Lattice Field Theory 2017 Problem set 5

Discussed on 6.11 and 13.11.

1. A representation of a group G is a set of matrices $R(U)$, for any $U \in G$, which obey $R(U_1)R(U_2) = R(U_1 U_2)$. The fundamental representation of SU(N) is the set of unitary $N \times N$ matrices with det $= 1$.

   Let now T^a, $a = 1 \ldots (N^2 - 1)$, be the generators of the fundamental representation of SU(N), i.e. they are traceless and Hermitean N^2 matrices which obey the algebra $[T^a, T^b] = i f^{abc} T^c$ and are normalised as $\mathrm{Tr}\, T^a T^b = \frac{1}{2}\delta^{ab}$. Show that the $(N^2 - 1)^2$-matrices t^a with $(t^a)^{bc} = -i f^{abc}$ are also generators of the group, i.e. they obey the same commutation relations. (It can be useful to consider the expression $[[T^a, T^b], T^c] = [T^a, [T^b, T^c]] - [T^b, [T^a, T^c]]$).

   t^a's generate the *adjoint* representation of SU(N), i.e. if $U = e^{i\theta^a T^a}$ is a fundamental representation matrix, $\tilde{U} = e^{i\theta^a t^a}$ is the corresponding adjoint representation matrix.

   Show that the djoint representation matrix can also be obtained through

   $$\tilde{U}^{ab} = 2\mathrm{Tr}\,[T^a U T^b U^\dagger]$$

   (it is sufficient to show this to 1st order in θ).