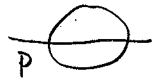
Renormalization

Thus, we found that

$$-\Pi(p) = -\frac{1}{2}\lambda\frac{1}{a^2}\left(c_0 - c_2(am)^2 + \frac{1}{16\pi^2}(am)^2\ln(am)^2\right) + \lambda\, O(a^2)$$
$$+ O(\lambda^2)$$

To this order this is momentum independent.
(tadpole)

At next order we have



$$\lambda^2\int_k\int_q \frac{1}{k^2+m^2}\,\frac{1}{q^2+m^2}\,\frac{1}{(k+q-p)^2+m^2}$$
$$= \lambda^2\frac{1}{a^2}\left(A + B(am)^2 + C(ap)^2 + O((ap)^4,(am)^4)\right)$$

contains $\ln(am)^2$

finite

This is generic (in 4d):

$\Pi =$  $= \frac{1}{a^2}\left(A + B(am)^2 + C(ap)^2\right)$ + contributions which $\to 0$, as $a \to 0$.

For $\lambda\varphi^4$, C is $O(\lambda^2)$. Consider now

$$G^{-1}(p) = p^2 + m^2 + \Pi =$$

we can now redefine our bare mass $m^2 \equiv m_0^2$ as

$$m^2 = m_R^2 + \delta m^2$$

where δm^2 is $O(\lambda)$ and contains the counterterms for divergences. (i.e. action now contains $\frac{1}{2}m_R^2\varphi^2 + \frac{1}{2}\delta m^2\varphi^2$)

In our case,

$$m_R^2 = m^2 - \delta m^2 = m^2 + \Pi\Big|_{\text{div. part at } p=0}$$

$$= m^2 + \frac{1}{2}\lambda\frac{1}{a^2}\left(c_0 - c_2 (am)^2 + \frac{1}{16\pi^2}(am)^2 \ln (am)^2\right) + O(\lambda^2)$$

we can invert this to obtain

$$m^2 = m_0^2 = m_R^2 - \frac{1}{2}\lambda\left(\frac{c_0}{a^2} - c_2 m_R^2 + \frac{1}{16\pi^2} m_R^2 \ln (am_R)^2\right) + O(\lambda^2)$$
$$= m_R^2 + \delta m^2$$

Thus, if we write the mass term in the action as $\frac{1}{2}m_R^2\varphi^2 + \frac{1}{2}\delta m^2\varphi^2$

counterterm

the counterterm (at 0-loop order) cancels the divergences at 1-loop order! Generalizes to higher loops $\to$ Renormalized perturbation theory.

What about the p^2 - contributions in Π?

$$G(p) = \frac{1}{p^2 + m^2 + \Pi} = \frac{1}{(1-C)p^2 + m_R^2}$$

$$= \underbrace{\frac{Z}{p^2 + m_R^2}}$$

Z = <u>wave function renormalization</u> $= 1 + C$

(C from p. 7b)

For $\lambda\varphi^4$-theory, $Z = 1 + O(\lambda^2)$

(note: this changes the definition of m_R^2, but at order λ^3)

This naturally can be absorbed with

$$Z \frac{1}{2} (\partial_\mu \varphi)^2 \rightarrow \underset{\substack{\uparrow \\ 1+\delta Z}}{Z} \frac{1}{2} \tilde{\varphi}_k \frac{1}{k^2} \varphi_k$$

1 ↑ tree-level, δZ ↑ counterterm

Thus, define renormalized field

$$\varphi_R = Z^{-1/2} \varphi \qquad \rightarrow \qquad (\partial_\mu \varphi)^2 = Z (\partial_\mu \varphi_R)^2$$

which thus containing counterterms.

We can also define renormalized 2-pt. function

$$G_R(p) = \langle \tilde{\varphi}_R^*(p) \tilde{\varphi}_R(p) \rangle = Z^{-1} G(p) = \frac{1}{p^2 + m_R^2}$$

Likewise, renormalized n-pt. function is

$$G_R^{(n)}(p_1 .. p_n) = Z^{-(n/2)} G^{(n)}(p_1 .. p_n)$$

For amputated n-pt. functions ("vertices")

$$\Gamma_R^{(n)}(p_1 .. p_n) = G^{(n)}_{R,\text{amp.}}(p_1 .. p_n) = G_R^{-1}(p_1) .. G_R^{-1}(p_n) G_R^{(n)}(p_1 .. p_n)$$

$$= Z^{n/2} G^{(n)}_{\text{amp.}}(p_1 .. p_n) = Z^{n/2} \Gamma^{(n)}(p_1 .. p_n)$$

4-pt. function:

Renormalized coupling is defined by

$$\lambda_R = -G^{(4)}_{R,\mathrm{amp}}(0,0,0,0) = -\Gamma^{(4)}_R(0,0,0,0) \qquad (\text{note } -\lambda \text{ at L.O.})$$

Now $\int_0^1 dx \ln A = \int_0^1 dx \ln (am)^2 = \ln (am)^2$, and permutations of external momenta give the same result.

(p. 69) $\Rightarrow$ (note $Z = 1 + O(\lambda^2)$)

$$\lambda_R = \lambda_0 + \frac{3}{2}\lambda_0^2 \frac{1}{16\pi^2}\left\{ \ln (am)^2 + 1 - 16\pi^2 C_2 \right\} + O(\lambda^3)$$

The result is, of course, λ_0 w. counterterms:

$$\lambda_R = \lambda_0 - \delta\lambda$$

Inverting we obtain

$$\lambda_0 = \lambda_R - \frac{3}{2}\lambda_R^2 \frac{1}{16\pi^2}\left\{ \ln (a^2 m_R^2) + 1 - 16\pi^2 C_2 \right\} + O(\lambda_R^3)$$

Thus, the p-dep. 4-pt vertex ($A = a^2 m_R^2 + x(1-x) a^2 p^2$)

$$\Gamma^{(4)}_R(p_1 \ldots p_4) = -\lambda_R - \frac{1}{2}\lambda_R^2 \frac{1}{16\pi^2}\left\{ \int_0^1 dx \ln \frac{m_R^2 + x(1-x)(p_1+p_2)^2}{m_R^2} \right.$$

$$\left. + (p_1+p_2) \to (p_1+p_3) + (p_1+p_2) \to (p_1+p_4) \right\}$$

Note that a-dep. cancels, as it should for G_R, Γ_R.

Also, all lattice integrals C_0, C_2 are absent.

no lattice dependence.

Let us define "running coupling" as

$$\bar{\lambda}(\mu) = -\Gamma_R^{(4)}(p_i)\Big|_{p_1+p_2 = p_1+p_3 = p_1+p_4 = \mu}$$

special, "symmetric" point.

$$= \lambda_R + \frac{3}{2}\lambda_R^2 \frac{1}{16\pi^2}\int_0^1 dx \ln\frac{m_R^2 + x(1-x)\mu^2}{m_R^2}$$

$\mu \ll m_R$: $\quad \approx \lambda_R$,

$\mu \gg m_R$: $\quad \approx \lambda_R + \frac{3}{2}\lambda_R^2 \frac{1}{16\pi^2}\int_0^1 dx\left[\ln x(1-x) + \ln\frac{\mu^2}{m_R^2}\right]$

$$= \lambda_R + \frac{3}{2}\lambda_R^2\frac{1}{16\pi^2}\left[\ln\frac{\mu^2}{m_R^2} - 2\right]$$

This indicates that as $\mu \to$ large, interactions grow!

In general, $\Gamma_R^{(4)}(p_i)$ at the limit $p_1+p_2 \gg m_R$

contains $\quad \lambda_R^2 \ln\frac{(p_1+p_2)^2}{m_R^2} \xrightarrow[\text{subst.}]{} \bar{\lambda}(\mu)\ln\frac{(p_1+p_2)}{\mu^2}$

If $(p_1+p_2)^2$ is of order μ^2, ln-correction is small and $\bar{\lambda}$ describes the interaction well.

Renormalization group and β-functions

Approach to continuum: 2 viewpoints:

- keep m_R^2, λ_R constant as $a \to 0$

  $\Rightarrow$ λ_0 must increase (p. 73)

  m_0^2 must decrease and become negative (p. 41)

  Physics is kept constant; the most intuitive view.

- keep λ_0, m_0^2 constant ($(m_0 a)^2$ decreases)

  $\Rightarrow$ λ_R must decrease

Let us look at the 1st case.

Evolution can be described by the differential equation, Callan-Symanzik eqn:

$$-\left[a \frac{d\lambda_0}{da}\right]_{\lambda_R, m_R^2} \equiv \beta(\lambda_0) \qquad \beta\text{- function}$$

Using $\lambda_0 = \lambda_R + \frac{3}{2} \lambda_R^2 \frac{1}{16\pi^2}\left[\ln(a m_R^2) + 1 - 16\pi^2 C_2\right]$

we get

$$-\left[a \frac{d\lambda_0}{da}\right]_{\lambda_R, m_R^2} = \frac{3}{2} \lambda_R^2 \frac{1}{16\pi^2} + O(\lambda_R^3)$$

and, using $\lambda_R = \lambda_0 + O(\lambda_0^2)$,

$$-\left[a \frac{d\lambda_0}{da}\right]_{\lambda_R, m_R^2} = \frac{3}{2} \lambda_0^2 \frac{1}{16\pi^2} + O(\lambda_0^3)$$

In general, the β-function can be arranged as a (pure) power series in λ_0; e.g. $\ln(am)$-terms can be absorbed.

thus,

$$\beta(\lambda_0) = \beta_1 \lambda_0^2 + \beta_2 \lambda_0^3 + \dots$$

and $\beta_1 = \dfrac{3}{32\pi^2}$.

Solving the dy: $-\dfrac{d\lambda_0}{\lambda_0^2} = \beta_1 \dfrac{da}{a}$

$$\Rightarrow \frac{1}{\lambda_0} = \beta_1 \ln a\Lambda \Rightarrow \lambda_0 = \frac{1}{\beta_1 \ln a\Lambda}$$

Λ integration constant. When $a \to \frac{1}{\Lambda}$, λ_0 diverges! Naturally our weak coupling analysis also fails. (Landau pole)

Thus, $\frac{1}{\Lambda}$ = smallest lattice spacing.

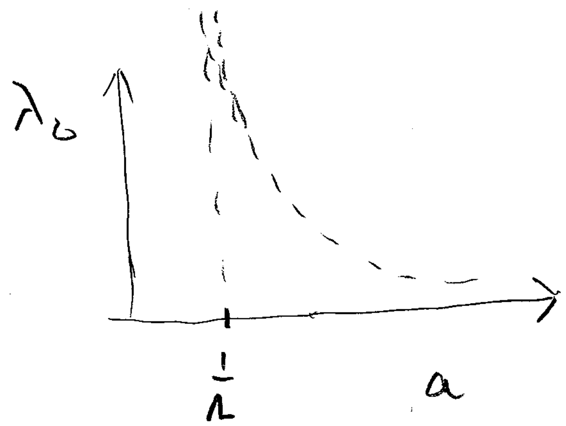
Using the fact that

$$\lambda_0 = \lambda_R + O(\lambda^3) \quad , \text{when } \ln(a^2 m_R^2) + 1 - 16\pi^2 c_2 = 0$$

the solution can be written as

$$\lambda_0 = \frac{\lambda_R}{1 + \lambda_R \beta_1 \left(\ln(a m_R) + \frac{1-16\pi^2 c_2}{2}\right)} + O(\lambda^2)$$

which is now in units of the input parameters.



The second approach:

$$-\left[a \frac{d\lambda_R}{da}\right]_{\lambda_0, m_0} = \beta_R(\lambda_R)$$

$$= -\beta_1 \lambda_0^2 + O(\lambda_0^3) = -\beta_1 \lambda_R^2 + O(\lambda_R^3)$$

Note different sign!