LFT Exam

1

$$S_L = \sum_x a \frac{1}{2} \left(\frac{\phi(x+a) - \phi(x)}{a} \right)^2$$

a)

$$= \sum_x a \frac{1}{2} \left[\frac{\phi(x+a)^2}{a^2} + \frac{\phi(x)^2}{a^2} - 2 \frac{\phi(x+a)\phi(x)}{a^2} \right]$$

$$= \sum_x a \frac{1}{2} \left[2 \frac{\phi(x)^2}{a^2} - \frac{\phi(x)\phi(x+a) + \phi(x)\phi(x-a)}{a^2} \right]$$

$$= - \sum a \frac{1}{2} \phi(x) \frac{\phi(x+a) - 2\phi(x) + \phi(x-a)}{a^2}$$

b) Expand $\phi(x \pm a) = \phi(x) \pm a \phi'(x) + \frac{1}{2} a^2 \phi''(x) \pm \frac{1}{3!} a^3 \phi^{(3)} + \frac{1}{4!} \phi^{(4)} + \dots$

$$\Rightarrow \quad S_L \simeq - \sum a \frac{1}{2} \phi(x) \frac{\frac{1}{2} a^2 \phi''(x) + \frac{1}{2} a^2 \phi''(x) + O(a^4)}{a^2}$$

$$= - \sum a \frac{1}{2} \phi \phi''(x) + \underline{O(a^2)}$$

2

Now $S_L \simeq - \sum a \frac{1}{2} \phi(x) \frac{A\phi(x+2a) + B\phi(x+a) + C\phi(x) + B\phi(x-a) + A\phi(x-2a)}{a^2}$

$$= - \sum a \frac{1}{2} \phi(x) \left[\frac{(2A + 2B + C)\phi(x) + \left(2 \frac{(2a)^2}{2} A + 2 \frac{a^2}{2} B\right) \phi''(x)}{a^2} + \frac{\left(2 \frac{(2a)^4}{4!} A + 2 \frac{a^4}{4!} B\right) \phi^{(4)} + O(a^6)}{a^2} \right]$$

works, if $\begin{cases} 2A + 2B + C = 0 \\ 4A + B = 1 \\ 16A + B = 0 \end{cases}$

$$\Rightarrow \begin{cases} B = -16A & \Rightarrow B = \frac{16}{12} = \frac{4}{3} \\ -12A = 1 & \Rightarrow A = -\frac{1}{12} \\ C = -2A - 2B = \frac{2}{12} - \frac{32}{12} = -\frac{30}{12} = -\frac{5}{2} \end{cases}$$

Now $S_L = -\sum a \frac{1}{2} \phi(x) \left(2A\phi(x+2a) + 2B\phi(x+a) + C\phi(x)\right) \cdot \frac{1}{a^2}$

$$= -\sum a^d \frac{1}{2a^2} \left(C\phi(x)^2 + 2B\,\phi(x)\phi(x+a) + 2A\,\phi(x)\phi(x+2a)\right)$$

$$= +\sum \frac{1}{a}\frac{1}{2}\left(+\frac{5}{2}\phi(x)^2 - \frac{8}{3}\phi(x)\phi(x+a) + \frac{1}{6}\phi(x)\phi(x+2a)\right)$$

③ Ising gauge

$$Z = \sum_{s_i} e^{+\beta \sum_\square s_\square}$$

, where $s_{\square_{ij}}(x) = s_i(x)\, s_j(x+\hat{i}) \times s_i(x+\hat{j})\, s_j(x)$

$s_i = \pm 1$

$\underline{\underline{s_\square = \pm 1}}$

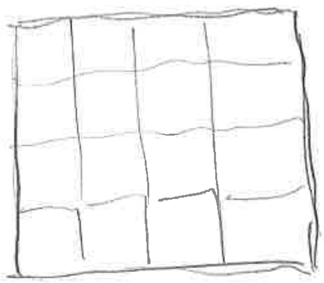
Gauge transformations

$g(x) = \pm 1$, and

$$s_i(x) \to g(x)\, s_i(x)\, g(x+\hat{i})$$

all closed loops are gauge invariant

Wilson loop:



$W = \prod s_i(x)$

x, i along the edge!

3 plaqs/site in 3d

3 links/site

Strong coupling: β small

$$Z = \sum_{s_i} e^{+\beta \sum_\square s_\square} = \sum_{s_i} \prod_\square e^{\beta s_\square} = \sum_{s_i} \prod_\square (\cosh\beta + s_\square \sinh\beta)$$

$$= \sum_{s_i} \prod_\square \overbrace{\cosh\beta}^{a}(1 + s_\square \overbrace{\tanh\beta}^{b}) = a^{\overset{\text{plaqs}}{3V}} \times 2^{\overset{\text{links}}{3V}}(1 + \ldots)$$

now need prod of $s_\square$'s so that each link appears twice
$\to$ closed surfaces!

Graphically,

$1 +$  $+$  $+ \ldots$

$$= 1 + Vb^6 + 3Vb^{10} + O(b^{12}),$$

thus,

$$Z = a^{3V} \times 2^{3V} \left(1 + V b^6 + 3Vb^{10} + O(b^{12})\right)$$

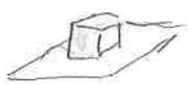
Wilson:

$$\langle W \rangle = \frac{1}{Z} \sum_{S_i} \Big(\prod_{x \in loop} S_i(x_e)\Big) \Big(\prod_{\square} a(1 + S_\square b)\Big)$$

Again, leading contrib when we <u>tile</u> the loop: each link within the loop comes twice

$$\langle W \rangle = \frac{1}{Z} a^{3V} 2^{3V} \left(b^{RT} + 2RT\, b^{RT+4} + \text{next-to-next} + \ldots \right.$$

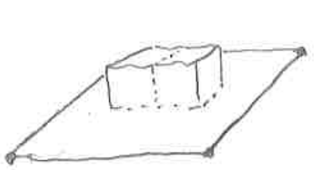
Next contrib. comes from a "lump"



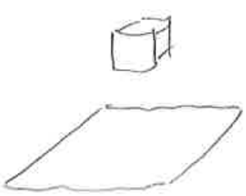
$R \times T$ places, 2 dirs

$R \times T + 4$ plaqs

And next-to-next order (not asked) from a 1×2 lump; or a loose cube: (6 new plaqs)



$$2\big((R-1)T + R(T-1)\big)$$
$$= 4RT - 2T - 2R$$



$V - 2RT$

↑

cannot share plaq

Thus,

$$\langle W \rangle = \frac{1}{1+Vb^6+O(b^{10})}\left(b^{RT} + 2RT\,b^{RT+4} + (V+2RT-2T-2R)\,b^{RT+6} + O(b^{RT+8})\right)$$

$$= \underset{\substack{\uparrow\\ \text{LO}}}{b^{RT}}\Big(1 + \underset{\substack{\uparrow\\ \text{NLO}}}{2RT\,b^4} + \underset{\substack{\uparrow\\ \text{NNLO}}}{(2RT-2T-2R)\,b^6} + O(b^8)\Big)$$

String tension: $\langle W \rangle \propto e^{-TV(R)}$, V "potential"

$$\Rightarrow \quad V(R) = -\frac{1}{T}\ln\langle W\rangle + \text{const.}$$

$$= -\frac{1}{T}RT\ln b - \frac{1}{T}\ln\left(1+2RT\,b^4 + (2RT-2T-2R)\,b^6 + \dots\right)$$

$$= -R\ln b - 2Rb^4 - \left(2R-2-2\frac{R}{T}\right)b^6 + O(b^8)$$

String tension: linear in R bit when $T\to\infty$:

$$\underline{\sigma = -\ln b - 2b^4 - 2b^6} + O(b^8)$$

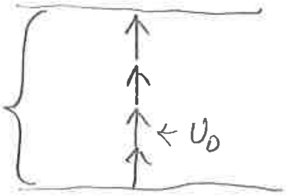
$b = \tanh\beta$, note β small $\Rightarrow -\ln b > 0$.

4

Polyakov loop

$$P(x) = \mathrm{Tr}\left[U_0(x,1)\,U_0(x,2)\ldots U_0(x,N_t)\right]$$

$aN_t = \frac{1}{T}$



a) Tr of a closed loop is gauge invariant.

Or : $U_0(\bar{x},i) \to g(\bar{x},i)\,U_0(\bar{x},i)\,g^+(x,i+1)$

where $g \in SU(N)$.

Now $P(x) \to \mathrm{Tr}\left[g(\bar{x},1)\,U_0(x,1)\,g^+(\bar{x},2)\,g(\bar{x},2)\,U_0(x,2)\,g^+(\bar{x},3)\ldots\right.$

$= P(x)$.

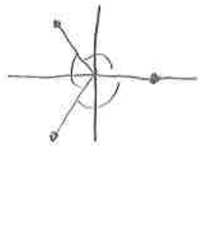
b) <u>Center</u> elements : $\propto \mathbb{I}$ . . $c\mathbb{I}$, $c \in \mathbb{C}$,

clearly commute with any $U \in SU(N)$.

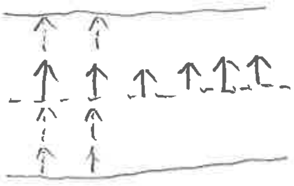
$$(c\mathbb{I})^+ = (c\mathbb{I})^{-1} \Rightarrow c^* = \frac{1}{c} \Rightarrow c^*c = 1 \Rightarrow |c| = 1$$

$$\det(c\mathbb{I}) = c^N = 1 \Rightarrow \underline{\underline{c = e^{i2\pi n/N}}}, \quad n = 0, 1, \ldots (N-1)$$

$$\begin{cases} SU(2) : c = \pm 1, & \text{center} = \{\mathbb{I}, -\mathbb{I}\} \\ SU(3) : c = 1, e^{i2\pi/3}, e^{i4\pi/3} = e^{-i2\pi/3} \end{cases}$$

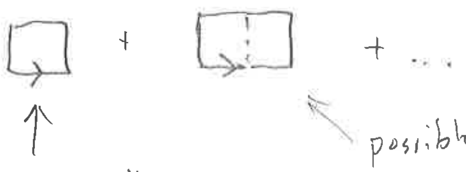


c)



$\leftarrow U_0 \to zU_0$ on this slice

Gauge action consists of (small) <u>closed loops</u> (traced) :



plaquette action

possible contrib. to improved actions

Clearly, if a loop includes $U_0(x, t_{fix})$, it also includes $U_0^+(y, t_{fix})$

$$\mathrm{Tr}\left[U_1 U_2 .. U_0(x,t_{fix}) \ldots U_0^+(y, t_{fix}) \ldots\right] \to$$
$$\mathrm{Tr}\left[.. \, z U_0(x,t_{fix}) .. U_0^+(y,t_{fix}) z^+ \ldots\right] = \mathrm{Tr}\left[\ldots U_0(x,t) .. U_0(y,t)\right]$$

<u>Action is invariant.</u>

$$P(x) = \mathrm{Tr}\left[\ldots U_0(x,t_{fix}) \ldots\right] \to \mathrm{Tr}\left[\ldots z U_0(x, t_{fix}) \ldots\right]$$
$$= c\, \mathrm{Tr}\left[\ldots U_0(x,t_{fix}) \ldots\right] = c\, P(x)$$

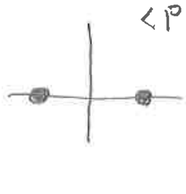
where $\underline{z = c\mathbb{1}}$. Thus, $P(x)$ is <u>rotated</u> in complex plane

d) $\langle P \rangle \neq 0$ at high T. Because $U_0(x,t_{fix}) \to z U_0(x,t_{fix})$ (c))

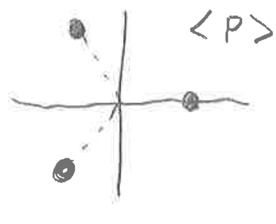
is a symmetry of the action, states with $\langle P \rangle$ and $z\langle P \rangle$ are equally likely.

$\Rightarrow$ as many broken states as elements of center
(could be more, but no reason to have)

$SU(2)$: 2



$SU(3)$: 3



e) If

$$e^{-F_q/T} \propto |\langle P\rangle| \quad , \quad F_q : \text{free energy of a test quark}$$

At low T, $\langle P\rangle = 0 \Rightarrow F_q/T = \infty \Rightarrow$ free quarks do not exist (confinement)

At high T, $|\langle P\rangle| > 0$ and F_q finite $\Rightarrow$ free quarks ? Quark-gluon plasma

If $\langle P\rangle = c$ = const. at high T phase,

then
$$\langle P(\bar{x})^* P(\bar{x}+\bar{r})\rangle \to \langle P\rangle^* \langle P\rangle = |c|^2 \text{ as } |\bar{r}| \to \infty$$
$$\propto e^{-V_{\bar{q}q}(r)/T}$$

$$\Rightarrow V_{\bar{q}q}(r) \to \text{const as } r \to \infty.$$

No linearly growing potential, no confinement

If, at low T,

$$\langle P(\bar{x})^* P(\bar{x}+\bar{r})\rangle \propto e^{-ar} \quad \text{as } r \to \infty$$
$$\propto e^{-V_{\bar{q}q}(r)/T}$$

$$\Rightarrow V_{\bar{q}q}(r) \propto aT \times r$$

linear potential, string tension aT

bonus: P transition

Low T   high T

SU(2) :  $\underline{\text{3d Ising}}$

2nd order

2 degenerate

SU(3) :  $\underline{\text{3d Potts}}$

1st order

3 degenerate

3d , because 4th dim size $= \frac{1}{T} \ll$ spatial size

and $P(\bar{x})$ is a 3d observable

Note: $\begin{cases} \text{low } T \\ \text{high } T \end{cases} \longleftrightarrow \begin{cases} \text{high } T \\ \text{low } T \end{cases}$

SU(N)   spin model