2.9.1 The behaviour of the free propagator

Let us now calculate the free propagator in coordinate time. For concreteness we choose $d = 4$, and we look at the propagator at fixed spatial momentum $\boldsymbol{k}$:

$$G(\boldsymbol{k}, t) = \int_{-\pi/a}^{\pi/a} \frac{dk_0}{2\pi} e^{ik_0 t} \tilde{G}(\boldsymbol{k}, k_0)$$

Denoting $x = k_0 a$, $\tau = t/a$, and using

$$2 \sin^2 \frac{x}{2} = 1 - \cos x,$$

we obtain

$$G(\boldsymbol{k}, \tau a) = a \int_{-\pi}^{\pi} \frac{dx}{2\pi} \frac{e^{ix\tau}}{2b - 2\cos x}$$

where

$$b = 1 + \frac{(ma)^2}{2} + \sum_{i=1}^{3} \frac{4}{2} \sin^2 \frac{k_i a}{2}.$$

Substituting now $z = e^{ix}$, $dz = ie^{ix} dx$,

$$G(\boldsymbol{k}, \tau a) = \oint_C \frac{idz}{2\pi} \frac{z^\tau}{z^2 - 2bz + 1}$$

where the integration contour C is a unit circle to positive direction. Poles of the denominator are at

$$z_\pm = b \pm \sqrt{b^2 - 1} \in R, \quad 0 < z_- < 1, \quad z_+ > 1, \quad z_- z_+ = 1$$

Thus, only the pole at z_1 is inside the unit circle. The residue gives

$$G(\boldsymbol{k}, \tau a) = -a \frac{z_-^\tau}{z_- - z_+}$$

Thus, this is an exponentially decreasing function. Denoting

$$z_- = 1/z_+ = e^{-\omega}, \quad \cosh \omega = \frac{1}{2}(z_+ + z_-) = b,$$

we finally obtain

$$G(\boldsymbol{k}, \tau a) = a \frac{e^{-\omega\tau}}{2 \sinh \omega}$$

where, using $\cosh\omega = 2\sinh^2\frac{\omega}{2} + 1$, we can express

$$\omega = 2\sinh^{-1}\sqrt{(b-1)/2} = 2\sinh^{-1}\sqrt{\frac{(ma)^2}{4} + \sum_{i=1}^{3}\sin^2\frac{k_i a}{2}}.$$

Thus, the propagator decreases exponentially as $e^{-k_0 t}$, where $k_0 = \omega/a$ is the pole of the propagator with the substitution $k_0 \to ik_0$ (section 2.9). In the continuum ($a \to 0$, $k_i a \ll 1$) this approaches

$$G(\boldsymbol{k}, t) = \frac{e^{-\omega_c t}}{2\omega_c}$$

with $\omega_c = \sqrt{m^2 + \boldsymbol{k}^2}$, the familiar continuum expression. Note that the Minkowski space expression can be obtained from here by substituting $t_E \to it_M$, and we obtain the propagator which oscillates with its characteristic frequency ω/a.
If we now go ahad and do the full real space propagator (transform also the $\boldsymbol{k}$-component), we obtain

$$G(x) = a\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \frac{e^{-\omega x_0/a}}{2\sinh\omega}$$

Here, and what follows, we often denote lattice sums or integrals by symbol

$$\begin{aligned}\sum_k &= \int_{-\pi/a}^{\pi/a} \frac{d^d k}{(2\pi)^d} \\ &= \frac{1}{(Na)^d}\sum_n\end{aligned}$$

where either the first or second form is used, depending on whether the system is of infinite size or not.

It should be noted that the Euclidean propagators, in the lattice or in the continuum, are analytical continuations of the *Feynman* propagators in Minkowski spacetime. If we do a continuous rotation $k_0 \to e^{i\theta}k_0$, $\theta = 0$ Minkowski, $\theta = \pi/2$ Euclidean, and denote $E = \sqrt{\boldsymbol{k}^2 + m^2}$,

$$\tilde{G}_E = \frac{1}{(k_0)^2 + E^2} = \frac{1}{-(e^{i\pi/2}k_0)^2 + E^2} \to \frac{-1}{(k_0)^2 - E^2 + i\epsilon} \simeq \tilde{G}_{M,\text{Feyn.}}$$

where now the $i\epsilon$ takes care that the pole at $k_0 = +E$ is circumvented from above and the pole at $k_0 = -E$ from below. This is naturally the Feynman prescription.

3 Gauge fields

• Gauge field lagrangian in (Euclidean) continuum:

$$\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} = \frac{1}{2}\mathrm{Tr}\,(F_{\mu\nu}F_{\mu\nu})$$

• Field tensor

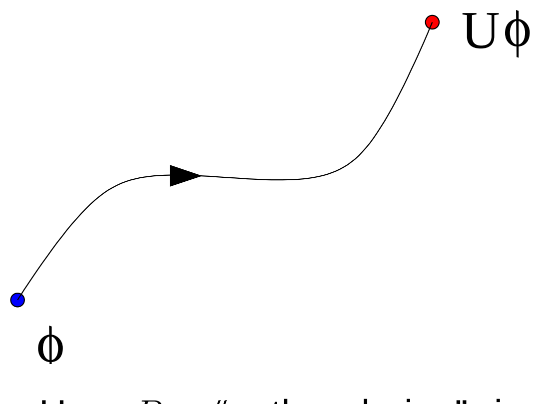
$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

where $A_\mu = A^a_\mu \lambda^a$, and λ^a are the group generators. We shall consider only unitary groups U(1) [QED] and SU(N) [QCD, Electroweak]. For SU(N), the generators are orthonormalized

$$\mathrm{Tr}\,\lambda^a\lambda^b = \frac{1}{2}\delta^{ab}.$$

• Put gauge potential A_μ directly on a lattice? Difficult to maintain gauge invariance.

• Good starting point for gauge field on a lattice is to consider consider gauge fields acting in a *parallel transport:* As field $\phi \in$ SU(N) is "parallel transported" along a path p, parametrized by $x_\mu(s)$, $s \in [0,1]$, gauge fields rotate it: $\phi \to U(s)\phi$



• Define $U(s)$ differentially via

$$\frac{dU(s)}{ds} = \frac{dx_\mu}{ds} igA_\mu U(s),$$

which can be formally solved:

$$U(s) = P\exp\left[ig\int_p ds\frac{dx_\mu}{ds}A_\mu\right]$$

• Here $P =$ "path ordering": in the power series expansion of the exponential one always takes $A(x)$ in the order they are encountered along the path.

• For U(1) (or any other Abelian group) path ordering is irrelevant.

• Alternatively, if we divide the path into N finite intervals of length Δs, and $x^n = x(n\Delta s)$, $n = 0 \ldots N-1$:

$$U(s) \approx P \exp\left[ig \sum_n \Delta s \frac{dx^n_\mu}{ds} A_\mu(x^n)\right] = \prod_i \exp\left[ig\Delta s \frac{dx^n_\mu}{ds} A_\mu(x^n)\right]$$

3.1 Gauge transformations

• Gauge transformation $\Lambda(x)$ is a (SU(N)) group element defined at every point. Gauge potential transforms as

$$A_\mu \to \Lambda A_\mu \Lambda^{-1} + \frac{i}{g}\Lambda \partial_\mu \Lambda^{-1}$$

• Field ϕ:

$$\phi(x) \to \Lambda(x)\phi(x)$$

• Path p:

$$U(p) \to \Lambda(x_1) U(p) \Lambda^{-1}(x_0),$$

where x_0 and x_1 are the beginning and end of path.

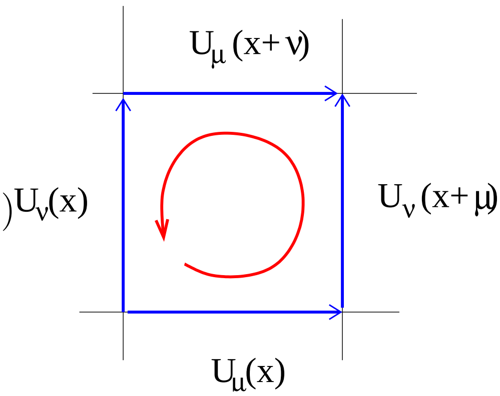
• Closed loops: $U(C) \to \Lambda(x_0) U(C) \Lambda^{-1}(x_0)$

• Trace of a closed loop: $\mathrm{Tr}\, U(C)$ is gauge invariant!

3.2 Lattice gauge fields

• Variables: parallel transporters from one lattice site to a neighbouring one, *Links:*

$$\begin{aligned} U_\mu(x) &= P \exp\left[ig \int_x^{x+\mu} dx_\mu A_\mu\right] \\ &= \exp\left[igaA_\mu(x + \tfrac{1}{2}\mu)\right] + O(ga^3) \end{aligned}$$



• The lattice action has to be gauge invariant. Only traces of closed loops are gauge invariant; the simplest one is *plaquette*

$$\mathrm{Tr}\, U_\Box = \mathrm{Tr}\ U_\mu(x) U_\nu(x+\mu) U^\dagger_\mu(x+\nu) U^\dagger_\nu(x)$$

- When a small (for SU(N)),

$$\mathrm{Re}\,\mathrm{Tr}\,U_\Box = N - \frac{a^4 g^2}{4} F^a_{\mu\nu} F^a_{\mu\nu} + O(g^4 a^6)$$

- $\to$ Wilson gauge action (K. Wilson, 1974):

$$\begin{aligned} S_g &= \frac{2N}{g^2 a^{4-d}} \sum_\Box \left[1 - \frac{1}{N}\mathrm{Re}\,\mathrm{Tr}\,U_\Box\right] \\ &= \int d^d x \tfrac{1}{2} \mathrm{Tr}\,[F_{\mu\nu}F_{\mu\nu}] + O(a^2 g^2) \end{aligned}$$

Partition function

$$Z = \int [\prod_{x,\mu} dU_\mu(x)] e^{-S_g}$$

Here the integral is over group elements U_μ (compact), *not* over algebra A_μ

Common notation:

$$\beta = \beta_G \equiv \frac{2N}{g^2} \quad \text{for SU(N),} \qquad \beta = \frac{1}{g^2} \quad \text{for U(1).}$$

3.3 Continuum limit for U(1) (in 4d)

- $U_\mu(x) = \exp iga A_\mu(x)$
- The Wilson action for U(1) is

$$\begin{aligned} S &= \frac{1}{g^2} \sum_{x;\mu<\nu} \Big(1 - \mathrm{Re}\,U_\mu(x) U_\nu(x+\mu) U^\dagger_\mu(x+\nu) U^\dagger_n u(x)\Big) \\ &= \frac{1}{g^2} \sum_{x;\mu<\nu} (1 - \mathrm{Re}\,\exp\,[iga(A_\mu(x) + A_\nu(x+\mu) - A_\mu(x+\nu) - A_\nu(x))]) \end{aligned}$$

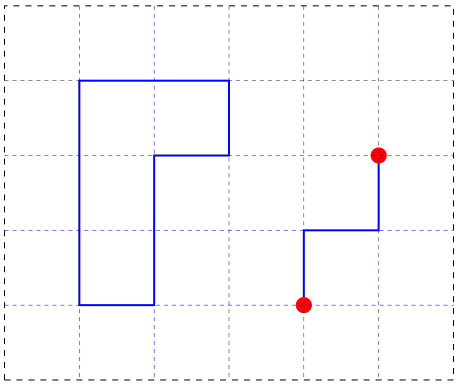
Expanding $A_\nu(x+\mu) = A_\nu(x) + a\partial_\mu A_\nu(x) + \frac{1}{2}a^2 \partial^2_\mu A_\nu(x) + \ldots$

$$\begin{aligned} S &= \frac{1}{g^2} \sum_{x;\mu<\nu} \Big(1 - \mathrm{Re}\,\exp\Big[iga^2(\partial_\mu A_\nu - \partial_\nu A_\mu) + O(a^4)\Big]\Big) \\ &= \frac{1}{g^2}\frac{1}{2} \sum_{x;\mu\neq\nu} \left[\frac{1}{2} g^2 a^2 F^2_{\mu\nu} + O(g^2 a^6)\right] \\ &= \frac{1}{4} \int d^4 x F^2_{\mu\nu} \end{aligned}$$

- Error in lagrangian $O(a^2)$
- Non-Abelian continuum limit comes in a similar way, but now one has to be careful with the commutators. Check it! (Campbell-Baker-Hausdorff formula $e^A e^B = e^{A+B-[A,B]+\cdots}$ might help, but it also comes directly from the expansion of the exponents.)

3.4 Gauge transformations:

- Gauge transform $\Lambda(x) \in$ SU(N) lives on lattice *sites*
- Link variable: $U_\mu(x) \to U'_\mu(x) = \Lambda(x) U_\mu(x) \Lambda^\dagger(x+\mu)$
  $\mapsto$
  $A_\mu(x) \to \Lambda(x) A_\mu(x) \Lambda^\dagger(x) + \frac{i}{g}\Lambda(x)\partial_m u \Lambda^\dagger(x)$
- Trace of a closed loop is gauge invariant
- Fundamental matter: $\phi(x) \to \Lambda(x)\phi(x)$, where ϕ is a N-component complex vector: operator
  $\phi^\dagger(x) U_P(x \mapsto y)\phi(y) =$
  $\phi^\dagger(x) U_\mu(x) U_\nu(x+\mu) \ldots U_\rho(y-\rho)\phi(y)$
  is gauge invariant.



- We want *only* gauge invariant animals on a lattice: lattice action + observables must consist of *closed loops* (gauge only quantities) or $\phi^\dagger U_P \phi$ - "strings" (matter).
- For example, the matter field kinetic term

$$(D_\mu\phi)^\dagger(D_\mu\phi),$$

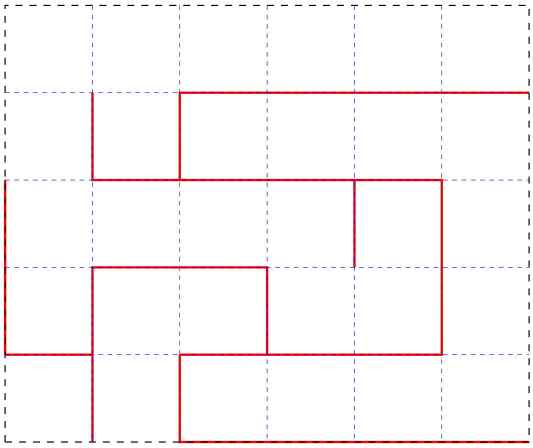
where $D_\mu = \partial_\mu + igA_\mu$, becomes

$$\frac{1}{a^2}\left[2d\phi^\dagger\phi - 2\sum_\mu \phi^\dagger(x) U_\mu(x)\phi(x+\mu)\right]$$

- *Elitzur's theorem:* expectation values of gauge non-invariant object = 0: $\langle U_\mu(x) \rangle = 0$.

3.5 Gauge fixing on the lattice

- Something you don't want to do
- Necessary for perturbative calculations
- Most of the (Euclidean) continuum gauges go over to lattice
- Special gauge: *maximal tree*



- A tree which connects every point of the lattice, but does not have closed loops.
- Link matrices $U_\mu(x)$ are *fixed* to arbitrary values (for example all $= 1$) in the tree
- Expectation values of gauge invariant quantities do not change (proof: easy)
- (Almost) complete gauge fixing

3.6 Confinement and Wilson loop

- Potential between static charge and anticharge, separated by R:

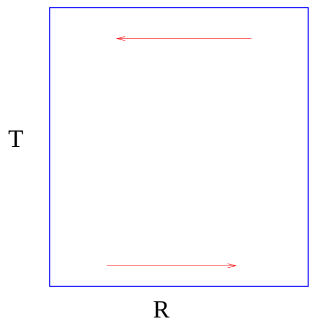
$$V(R \to \infty) \to \begin{cases} \infty: & \text{confinement} \\ \text{finite:} & \text{no confinement} \end{cases}$$

- Standard probe: Wilson loop W_{RT}. Let W be a rectangular path of size $R \times T$ along x_1 (say) and $x_0 = \tau$ directions.

$$W_{RT} = \mathrm{Tr}\, P \exp ig \oint_W ds_\mu A_\mu = \mathrm{Tr} \prod_{<xy>\in W} U_{xy}.$$

  Now $-\log W_{RT}$ gives the "free energy" of a static charge-anticharge ("quark-antiquark") system separated by R and which evolves for time T:

$$-\log\langle W_{RT}\rangle = V(R)T \qquad \text{valid as } T \to \infty$$



- *Perimeter law:* $-\log\langle W_{RT}\rangle \sim m(2R + 2T)$

  Free charges, m: "mass" of the charge due to gauge field

- *Area law:* $-\log\langle W_{RT}\rangle \sim \sigma RT$; $V(R) = \sigma R$

  Charges confined with linear potential, σ: string tension

- In general, $\frac{1}{T}\log\langle W\rangle \sim \sigma R +$ const. $+ c/R + \ldots$

3.6.1 *Motivation:*

- In temporal gauge $A_0 = 0 \quad \rightarrow \quad U_0 = 1$ we can define gauge field Hamiltonian $\hat{H}$ [Kogut-Susskind]
- Ψ: (trial) wave function of $q\bar{q}$ pair at $\bar{x}$, $\bar{y}$; spatial gauge transformations

  $\Psi_{ab} \rightarrow \Lambda_{ac}(\bar{x})\Lambda^{\dagger}_{bd}(\bar{y})\Psi_{cd}$
- $\hat{H}$ gauge invariant: $\langle\Psi'|e^{-T\hat{H}}|\Psi\rangle = 0$ unless Ψ, Ψ' have similar gauge transformation properties.

$$\begin{aligned} \langle\Psi|e^{-T\hat{H}}|\Psi\rangle &= \sum_n |\langle\Psi_n|\Psi\rangle|^2 e^{-E_n T} \\ &\rightarrow |\langle\Psi_0|\Psi\rangle|^2 e^{-E_0 T} \qquad \text{when } T \rightarrow \infty \end{aligned}$$

- $E_0 = V(R)$ ground state energy of static charges ($R = |\bar{x} - \bar{y}|$)
- Trial wave function: $\Psi_{ab} = U_{ab}(x \mapsto y)\Psi_{\text{vac}}$, where Ψ_{vac} is the (gauge invariant) vacuum wave function

$$\begin{aligned} \langle\Psi|e^{-T\hat{H}}|\Psi\rangle &= \frac{1}{Z}\int [dU]\,\mathrm{Tr}\,[U^{\dagger}(T;\bar{x}\mapsto\bar{y})\,U(0;\bar{x}\mapsto\bar{y})]e^{-S} \\ &= \langle W_{RT}\rangle \end{aligned}$$

- W_{RT} gauge invariant: can be measured in *any* gauge:

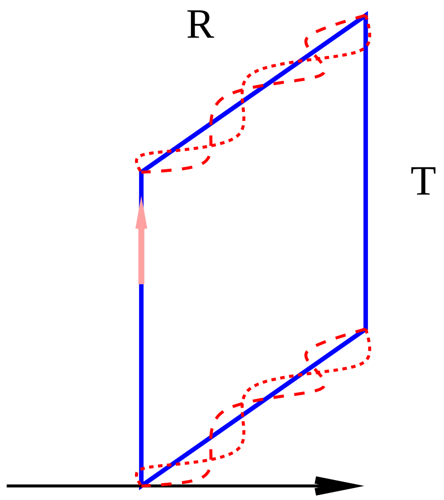
  $$-\frac{1}{T}\log\langle W_{RT}\rangle \rightarrow V(R)$$

  as $T \rightarrow \infty$.
- If we can guess/construct $\Psi \approx \Psi_0$, T need not be very large.

3.7 Measuring the string tension from Monte Carlo

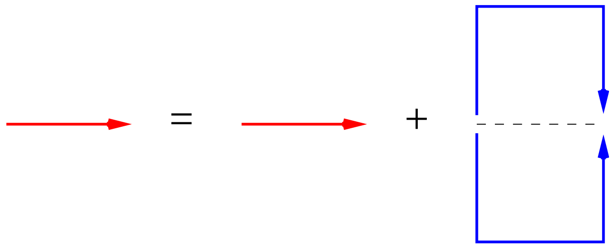
The Wilson loops become exponentially small ($\sim e^{-\sigma RT}$) as the size increases. However, the statistical noise is $\sim$ constant in magnitude independent of loop size! Thus, we need to increase the number of measurements exponentially as the loop size increases.

Smearing improves the situation: instead of using only straight const-T legs in the loop, take an averaged sum over paths around the straight one. The smearing is to mimic the wave function of the desired $q\bar{q}$ ground state. The hope is that now a smaller T-extent would be sufficient to get the asymptotic behaviour, i.e. look like the limit $T \to \infty$.



Smearing can be done recursively: let $i \perp t$, and

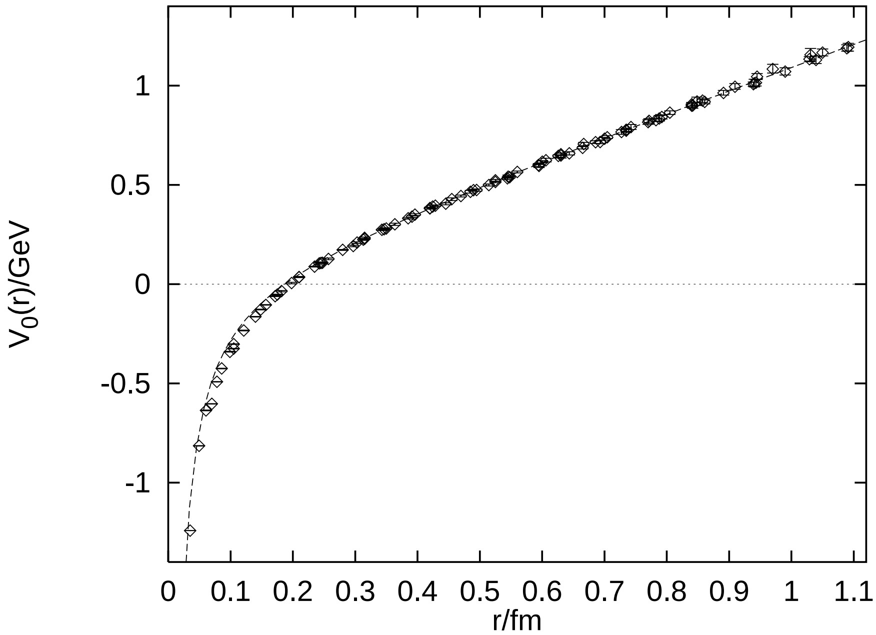
$$U_i(x) \leftarrow U_i(x) + c \sum_{j \perp i,t} U_j(x) U_i(x+\hat{j}) U_j^\dagger(x+\hat{i})$$



i.e. substitute link matrices on a plane $\perp$ to t by the weighted average of the link and the staples (which are also $\perp$ to t). This can be repeated a few times.

3.8 Wilson loop in pure gauge SU(3)

[Bali,Schilling, Wachter 1995] $48^3 \times 64$ quenched lattice, $\beta = 6.8$



For large loops

$$\langle W \rangle \sim e^{-V(r)T}$$

Phenomenological form

$$V(r) = V_0 + \sigma r - \frac{e}{r} + f\left[G_L(\vec{r}) - \frac{1}{r}\right]$$