7 Strong coupling expansion and the Wilson loop

- Perturbation theory (weak coupling, $g \ll 1$) is toothless: *no confinement*
- Strong coupling: $g \gg 1$. For simplicity, consider U(1) ($beta = 1/g^2 \ll 1$). The link variable is $U_\mu = \exp i\theta_\mu$.

$$
\begin{aligned}
S &= \beta \sum_\Box (1 - \mathrm{Re}\, e^{i\theta_\Box}) \\
Z &= \int_0^{2\pi} \left[\prod_{x,\mu} \frac{d\theta_\mu}{2\pi} \right] \prod_\Box \exp[\beta \cos\theta_\Box]
\end{aligned}
$$

Here $\theta_\Box = \theta_1 + \theta_2 - \theta_3 - \theta_4$ around the plaquette $\Box$.

- Expand in powers of β? OK, but more convenient is:

$$
\begin{aligned}
e^{\beta\cos\theta} &= \sum_{n=-\infty}^{\infty} I_n(\beta) e^{in\theta} \\
&= A \left[1 + \sum_{n=1}^{\infty} f_n \cos n\theta \right]
\end{aligned}
$$

where

$$
\begin{aligned}
A &= I_0(\beta) = 1 + \frac{1}{4}\beta^2 + \frac{1}{64}\beta^4 + \ldots \\
f_n &= 2I_n(\beta)/I_0(\beta) = \frac{1}{2^{n-1}n!}\beta^n + O(\beta^{n+2}) \\
f_1 &= \beta - \frac{1}{8}\beta^3 + \frac{1}{48}\beta^5 + \ldots
\end{aligned}
$$

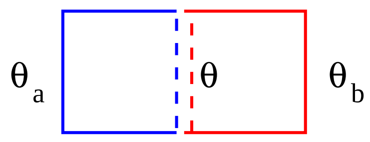
The partition function becomes

$$
Z = \int [\frac{d\theta}{2\pi}]\, A^{N_\Box} \prod_\Box (1 + f_1 \cos\theta_\Box + f_2 \cos 2\theta_\Box + \ldots)
$$

here $N_\Box = 6V$ is the number of plaquettes (in 4d).

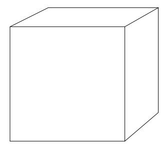
- Integration: $\int d\theta_a \cos n\theta_\Box = 0$, if $\theta_a \in \Box$.
  2 adjacent plaquettes:

$$\int \frac{d\phi}{2\pi} \cos(\theta + \theta_a) \cos(-\theta + \theta_b) = \tfrac{1}{2} \cos(\theta_a + \theta_b)$$



  (also for $\cos n\theta$)

- Non-zero contributions only from *closed surfaces*: lowest non-trivial comes from 3-d cube:

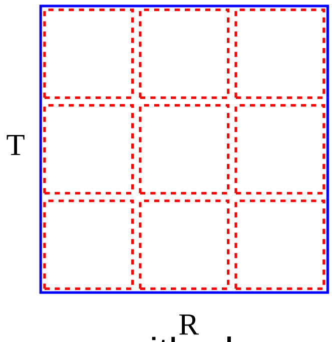


$$Z = A^{N_\Box}[1 + 4V(\tfrac{1}{2}f_1)^6 + O(\beta^{10})]$$

- Each plaquette on the surface contributes $\frac{1}{2}f_i$

- Expansion of free energy $F = -\log Z$ contains only connected graphs – *cluster expansion*

7.1 Wilson loop:



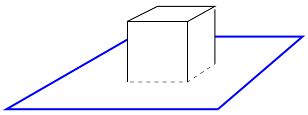
$$\langle W_{RT}\rangle = \frac{A^{N_\Box}}{Z}\int[\frac{d\theta}{2\pi}]\prod_{a\in W} e^{i\theta_a}\prod_{\Box}[1+\tfrac{1}{2}f_1(e^{i\theta_\Box}+e^{-i\theta_\Box})]$$

- First contribution comes when we "tile" the loop area with plaquettes:

$$\langle W_{RT}\rangle \approx (\tfrac{1}{2}f_1)^{RT} = (\tfrac{1}{2}\beta)^{RT} + O(\beta^{RT+2})$$

- We see *confinement:* $-\log\langle W\rangle/T = \log(\frac{1}{2}f_1)R = \sigma R$

- Next order: "bump"



$$\langle W_{RT}\rangle \approx (\tfrac{1}{2}f_1)^{RT}[1+4RT(\tfrac{1}{2}f_1)^4]$$

which gives

$$-a^2\sigma = -\frac{1}{RT}\log\langle W_{RT}\rangle = \log\tfrac{1}{2}f_1 + 4(\tfrac{1}{2}f_1)^4 + O([\tfrac{1}{2}f_1]^6)$$

- Order $[\frac{1}{2}f_1]^6$ gets contributions from the disconnected surfaces and Z.

- $-a^2\sigma = \log u + 4u^4 + \frac{176}{3}u^8 + \frac{10\,936}{405}u^{10} + \frac{1\,532\,044}{1\,215}u^{12} + \frac{3\,596\,102}{5\,103}u^{14}$, where $u = (\frac{1}{2}f_1)$ (see Montvay+Münster, for example)

- For U(1) this is strong coupling artefact! When $g\ 1$, there $\exists$ a phase transition. As $g\to 0$ we regain free theory.

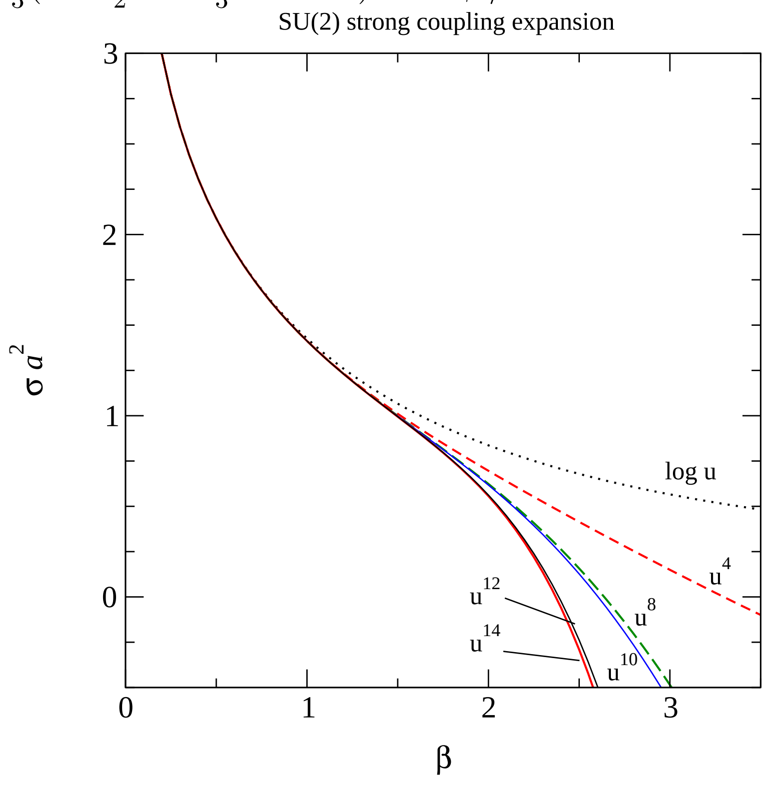
7.2 Non-Abelian gauge fields

- Same, but more difficult: use *character expansion*

$$e^{1/N\beta \mathrm{Re\,Tr}\,U} = A_\beta[1 + \sum_R b_R(\beta)\,\chi_R(U)]$$

  which gives "orthogonal" integration rules.

- Graphs are again surfaces – with complications!
- SU(2): $-a^2\sigma = \log u + 4u^4 + \frac{176}{3}u^8 + \ldots$,
  with $u = I_2(\beta)/I_1(\beta)$
- SU(3): $-a^2\sigma = \log u + 4u^4 + 12u^5 - 10u^6 - \ldots$
  with $u = \frac{1}{3}(x - \frac{1}{2}x^2 + \frac{2}{3}x^3 + \ldots)$, $x = \beta/6$.



- Convergence is good in *finite* range $\beta < \beta_{\max}$.
- *Mass gap:* plaquette-plaquette correlation function decays exponentially. Can be calculated using strong coupling, not in perturbation theory.

- *Roughening transition* for Wilson loops as β increases?
- In the group integration, we need the result

$$\int dU \chi_R^*(U)\chi_{R'}(U_a U) = \frac{1}{d_R}\delta_{R,R'}\chi_R(U_a)$$

where d_R is the dimensionality of the representation R.