Lattice Field Theory 2017 Home exam 12.12–15.12

Return before 14.00 on 15.12 to Kari Rummukainen's mail box (3rd floor, C-wing). The bonus problems are more difficult, they are not necessary to get full points from the exam, but are taken into account in grading.

1. 1-dimensional massless free scalar theory ($\phi \in R$) can be described with the action

$$S_{\text{cont.}} = \int dx \frac{1}{2}(\partial\phi)^2 \longrightarrow S_{\text{latt}} = \sum_x a \frac{1}{2}\left(\frac{\phi(x+a)-\phi(x)}{a}\right)^2$$

   a) Show that the lattice action can be written equivalently in form

$$S = -\sum_x a \frac{1}{2}\phi(x)\frac{\phi(x+a)-2\phi(x)+\phi(x-a)}{a^2}$$

   (Note that $\sum_x f(x+a) = \sum_x f(x)$ etc.) This corresponds to partial integration of the continuum action, i.e. $\int dx \frac{1}{2}(\partial\phi)(\partial\phi) = -\int dx \frac{1}{2}\phi\partial^2\phi$.

   b) Show that the leading discretization errors of the lattice action are $O(a^2)$. Hint: consider the second expression, and Taylor expand $\phi(x \pm a)$ to sufficient order.

2. *Improved action:* Let us now attempt to cancel the $O(a^2)$ errors of the action in the above question. It is convenient to start from the second expression of the lattice action and improve the 2nd derivative discretization. Let us thus try

$$S_{\text{imp.}} = -\sum_x a \frac{1}{2}\phi(x)\frac{A\phi(x+2a)+B\phi(x+a)+C\phi(x)+B\phi(x-a)+A\phi(x-2a)}{a^2}$$

   Here we used symmetry between $x+a$ and $x-a$.

   What are the values of the constants A, B and C in order for the $O(a^2)$ error to vanish? What is the order of the leading error?

   Give the resulting action also in form $\sum_x a\,[C_1\phi(x)^2 + C_2\phi(x)\phi(x+a) + C_3\phi(x)\phi(x+2a)]$.

3. Consider the **Ising gauge theory** (or Z_2 gauge theory) in 3 dimensions. The partition function is

$$\begin{aligned} Z &= \sum_{s_i(x)} \exp[-\beta S] \\ S &= -\sum_{x,i<j} s_\square(i,j;x); \qquad s_\square(i,j;x) = s_i(x)s_j(x+\hat{i})s_i(x+\hat{j})s_j(x) \end{aligned}$$

   where x is 3-dimensional coordinate (with periodic boundary conditions), directions $i,j = 1,2,3$, the link variable $s_i(x) = \pm 1$ lives on the link connecting points x and $x+\hat{i}$, and finally $s_\square$ is a "plaquette" formed from the link variables. What are the gauge transformations in this model?

   Use the strong coupling (small β) expansion to calculate the Wilson loop and the string tension to a) leading and b) next to leading order. (Hint: use the "character" expansion of the action.)

4. *Polyakov loops and confinement*: Let us consider a SU(N) gauge field theory (no fermions) on an Euclidean lattice at non-zero temperature. In this case the lattice extent to imaginary time t direction is $1/T = aN_t$, and the boundary conditions to all directions are periodic. Let us assume the action is the standard single-plaquette gauge action discussed on the lectures.

   The *Polyakov loop* is defined as the trace of the product of the link matrices along a closed path around the t-direction (0-direction):

   $$P(\boldsymbol{x}) = \mathrm{Tr}[U_0(\boldsymbol{x}, 1)U_0(\boldsymbol{x}, 2) \dots U_0(\boldsymbol{x}, N_t)]$$

   (Answers to most points below are very short, almost one-liners and often qualitative.)

   a) Show that $P(\boldsymbol{x})$ is gauge invariant.

   b) The *center* of a group is defined as the set of group elements which commute with all elements of the group. What are the centers of SU(2) and SU(3)? (recall: center elements must be proportional to the unit matrix, and belong to the group.)

   c) Let z be a member of the center of the gauge group. Argue that the gauge action is invariant under the transformation

   $$U_0(\boldsymbol{x}, t) \mapsto zU_0(\boldsymbol{x}, t) \quad \text{for all } \boldsymbol{x} \text{ and some fixed } t.$$

   How does $P(\boldsymbol{x})$ transform under this transformation?

   d) In non-Abelian gauge theories, turns out that the above center symmetry is spontaneously broken at high temperatures, but restored at low temperatures. Polyakov loop is the order parameter of the transition, with $\langle P \rangle = 0$ at low T, $\langle P \rangle \neq 0$ at high T. (In real QCD, this transition happens around $T \approx 170\,\mathrm{MeV}$.) Considering the center transformation, how many degenerate broken phases are there in SU(2) and SU(3) gauge theories?

   e) Somewhat naively, the interaction of a static (heavy) quark with the gauge field can be related to the Polyakov line:

   $$e^{-F_q/T} \propto |\langle P \rangle|$$

   where F_q is the quark free energy. What does this indicate about the existence of single quarks at low T?

   More physically, the correlation function of two Polyakov lines behaves (almost) like a Wilson loop and is related to an effective potential between a static color charge and anticharge:

   $$\langle P^*(\boldsymbol{x})P(\boldsymbol{x} + \boldsymbol{r})\rangle \propto e^{-V_{\bar{q}q}(r)/T}$$

   What does this indicate about the color confinement in the high-T phase? Assuming the above correlation function vanishes exponentially in r in the low T-phase, what is the long distance behaviour of $V_{\bar{q}q}$?

   *Bonus (1p): using the symmery properties between 3d discrete spin models and Polyakov line expectation values, we can argue that the phase transitions in SU(2) and SU(3) theories are in the same universality class than certain simple 3-dimensional discrete spin models - which ones? (*Svetitsky-Yaffe *conjecture.) What does this say about the 1st/2nd order nature of the transition in these gauge theories?*