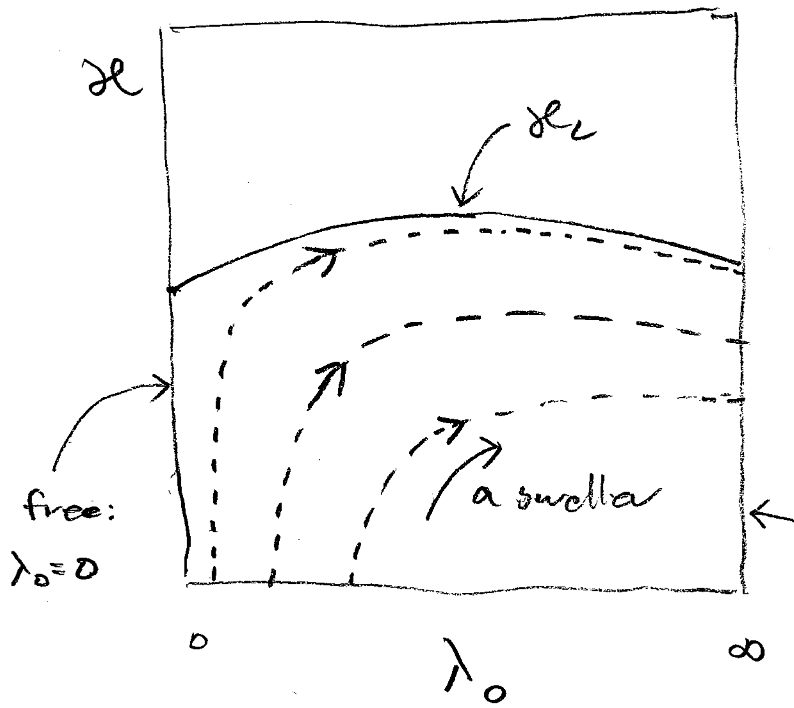
OR, on the lattice: w. fixed bare coupling $\lambda_0$, $a \to 0$ $\Rightarrow$ $\lambda_R \to 0$. Theory is free.

Lüscher-Weisz solution:

Lüscher & Weisz have combined weak coupling β-functions and strong coupling analysis to construct <u>lines of constant physics</u>

Parametrizing

$$\mathcal{L} = -\varkappa \sum_{x,\mu} \varphi_x \varphi_{x+\hat{\mu}} + \varphi_x^2 + \lambda_0(\varphi^2-1)^2$$




$\varkappa_c(\lambda_0)$: line where mass $= 0$ (critical line, phase transition: below: symmetric above: broken phase)

4d Ising : $\lambda_0 \to \infty$

Dotted lines: constant $\lambda_R$, a varies

- - > - direction where a decreases.

Smallest lattice spacing: Ising model!

$\lambda \varphi^4$ model in 3 dimensions

$$\mathcal{L} = \frac{1}{2}(\partial_i \varphi)^2 + \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4!}\varphi^4$$

Now $S = \int d^3x\, \mathcal{L}$ $\Rightarrow$ $d^3x\,(\partial_i \varphi)^2$ dimensionless

$\Rightarrow [\varphi] = \ell^{-1/2}$

$\Rightarrow [\lambda] = \frac{1}{\ell}$

Thus, $\lambda$ is dimensionful

Theory is <u>superrenormalizable</u> : finite number of divergent diagrams

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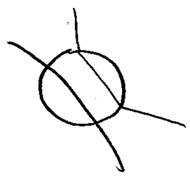
<u>Apparent divergence:</u>



Consider a diagram: (1PI)

- $E$ : external lines
- $V$ : vertices
- $I$ : internal lines ; $E + 2I = 4V$ (↑ lines/vert)
- $L$ : loops

Each $I$-line gives $\int_p$, each vertex a $\delta(\sum p)$. 1 $\delta(p)$ remains for overall mom. conservation. Thus,

$$L = \text{number of } \int_p\text{-integrals} = I - V + 1 = \frac{4V-E}{2} - V + 1 = \underline{V - \frac{E}{2} + 1}$$

Apparent divergence: each loop gives $\int d^d p$ ;
each internal line $p^{-2}$.

divergence $p^D = [\int d^d p]^L [p^{-2}]^I = p^{dL-2I}$

$$= p^{(d-4)V + (1-\frac{d}{2})E + d}$$

$$\underline{D = (d-4)V + (1-\tfrac{d}{2})E + d}$$

If $D=0$, diagram usually has log-divergence.

<u>$d=4$</u>: $D = 4 - E$ : 2-pt diagrams diverge $\Lambda^2$
4-pt $\log \Lambda$
6, 8, ... not divergent!
(may still contain <u>subdivergences</u>, e.g.  )

<u>$d=3$</u>: $D = 3 - \frac{E}{2} - V$

2-pt $E=2$: $D = 2 - V$ :  $D=1$ $\frac{1}{a}$

 $D=0$ $\log a$

no other divergences

$E=4$: $D = 1 - V$ : <u>no</u> divergences,
diagram w. $V=1$ does not have loops

Only 2-pt function is divergent at 1 and 2 loop order.

1-loop p. 65:

$$\underset{p}{\overset{q}{-\!\!\bigcirc\!\!-}} = I = \int_{-\pi/a}^{\pi/a} \frac{d^3 q}{(2\pi)^3} \frac{1}{\sum_i \left(\frac{2}{a}\sin\frac{q_i a}{2}\right)^2 + m^2}$$

$s=\frac{1}{2}$

$$= a^{-1} \int_{-\pi}^{\pi} \frac{d^3 x}{(2\pi)^3} \frac{1}{\sum_i \left(2\sin\frac{x_i}{2}\right)^2 + (ma)^2} \equiv a^{-1} I'(ma)$$

$$= c_0 \frac{1}{a} + c_1 m + c_2 a m^2 + \ldots \quad \text{(dimensional args.)}$$

$c_0$ can be obtained by setting $ma = 0$.

The resulting integral can be evaluated numerically or expressed as a complete elliptic integral of the second kind, with a result

$$c_0 = \frac{1}{4\pi} \Sigma, \quad \Sigma = 3.17591153562 5\ldots$$

$c_1$-term arises because the integral

$$\frac{d}{d(am)} I'(am) = \int \frac{d^3 x}{(2\pi)^3} \frac{am}{\left[\sum_i \hat{x}_i^2 + (am)^2\right]^2}$$

becomes divergent at $x \to 0$ as $(am) \to 0$, and the limit exists.

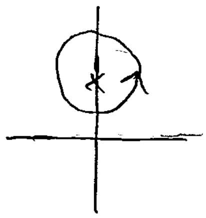
Because the divergence is at $x \simeq 0$, we can substitute $\hat{x} \to x$. Somewhat miraculously, $c_1$ can be calculated by the continuum integral

$$I = \int_{-\infty}^{\infty} \frac{d^3 q}{(2\pi)^3} \frac{1}{q^2+m^2} = \text{div} + c_1 m$$

by ignoring the divergence (included in $c_0$)

$$= \frac{4\pi}{8\pi^3} \int_0^{\infty} dq \, \frac{q^2}{q^2+m^2} = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dq \, \frac{q^2}{q^2+m^2}$$

$$= \frac{1}{4\pi^2} \cdot 2\pi i \, \frac{(im)^2}{2im} = -\frac{1}{4\pi} m$$



Thus, $c_1 = -\frac{1}{4\pi}$.

2-loop integral  gives $\log(am)$-contribution which is computable. Is there $p^2$-dep. divergent piece? No, taking $\frac{d}{dp^2}$  makes the diagram more convergent $\to$ no divergence.

$\Rightarrow$ $Z$ does not contain div. parts, can be set to $= 1$.

Outcome in 3d:

$$\lambda_R = \lambda_0$$

$$m_R^2 = m_0^2 + \frac{1}{2}\lambda\left(\frac{\Sigma_1}{4\pi}\frac{1}{a} - \frac{1}{4\pi} + \#\log(a)\text{-part}\right)$$

$$Z = 1$$

In this case there is interacting continuum limit. Counterterms are known.

This is relevant for e.g. Standard model at high temperatures: the effective theory is 3D SU(2)+Higgs theory, with computable coefficients. It is superrenormalizable and counterterms are known.