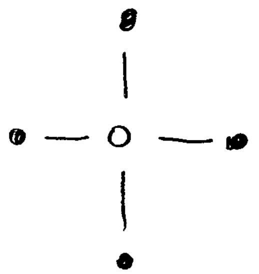
Mean field solution of the Ising model



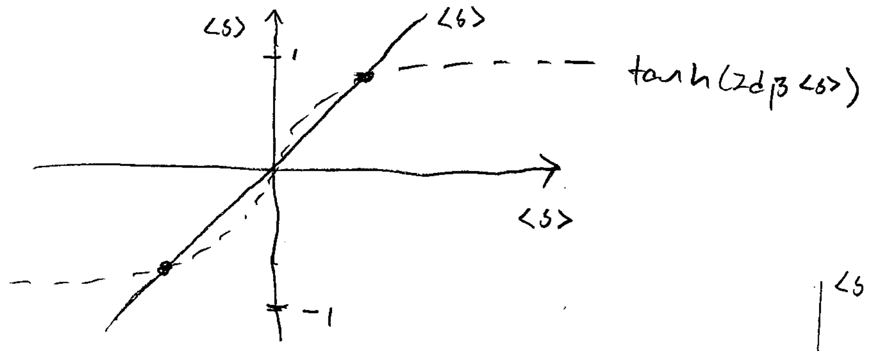
Substitute neighboring spins with the average $s_i \to \langle s \rangle$

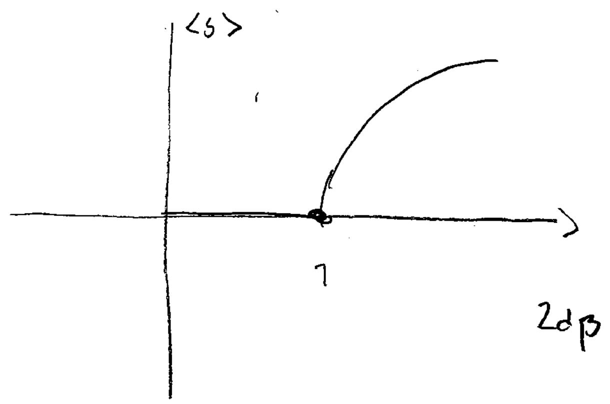
Now $E(s) = -2d\langle s \rangle s$ in d-dimensions.

Consistency:

$$\langle s \rangle = \frac{\sum_s s\, e^{-E(s)\cdot\beta}}{\sum_s e^{-E(s)\cdot\beta}} = \frac{\sinh 2d\beta\langle s \rangle}{\cosh 2d\beta\langle s \rangle} = \tanh 2d\beta\langle s \rangle$$

This has solution $\langle s \rangle \neq 0$ when $2d\beta > 1$:





When $\langle s \rangle$ small

$$\langle s \rangle \simeq 2d\beta\langle s \rangle - \frac{1}{3}(2d\beta\langle s \rangle)^3 \Rightarrow \langle s \rangle = 0 \quad \text{or}$$

$$\Rightarrow \langle s \rangle = \sqrt{\frac{3(2d\beta-1)}{(2d\beta)^3}}$$

Thus, let $\tau = \beta - \beta_c = \beta - \frac{1}{2d}$

$$\langle s\rangle = \sqrt{\frac{6d\tau}{(2d\tau+1)^3}} \simeq \sqrt{6d}\,\tau^{1/2} \quad \Rightarrow \quad \underline{\beta = 1/2} \;\; (p.13)$$

(p. 13 had this erroneous δ !)

$$\frac{E}{V} = \frac{\sum_s (-ds\langle s\rangle)\, e^{-\beta E(p)}}{\sum_s e^{-\beta E(p)}} = -d\langle s\rangle \tanh(2d\beta\langle s\rangle)$$

$$\simeq -2d^2\beta\langle s\rangle^2 \propto \tau$$

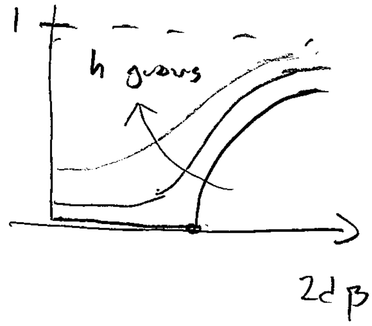
Thus, $\chi = \frac{1}{V}\frac{\partial E}{\partial \tau} = \text{const} \Rightarrow \underline{\alpha = 0}$

Mean field solution gives the correct critical exponents when $\underline{d \geq 4}$. At lower d it fails

We would obtain γ if we used ext. h. in the original ansatz :

$$E(s) = -2d\langle s\rangle s - hs \Rightarrow$$

$$\langle s\rangle = \tanh(2d\beta\langle s\rangle + h) \Rightarrow \langle s\rangle =$$



$\langle s\rangle, h$ small: $\langle s\rangle = 2d\beta\langle s\rangle + h$

$$\Rightarrow \langle s\rangle = \frac{h}{1-2d\beta}$$

$$\Rightarrow \frac{\partial\langle s\rangle}{\partial h} = \frac{1}{1-2d\beta} \propto |\tau|^{-1} \Rightarrow \underline{\gamma = +1}$$

Ising critical exp.

d	α	β	γ	ν	δ
2	0	$1/8$	$7/4$	1	15
3	0.110(1)	0.3265(3)	1.2372(5)	0.6301(4)	4.789(2)
4+	0	$1/2$	1	$1/2$	3

← arxiv: cond-mat/001216

Here $\langle s \rangle = h^{1/\delta}$ at $\tau = 0$ (critical point)

Weak coupling expansion for the scalar theory

- Weak coupling works pretty much as in continuum, except:
  - propagators are different:

$$(\text{scalar}) \quad \frac{1}{k^2+m^2} \to \frac{1}{\hat{k}^2+m^2}\,; \quad \hat{k}_\mu = \frac{2}{a}\sin\frac{k_\mu a}{2}$$

  - Vertices can become complicated (gauge fields, to be discussed)
  - Momenta restricted to Brillouin zone:

$$\int_{-\pi/a}^{\pi/a} \frac{d^d k}{(2\pi)^d}\,, \quad \sum_{k_\mu = \frac{2\pi n_\mu}{aN}} \frac{1}{(aN)^d} \leftarrow \text{if finite box } N^d$$

$\Rightarrow$ UV regulator is in place & well-defined, calculations clumsy.

For simplicity, let us consider real scalar field:

$$S = \sum_x{}' a^d \left[\frac{1}{2} \sum_\mu{}' \frac{(\varphi_{x+\hat\mu} - \varphi_x)^2}{a^2} + \frac{1}{2} m^2 \varphi_x^2 + \frac{1}{4!} \lambda \varphi_x^4 \right]$$

$$= \sum_x a^d \left[\frac{1}{2} \varphi_x \Box \varphi_x + \frac{1}{2} m^2 \varphi_x^2 + \frac{1}{4!} \lambda \varphi_x^4 \right]$$

where $\Box \varphi_x = \sum_\mu \frac{1}{a^2} (2\varphi_x - \varphi_{x+\hat\mu} - \varphi_{x-\hat\mu})$

A) For free theory,

$$Z_0 = \int [d\varphi]\, e^{-S_0 + J^T\varphi} \qquad J^T\varphi = \sum_x a^4 J_x \varphi_x$$

$$= \int [d\varphi]\, e^{-\frac{1}{2}\varphi^T M \varphi + J^T \varphi} = \int [d\varphi]\, e^{-\frac{1}{2}(\varphi^T - J^T M^{-1}) M (\varphi - M^{-1}J)} \times e^{\frac{1}{2} J^T M^{-1} J}$$

$$= \sqrt{\det M}\, (2\pi)^{N^d/2}\, e^{\frac{1}{2} J^T M^{-1} J}$$

where $M = \Box + m^2$; $\quad M^{-1} = \frac{1}{\Box + m^2} = \int_{-\pi/a}^{\pi/a} \frac{d^d p}{(2\pi)^d} e^{ip(x-y)} \frac{1}{\hat p^2 + m^2}$

$$= G(x-y).$$

Free theory n-pt. functions are generated by

$e^{-W} = Z \Rightarrow W = -\ln Z$

$$\langle \varphi_1 \cdots \varphi_n \rangle_0 \equiv \frac{\int [d\varphi]\, \varphi_1 \cdots \varphi_n e^{-S}}{Z[J=0]} = \frac{\frac{\delta}{\delta J_1} \cdots \frac{\delta}{\delta J_n} Z[J] \big|_{J=0}}{Z[0]}$$

$$= \frac{\delta}{\delta J_1} \cdots \frac{\delta}{\delta J_n} \, e^{\frac{1}{2} J^T G J} \Big|_{J=0}$$

$$= G_{12}\, G_{34} \cdots G_{(n-1)n} \quad + \text{ permutations.}$$

Thus, e.g. 4-pt. function (Wick theorem)

$$\langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle = \underbrace{\varphi_1 \varphi_2}\,\underbrace{\varphi_3 \varphi_4} + \varphi_1 \varphi_2 \varphi_3 \varphi_4 + \underbrace{\varphi_1 \overbrace{\varphi_2 \varphi_3} \varphi_4}$$

where $\underbrace{\varphi_a \varphi_b} = G(a-b)$.

Note that

$$S_0 = \frac{1}{2} \varphi^T M \varphi = \sum_x a^d \frac{1}{2} \varphi_x (\Box + m^2) \varphi_x$$

Let $\varphi_x = \int_p e^{ipx} \tilde{\varphi}_p \;\Rightarrow$

$$S_0 = \sum_x a^d \int_p \int_{p'} \frac{1}{2} e^{i(p+p')x} \tilde{\varphi}_p \tilde{\varphi}_{p'} \left(\sum_N \frac{2 - 2\cos(p_N a)}{a^2} + m^2 \right)$$

$$= \int_p \frac{1}{2} \tilde{\varphi}_{-p} \left(\hat{p}^2 + m^2 \right) \hat{\varphi}_p = \int_p \frac{1}{2} \tilde{\varphi}^*_p \left(\hat{p}^2 + m^2 \right) \hat{\varphi}_p$$

because $\varphi \in \mathbb{R}$.

In momentum space:

$$\underbrace{\tilde{\varphi}^*_k \tilde{\varphi}_p}_{} = \sum_{y,x}{}' a^d a^d e^{iky} e^{-ipx} G(x-y)$$

$$= \sum_{y,x}{}' a^{2d} e^{iky-ipx} \int \frac{d^d q}{(2\pi)^d} e^{iq(x-y)} \frac{1}{\hat{q}^2+m^2}$$

[Note: $\displaystyle\int_{-\pi/a}^{\pi/a} \frac{d^d p}{(2\pi)^d} e^{ipx} = \frac{1}{a^d} \delta_{x,0}$ because x discrete $x = na$

$$\sum_x{}' a^d e^{-ipx} = (2\pi)^d \delta^{(d)}(p)$$]

$$= \sum_y{}' a^d e^{iky-ipy} \frac{1}{\hat{p}^2+m^2}$$

$$= (2\pi)^d \delta(k-p) \underbrace{\frac{1}{\hat{p}^2+m^2}}_{\tilde{G}(p)} .$$

B) Interactions: use $S = S_0 + S_I$, $S_I = \sum_x a^d \frac{1}{4!} \lambda \varphi_x^4$

Now

$$\langle \varphi_1 \cdots \varphi_n \rangle = \frac{\int [d\varphi]\, \varphi_1 \cdots \varphi_n\, e^{-S_0} e^{-S_I}}{\int [d\varphi]\, e^{-S_0} e^{-S_I}}$$

$$= \frac{\langle \varphi_1 \ldots \varphi_n e^{-S_I} \rangle_0}{\langle e^{-S_I} \rangle_0} = \frac{\langle \varphi_1 \ldots \varphi_n (1 - S_I + \frac{1}{2} S_I^2 + \ldots) \rangle_0}{\langle (1 - S_I + \frac{1}{2} S_I^2 + \ldots) \rangle_0}$$

$$= \left(\langle \varphi_1 \ldots \varphi_n \rangle_0 - \langle \varphi_1 \ldots \varphi_n S_I \rangle_0 + \ldots \right) \left(1 + \langle S_I \rangle_0 - \ldots \right)$$

$$= \langle \varphi_1 \ldots \varphi_n \rangle_0 \underbrace{- \langle \varphi_1 \ldots \varphi_n S_I \rangle_0 + \langle \varphi_1 \ldots \varphi_n \rangle_0 \langle S_I \rangle_0}_{O(\lambda)} + O(\lambda^2)$$

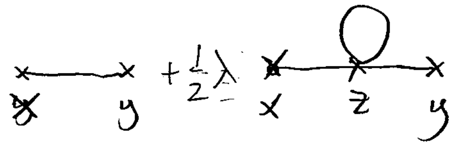
E.g. $\langle \varphi_x \varphi_y \rangle = \langle \varphi_x \varphi_y \rangle_0$

$$- \sum_z a^d \frac{1}{4!} \lambda \left(\langle \varphi_x \varphi_y \varphi_z \varphi_z \varphi_z \varphi_z \rangle_0 - \langle \varphi_x \varphi_y \rangle_0 \langle \varphi_z^4 \rangle_0 \right)$$

perms cancel

what remains is $\varphi_x \varphi_y \varphi_z \varphi_z \varphi_z \varphi_z$ 4×3 options

$$= G_{xy} - \sum_z a^d \frac{1}{2} \lambda G_{xz} G_{yz} G_{zz} + O(\lambda^2)$$

$=$  $+ O(\lambda^2)$

In momentum space

$$\langle \tilde{\varphi}^*_k \tilde{\varphi}_p \rangle = (2\pi)^d \delta(k-p)\, \tilde{G}(p)$$

$$-\frac{1}{2}\lambda \sum_{x,y} a^{2d} \sum_z a^d \int_{q_1} e^{iq_1(x-z)} \tilde{G}(q_1) \int_{q_2} e^{iq_2(y-z)} \tilde{G}(q_2) \times$$

$$\times \int_{q_3} e^{iq_3(z-z)} \tilde{G}(q_3)\, e^{iky}\, e^{-ipx}$$

$$= (2\pi)^d \delta(k-p)\, \tilde{G}(p)$$

$$-\frac{1}{2}\lambda\, \tilde{G}(p)\, \tilde{G}(k)\, (2\pi)^d \delta(k-p) \times \int_{q_3} \tilde{G}(q_3)$$

$$+ O(\lambda^2)$$

Etc. Thus, in mom. space the Feynman rules

$\langle \tilde{\varphi}_{k_1} \cdots \tilde{\varphi}_{k_n} \rangle$

- Overall $(2\pi)^d \delta(k_1 + \dots + k_n)$ (note sign of k_i)
- Vertex : $-\lambda$ (because of e^{-S_I})
- lines : $\tilde{G}(p) = \dfrac{1}{\hat{p}^2 + m^2}$
- loops : $\displaystyle\int_q \equiv \int_{-\pi/a}^{\pi/a} \frac{d^d q}{(2\pi)^d}$
- Symmetry factor : $\frac{1}{4!}$ each vertex
  $\frac{1}{n!}$ n vertices
  $\times$ combinatorics

Recall: generating functions

perhaps more elegant way to generate diagrams

$$Z[J] = e^{-S_I[\frac{\delta}{\delta J}]} Z_0[J] = Z_0[0]\, e^{-S_I[\frac{\delta}{\delta J}]} e^{\frac{1}{2} J^T G J}$$

- $$\frac{\frac{\delta}{\delta J_1} \cdots \frac{\delta}{\delta J_n} Z[J]\Big|_{J=0}}{Z[0]}$$ generates n-pt. functions.

- $e^{-W} = Z[J] \Leftrightarrow W[J] = -\ln Z[J]$

  generates connected functions:

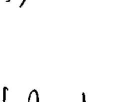
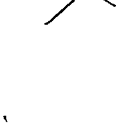
  $$\langle \varphi_1 \cdots \varphi_n \rangle_c = \frac{\delta}{\delta J_1} \cdots \frac{\delta}{\delta J_n} (-W[J]) \Big|_{J=0}$$

- Effective action

  $$\Gamma(\phi) \equiv W[J] - \sum_x a^d \phi_x J_x \qquad \text{(Legendre)}$$

  where $\phi(x) \equiv \dfrac{\delta W[J]}{\delta J(x)}$ is "classical field"

  Generates 1-particle irreducible (1PI) diagrams, i.e. diagrams which cannot be made disconnected by cutting 1 internal line

   is 1PI  not 1PI

- Amputated Green functions:

  leave out external legs

2-pt. function:

General 2-pt. function can be written as

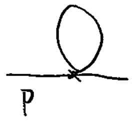
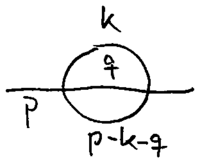
1PI blob, sum over all 1PI 2-pt. functions

$$= \tilde{G} + \tilde{G}(-\Pi)\tilde{G} + \tilde{G}(-\Pi)\tilde{G}(-\Pi)\tilde{G} + \ldots$$

$$= \tilde{G}\sum_{n=0}^{\infty}(-\Pi\tilde{G})^n = \tilde{G}\frac{1}{1+\Pi\tilde{G}} = \frac{1}{\tilde{G}^{-1}+\Pi}$$

$$= \underset{\sim\sim\sim}{\frac{1}{\hat{p}^2+m^2+\Pi}}$$

Π = sum of all 1PI amputated

$\Pi(p) =$  $+$ 

$+ O(\lambda^3)$

$$= -\frac{1}{2}\lambda\int_q \frac{1}{\hat{q}^2+m^2} + \frac{1}{6}\lambda^2\int_k\int_q \frac{1}{\hat{k}^2+m^2}\,\frac{1}{\hat{q}^2+m^2}\,\frac{1}{\widehat{(p-k-q)}^2+m^2}$$

Symmetry factors:

$\frac{1}{4!}\cdot 4\cdot 3 = \frac{1}{2}$

2n order of e^{-S_I}

$\frac{1}{4!}\,\frac{1}{4!}\,\frac{1}{2!}\,8\cdot 4\cdot 3\cdot 2 = \frac{1}{6}$

How to evaluate

$$\int_q \frac{1}{\hat{q}^2+m^2} = \int_{-\pi/a}^{\pi/a} \frac{d^d q}{(2\pi)^d} \frac{1}{\sum_\mu \left(\frac{2}{a}\sin\frac{q_\mu a}{2}\right)^2+m^2} \equiv I$$

This is where the difference to continuum comes.

Rescaling $x = qa$,

$$I = a^{2-d} \int_{-\pi}^{\pi} \frac{d^d x}{(2\pi)^d} \frac{1}{\sum_\mu (2\sin\frac{x_\mu}{2})^2+(am)^2} \equiv a^{2-d}\, I'(am)$$

when $a \to 0$, $I \propto a^{2-d} = \begin{cases} a^{-2} & \text{in } 4d \\ a^{-1} & \text{in } 3d \end{cases}$

UV divergent, regulated by a. We are interested in parts which diverge as $a \to 0$.

Take $d=4$:

- Clearly, the leading order result can be obtained by $am \to 0$. $I'(0)$ can be evaluated e.g. by numerical methods.

  $I'(0) \simeq 0.154933\ldots$

- What about higher orders? Note, we cannot expand $I'(am)$ in geometric series, because $x=0$ is smaller than am.

- $\frac{d}{da} I'(am)$ clearly leads to log-divergent integral $\int \frac{d^4x}{x^4}$. This usually indicates a $\ln a$ -behaviour

Try the following trick: divide integral in regions $|x| < \delta$ and $|x| > \delta$, δ small $\to 0$, but $am \ll \delta$

Now

$$I'_{|x|<\delta}(am) = \int_{|x|<\delta} \frac{d^4x}{(2\pi)^4} \frac{1}{(am)^2 + \sum_\mu \left(2\sin\frac{x_\mu}{2}\right)^2}$$

$$= \int_0^\delta \frac{dx\, x^3}{4(2\pi)^2} \frac{1}{(am)^2 + x^2}$$

$$= \frac{1}{16\pi^2}\left[\delta^2 - (am)^2 \ln\frac{(am)^2+\delta^2}{(am)^2}\right]$$

$$\simeq \frac{1}{16\pi^2}\left[-(am)^2\ln\delta^2 + (am)^2\ln(am)^2 + O(a^4,\delta^2)\right]$$

And

$$I'_{|x|>\delta}(am) = I'_{|x|>\delta}(0) + \frac{d}{d((am)^2)} I'_{|x|>\delta}\cdot(am)^2 + O(a^4)$$

$$\simeq I'(0) - \int_{-\pi,\,|x|>\delta}^{\pi} \frac{d^4x}{(2\pi)^4}\frac{1}{(\hat{x}^2)^2}(am)^2 + O(a^4,\delta^2)$$

δ-dep. behaviour $\displaystyle\int_\delta \frac{dx\,x^3}{4(2\pi)^2}\frac{1}{x^4}$

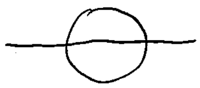
$$= -\frac{1}{16\pi^2}\ln\delta\,.$$

Cancels $\ln\delta$ from above!

Thus, $I'(am) = a^2 I(am) = C_0 + C_2(am)^2 + \frac{1}{16\pi^2}(am)^2\ln(am)^2 + O(a^4)$

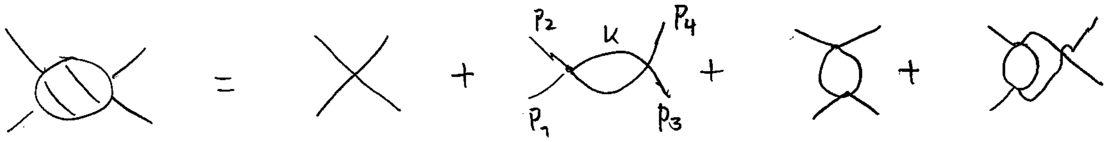
with $C_0 = I'(0) = 0.154933\ldots$

$$C_2 = -\lim_{\delta\to 0}\left[\int_{-\pi,\,|x|>\delta}^{\pi}\frac{d^4x}{(2\pi)^4}\frac{1}{(\hat{x}^2)^2} + \frac{1}{16\pi^2}\ln\delta^2\right] \simeq 0.0303\ldots$$

Next diagram,  $= \frac{1}{6}\lambda^2 \frac{1}{a^2}\left(\# + (am)^2 + \dots \right)$

Difficult to compute, but doable.

4-pt: 1PI amputated:



$+ O(\lambda^3)$

$$-\lambda \quad + \frac{1}{2}\lambda^2 \int_k \tilde{G}(k)\, \tilde{G}(p_1+p_2-k) + \dots$$

symmetry : $\frac{1}{2!}\frac{1}{4!}\frac{1}{4!}\, 8\cdot 3\cdot 4\cdot 3\cdot 2 = \frac{1}{2}$

Let now $p = p_1 + p_2$ and

$$J = \int_{-\pi/a}^{\pi/a} \frac{d^4 k}{(2\pi)^4} \frac{1}{\hat{k}^2 + m^2} \frac{1}{\widehat{(k-p)}^2 + m^2}$$

$$= \int_{-\pi}^{\pi} \frac{d^4 v}{(2\pi)^4} \frac{1}{\hat{v}^2 + (am)^2} \frac{1}{\widehat{(v-ap)}^2 + (am)^2} \qquad \hat{v}^2 = \sum_\mu \left(2\sin\frac{v_\mu}{2}\right)^2$$

Diverges logarithmically as $a \to 0$. Divergence is at $|\hat{v}| \to 0$.

Use again the trick of dividing integration volume

$$J = J_{|v|<\delta} + J_{|v|>\delta} \quad , \quad \delta \gg am$$

$$J_{|v|<\delta} = \int_{|v|<\delta} \frac{d^4 v}{(2\pi)^4} \frac{1}{(am)^2 + v^2} \frac{1}{(am)^2 + (v-ap)^2} + O(a^2)$$

$$J_{|v|>\delta} = \int_{-\pi,\, |v|>\delta}^{\pi} \frac{d^4 v}{(2\pi)^4} \frac{1}{(\hat{v}^2)^2} + O(a^2)$$

Use Feynman parameter to $J_{|v|<\delta}$

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[xA+(1-x)B]^2}$$

$$J_{|v|<\delta} = \int_0^1 dx \int_{|v|<\delta} \frac{d^4v}{(2\pi)^4} \frac{1}{[(am)^2 + v^2 - 2(1-x)v\cdot ap + (1-x)(ap)^2]^2}$$

Get rid of $v\cdot ap$ by $k = v-(1-x)ap$ (standard)

$$= \int_0^1 dx \int_{|k+(1-x)ap|<\delta} \frac{d^4k}{(2\pi)^4} \frac{1}{[(am)^2 + k^2 + x(1-x)(ap)^2]^2}$$

We can shift integration region back to $|k|<\delta$

The difference is of order a

$$= \int_0^1 dx \int_{|k|<\delta} \frac{dk\,k^3}{4(2\pi)^2} \frac{1}{[k^2+A]^2} \qquad A \equiv (am)^2 + x(1-x)(ap)^2$$

$$= \frac{1}{16\pi^2}\int_0^1 dx \left[\ln(A+\delta^2) - \ln A - \frac{\delta^2}{A+\delta^2}\right]$$

$$= \frac{1}{16\pi^2}\left[\ln\delta^2 - \int_0^1 dx \ln A - 1\right] + O(a^2)$$

$J_{|v|>\delta}$ is the same as $I'_{|x|>\delta}$ and gives

$$J_{|v|>\delta} = \underbrace{\left[\int_{-\pi,\,|v|>\delta}^{\pi} \frac{d^4v}{(2\pi)^4}\frac{1}{(\hat{v}^2)^2} + \frac{1}{16\pi^2}\ln\delta^2\right]}_{\to\, C_2 = 0.03034\ldots \text{ as } \delta\to 0} - \frac{1}{16\pi^2}\ln\delta^2$$

cancels w. $J_{|v|<\delta}$

Thus, $J(p) = C_2 - \frac{1}{16\pi^2} - \frac{1}{16\pi^2}\int_0^1 dx \ln A + O(a^2)$

where $A = a^2 m^2 + x(1-x)(ap)^2$

Thus, full 4-pt function is

$$\Gamma^{(4)}(p_1,p_2,p_3,p_4) = -\lambda + \frac{1}{2}\lambda\left[C_2 - \frac{1}{16\pi^2} - \frac{1}{16\pi^2}\int_0^1 dx \ln A(p_1+p_2)\right]$$
$$+ (p_1+p_2 \to p_1+p_3)$$
$$+ (p_1+p_2 \to p_1+p_4)$$

Is log-divergent as $a \to 0$.

The divergences can be absorbed in counterterms, which leads to renormalized m_R^2, λ_R. The bare masses and couplings are seen to diverge, as we shall see.