Physics on the lattice

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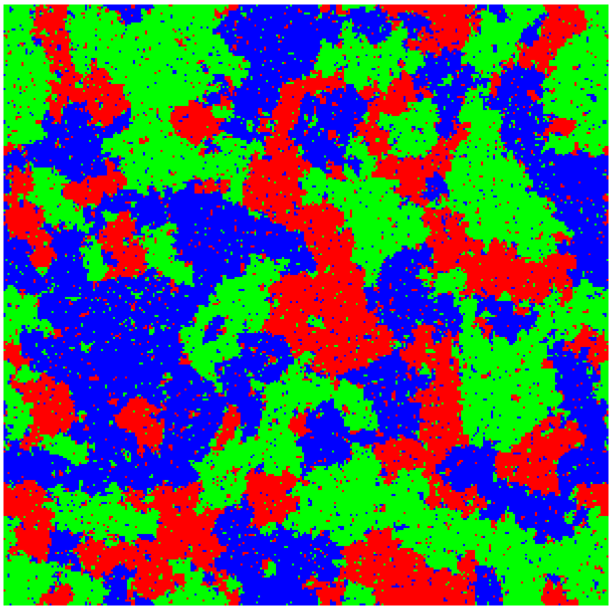


Intro

Lattice: formulation of a field theory, Feynman path integral, on a discrete space-time lattice (K. Wilson 1974). The lattice is usually chosen to be regular and (hyper)cubic, the simplest choice.
It's a trick, which makes the path integral mathematically well-defined. However, our space-time is not discrete $\rightarrow$ the results have to be extrapolated to the continuum limit, where the *lattice spacing* a goes to zero.
Nevertheless, it:

- provides an explicit ultraviolet cutoff, removing the UV divergences so common in field theories. These have to be renormalized before the continuum limit is taken.
- allows the use of (almost all of) the analytical tools used for continuum field theories – especially, the lattice perturbation theory is equivalent to the standard one (but more cumbersome to use).
- provides new analytical methods and tools, not possible for continuum field theories (strong coupling/"high temperature" expansions, duality relations etc.).
- emphasizes the connection between *field theory and statistical mechanics*: Feynman path integral $\equiv$ partition function. All of the tools developed for statistical mechanics are at our disposal!
- permits the evaluation of the path integral numerically, using *Monte Carlo calculations*. The integral is evaluated as-is, without expanding it in any small parameter ($\equiv$ *non-perturbative*).

Models studied in HEP context:

- *Lattice QCD:*

  Since around 1980 Quantum Chromodynamics (QCD) has been studied extensively on the lattice. The results have given invaluable *qualitative* insights (confinement); and, during the last ~ 5 years the results have also been *quantitatively* comparable to experiments.

  - Confinement mechanism, hadronic mass spectrum, matrix elements . . .
  - QCD phase transition at $T \approx 150$ GeV

- *Electroweak phase transition:*

  Electroweak theory at $T = 0$ is perturbative: coupling constants are small, and the gauge bosons W, Z are massive. However, as T increases, the "broken SU(2) symmetry" is restored: gauge bosons become massless, and IR singularites make the theory non-perturbative.

  Since $\sim$ 1994 EW theory (theories) have been studied extensively on the lattice, with very solid results $\Rightarrow$ we know how the EW phase transition behaves. Applied also to MSSM (Minimal Supersymmetric Standard Model)

- *Almost any other model . . .*

  - QED
  - Ginzburg-Landau theory
  - Quantum gravity
  - . . .

Units:

Standard HEP units ("natural", "God-given", depending on your source), where

$$c = \hbar = k_B = 1,$$

and thus

$$[length]^{-1} = [time]^{-1} = [mass] = [temperature] = [energy] = \text{GeV}.$$

Mass m = rest energy mc^2 = (Compton wavelength)$^{-1}$ $mc/\hbar$.

$$(1\,\text{GeV})^{-1} \approx 0.2\,\text{fm}$$

Books:

- M. Creutz, *Quarks, Gluons and Lattices*, Cambridge University Press 1983
- I. Montvay and G. Münster, *Quantum Fields on a Lattice*, Cambridge University Press 1994
- H.J. Rothe. *Lattice Gauge Theories: an Introduction*, World Scientific 1992
- ...

Fundamentals:

- Quantum field theory (QFT) can be defined using *Feynman's path integral* [R. Feynman, Rev. Mod. Phys. 20, 1948]:

$$Z = \int \Big[\prod_x d\phi(x)\Big] \exp[iS(\phi)] \tag{1}$$
$$S = \int d^4x \mathcal{L}(\phi, \partial_t \phi)$$

  Here $x = (x_0, x_1, x_2, x_3)$, and $g_{\mu\nu} = \text{diag}(+,-,-,-)$

- Physical observables can be evaluated if we add a source $S \to S + J_x\phi_x$. For example, 1- and connected 2-point functions are:

$$\langle \phi_x \rangle = \frac{\partial}{i\partial J_x}\bigg|_{J=0} \log Z = \frac{1}{Z}\int [d\phi]\phi_x \exp[iS]$$
$$\langle \phi_x \phi_y \rangle = \frac{\partial}{i\partial J_x}\frac{\partial}{i\partial J_y} \log Z = \frac{1}{Z}\int [d\phi]\phi_x\phi_y \exp[iS] - \langle \phi_x \rangle^2$$

- These can be computed using perturbation theory, if the coupling constants in S are small (see any of the oodles of QFT textbooks).

- *However:* if
  - 4-volume is *finite*, and
  - 4-coordinate x is *discrete* ($x \in aZ^4$, a lattice spacing),

  the integral in (1) has finite dimensionality and can, in principle, be evaluated by brute force (= computers)

  $\mapsto$ gives fully *non-perturbative* results.

- Need to extrapolate: $V \to \infty$, $a \to 0$ in order to recover continuum physics.

- But:
  - the dimensionality of the integral is (typically) huge
  - $\exp iS$ is complex+unimodular: every configuration $\{\phi\}$ contributes with equal magnitude

  $\mapsto$ extremely slow convergence in numerical computations (useless in practice).
- This is (largely) solved by using *imaginary time formalism* (Euclidean spacetime).
- Imaginary time also admits non-zero temperature T.

Path integral in imaginary time

Consider *scalar field theory* in Minkowski spacetime:

$$S = \int d^3x\, dt\, \mathcal{L}(\phi, \partial_t\phi) = \int d^3x\, dt\, \left[\tfrac{1}{2}\partial_\mu\phi\, \partial^\mu\phi - V(\phi)\right]$$

We obtain the (classical) *Hamiltonian* by Legendre transformation:

$$\begin{aligned} H &= \int d^3x\, dt\, \left[\pi\dot{\phi} - \mathcal{L}\right] \\ &= \int d^3x\, dt\, \left[\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\partial_i\phi)^2 + V(\phi)\right] \end{aligned}$$

Here π is canonical momentum for ϕ: $\pi = \delta\mathcal{L}/\delta\dot{\phi}$.

Quantum field theory: consider Hilbert space of states $|\phi\rangle$, $|\pi\rangle$, and *field operators* ($\phi \to \hat{\phi}$, $\pi \to \hat{\pi}$, $H \to \hat{H}$), with the usual properties:

- $\hat{\phi}(\bar{x})|\phi\rangle = \phi(\bar{x})|\phi\rangle$
- $\int [\prod_x d\phi(\bar{x})]|\phi\rangle\langle\phi| = 1 \qquad \int [\prod_x d\pi(\bar{x})/(2\pi)]|\pi\rangle\langle\pi| = 1$
- $[\hat{\phi}(\bar{x}), \hat{\phi}(\bar{x}')] = -i\delta^3(\bar{x} - \bar{x}')$
- $\langle\phi|\pi\rangle = \exp\left[i \int d^3x \pi(\bar{x})\phi(\bar{x})\right]$

Time evolution operator is $U(t) = e^{-i\hat{H}t}$

Feynman showed that the quantum theory defined by $\hat{H}$ and the Hilbert space is equivalent to the path integral (1). We shall now do this in imaginary time.

Let us now consider the quantum system in *imaginary time:*

$$t \to \tau = it\,, \qquad \exp[-i\hat{H}t] \to \exp[-\tau\hat{H}]$$

This makes the spacetime *Euclidean*: $g = (+ - - -) \to (+ + + +)$. We shall keep the Hamiltonian $\hat{H}$ as-is, and also the Hilbert space.

Let us also discretize space (convenient for us but not necessary at this stage), and use finite volume:

$$\begin{aligned}
\bar{x} &= a\bar{n}, \qquad n_i = 0 \ldots N_s \\
\int d^3x &\to a^3 \sum_x \\
\partial_i \phi &\to \frac{1}{a}[\phi(\bar{x} + \bar{e}_i a) - \phi(\bar{x})] \equiv \Delta_i \phi(\bar{x})
\end{aligned}$$

Thus, the Hamiltonian is:

$$\hat{H} = a^3 \sum_x \left[\tfrac{1}{2}[\hat{\pi}(x)]^2 + \tfrac{1}{2}[\Delta_i \hat{\phi}(\bar{x})]^2 + V[\hat{\phi}(\bar{x})]\right]$$

Let us now consider the amplitude $\langle \phi_B | e^{-(\tau_B - \tau_A)\hat{H}} | \phi_A \rangle$:

1) divide time in small intervals: $\tau_B - \tau_A = N_\tau a_\tau$. Obviously

$$\langle \phi_B | e^{-(\tau_B - \tau_A)\hat{H}} | \phi_A \rangle = \langle \phi_B | (e^{-a_\tau \hat{H}})^{N_\tau} | \phi_A \rangle.$$

2) insert 1-operators $\int [\prod_x d\phi_i] |\phi_i\rangle\langle\phi_i|$, $\quad \int [\prod_x d\pi_i] |\pi_i\rangle\langle\pi_i|$

$$\begin{aligned}
&\langle \phi_B | e^{-\tau\hat{H}} | \phi_A \rangle = \\
&\int \left[\prod_{i=2}^{N_\tau} \prod_x d\phi_i(x)\right] \left[\prod_{i=1}^{N_\tau} \prod_x d\pi_i(x)\right] \langle \phi_B | \pi_{N_\tau} \rangle \langle \pi_{N_\tau} | e^{-a_\tau \hat{H}} | \phi_{N_\tau} \rangle \\
&\times \langle \phi_{N_\tau} | \pi_{N_\tau - 1} \rangle \langle \pi_{N_\tau - 1} | e^{-a_\tau \hat{H}} | \phi_{N_\tau - 1} \rangle \ldots \times \langle \phi_2 | \pi_1 \rangle \langle \pi_1 | e^{-a_\tau \hat{H}} | \phi_A \rangle
\end{aligned}$$

Thus, we need to calculate

$$\begin{aligned}\langle\pi_i|e^{-a_\tau\hat{H}}|\phi_i\rangle &= \exp\left[-a^3 a_\tau \sum_x \left(\tfrac{1}{2}\pi_i^2 + \tfrac{1}{2}(\Delta_i\phi_i)^2 + V[\phi_i]\right)\right] \\ &\times \ \langle\pi_i|\phi_i\rangle + O(a_\tau^2)\end{aligned}$$

and

$$\begin{aligned}\langle\phi_{i+1}|\pi_i\rangle\langle\pi_i|\phi_i\rangle &= e^{ia^3\sum_x \pi_i(\bar{x})\phi_{i+1}(\bar{x})} e^{ia^3\sum_x \pi_i(\bar{x})\phi_i(\bar{x})} \\ &= \exp[ia^3 a_\tau \sum_x \pi_i(\bar{x})\Delta_0\phi_i(\bar{x})] \qquad (2)\end{aligned}$$

where we have defined $\Delta_0\phi_i = \dfrac{1}{a_\tau}(\phi_{i+1} - \phi_i)$.
We can now integrate over $\pi_i(\bar{x})$:

$$\begin{aligned}&\int [\prod_x \pi_i(\bar{x})] \exp[a^3 a_\tau \sum_x -\tfrac{1}{2}\pi_i^2 + (i\Delta_0\phi_i)\pi_i] \\ &= \left[\frac{2\pi}{a^3 a_\tau}\right]^{N_S^3/2} \times \exp[-a^3 a_\tau \sum_x \tfrac{1}{2}(\Delta_0\phi_i(\bar{x}))^2]\end{aligned}$$

Repeating this for $i = 1 \ldots N_\tau$, and identifying the time coordinate $x_0 = \tau = a_\tau i$, we finally obtain the path integral

$$\langle\phi_B|e^{-\tau\hat{H}}|\phi_A\rangle \propto \int [\prod_x d\phi] e^{-S_E}$$

where S_E is the *Euclidean* (imaginary time) action

$$\boxed{\begin{aligned}S_E &= a^3 a_\tau \sum_x (\tfrac{1}{2}(\Delta_\mu\phi)^2 + V[\phi]) \\ &\to \int d^4x (\tfrac{1}{2}(\partial_\mu\phi)^2 + V[\phi])\end{aligned}}$$

and we have the boundary conditions $\phi(\tau_A) = \phi_A$, $\phi(\tau_B) = \phi_B$.

The path integral is precisely in the form of a *partition function* for a 4-dimensional *classical* statistical system, with the identification $S_E \leftrightarrow H/(k_B T)$.
For convenience, we make the system *periodic* in time by identifying $\phi_A = \phi_B$ and integrating over ϕ_A.

We can now make a connection between the correlation functions of the "statistical" theory and the Green's functions of the quantum field theory. First, note that we can interpret $T_{\phi_{i+1},\phi_i} = \langle \phi_{i+1}| e^{-a_\tau \hat{H}} |\phi_i\rangle$ as a *transfer matrix*. In terms of T the partition function is

$$Z = \int [d\phi] e^{-S_E} = \mathrm{Tr}\,(T^{N_\tau})$$

Let us label the eigenvalues of T by $\lambda_0, \lambda_1 \ldots$, so that $\lambda_0 > \lambda_1 \geq \ldots$. Note that $\lambda_i = \exp - E_i$, where E_i are eigenvalues of $\hat{H}$. Thus, λ_0 corresponds to the state of lowest energy, *vacuum* $|0\rangle$. If we now take N_τ to be very large (while keeping a_τ constant; i.e. take $\Delta\tau$ to be large),

$$Z = \sum_i \lambda_i^{N_\tau} = \lambda_0^{N_\tau}[1 + O((\lambda_1/\lambda_0)^{N_\tau})]$$

For example, a 2-point function can be written as (let $i - j > 0$)

$$\langle \phi_i \phi_j \rangle = \frac{1}{Z} \int [d\phi] \phi_i \phi_j e^{-S_E} = \frac{1}{Z} \mathrm{Tr}\,(T^{N_\tau - i + j} \hat{\phi} T^{i-j} \hat{\phi}).$$

Taking now $N_\tau \to \infty$, and recalling $a_\tau (i - j) = \tau_i - \tau_j$,

$$\langle \phi_i \phi_j \rangle = \langle 0 | \hat{\phi} (T/\lambda_0)^{i-j} \hat{\phi} | 0 \rangle = \langle 0 | \hat{\phi}(\tau_i) \hat{\phi}(\tau_j) | 0 \rangle,$$

where we have introduced time-dependent operators

$$\hat{\phi}(\tau) = e^{\tau \hat{H}} \hat{\phi} e^{-\tau \hat{H}}.$$

Allowing for both positive and negative time separations of τ_i and τ_j, we can identify

$$\langle\phi(\tau_i)\phi(\tau_j)\rangle = \langle 0|\mathcal{T}[\hat{\phi}(\tau_i)\hat{\phi}(\tau_j)]|0\rangle,$$

where $\mathcal{T}$ is the time ordering operator.

Mass spectrum:

- Green's functions in time τ:

$$\begin{aligned}
\langle 0|\Gamma(\tau)\Gamma^\dagger(0)|0\rangle &= \frac{1}{Z}\int[d\phi]\Gamma(\tau)\Gamma^\dagger(0)e^{-S} \\
&= \langle 0|e^{\hat{H}\tau}\Gamma(0)e^{-\hat{H}\tau}\Gamma^\dagger(0)|0\rangle \\
&= \langle 0|\Gamma(0)\sum_n |E_n\rangle\langle E_n|e^{-\hat{H}\tau}\Gamma^\dagger(0)|0\rangle \\
&= \sum_n e^{-E_n\tau}|\langle 0|\Gamma(0)|E_n\rangle|^2 \\
&\rightarrow e^{-E_0\tau}|\langle 0|\Gamma(0)|E_0\rangle|^2 \qquad \text{as } \tau\rightarrow\infty
\end{aligned}$$

where $|E_0\rangle$ is the *lowest* energy state with non-zero matrix element $\langle 0|\Gamma(0)|E_0\rangle$.

$\rightarrow$ measure masses (E_0) from the exponential fall-off of correlation functions.

Finite temperature

Connection *Euclidean QFT* $\leftrightarrow$ *classical statistical mechanics* was derived for zero-temperature quantum system. However, this can be readily generalized to finite temperature:

Quantum thermodynamics w. the Gibbs ensemble:

$$Z = \mathsf{Tr}\, e^{-\hat{H}/T} = \int [d\phi] \, \langle \phi | e^{-\hat{H}/T} | \phi \rangle$$

Expression is of the same form as the one which gave us the Euclidean path integral for $T = 0$ theory! The difference here is

1) Finite + fixed "imaginary time" interval $1/T$
2) Periodic boundary condition: $\phi(1/T) = \phi(0)$.

Repeating the previous derivation, the partition function becomes

$$Z(T) = \int [d\phi] \, e^{-S_E} = \int [d\phi] \, \exp[-\int_0^{1/T} d\tau \int d^3x \mathcal{L}_{\mathcal{E}}]$$

Thus, a connection between:

– Quantum statistics in 3d: $Z = \mathsf{Tr}\, e^{-\hat{H}/T}$
– Classical statistics in 4d: $Z = \int [d\phi] e^{-S}$

Euclidean P.I. is a very common tool for finite T field theory analysis [J. Kapusta, *Finite Temperature Field Theory*, Cambridge University Press]

Some terminology:

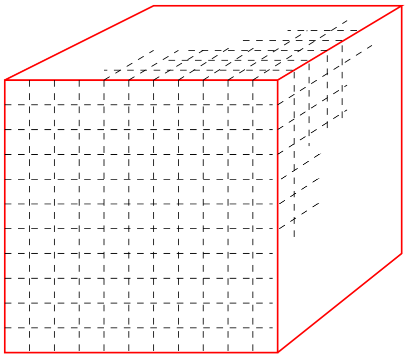
In numerical work, lattice is a finite box with finite lattice spacing a. In order to obtain continuum results, we should take 2 limits:

- $V \to \infty$ *thermodynamic limit*
- $a \to 0$ *continuum limit*

Both have to be controlled – expensive!

1) Perform simulations with fixed a, various V. Extrapolate $V \to \infty$.
2) Repeat 1) using different a's. Extrapolate $a \to 0$.
3) [In QCD, one often has to extrapolate $m_q \to m_{q,\text{phys.}}$.]

$T = 0$:



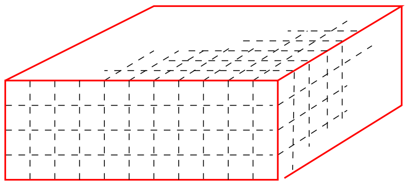
1) $V \to \infty$:

   $N_\tau, N_s \to \infty$, a constant.

2) continuum:

   $a \to 0$.

$T > 0$:



1) $V \to \infty$:

   $N_s \to \infty$, N_τ, a constant.

2) continuum:

   $a \to 0$, $\frac{1}{T} = aN_\tau$ constant.

Scalar field

Free scalar field on a finite d-dimensional lattice with periodic boundary conditions:

$$x_\mu = a n_\mu, \quad n_\mu \in Z$$

Action:

$$S = \sum_x a^d \left[\tfrac{1}{2} \sum_\mu \frac{1}{a^2} (\phi_{x+\mu} - \phi_x)^2 + \frac{1}{2} m^2 \phi^2 \right] = a^d \left[\tfrac{1}{2} \phi_x \Box_{x,y} \phi_y + \tfrac{1}{2} m^2 \phi_x^2 \right]$$

(implicit sum over x, y), and the lattice d'Alembert operator is

$$\Box_{x,y} \phi_y = -\Delta^2 \phi = \sum_\mu \frac{2\phi_x - \phi_{x+\mu} - \phi_{x-\mu}}{a^2} \delta_{x+\mu,y}$$

The action is of form $S = \frac{1}{2} \phi_x M_{x,y} \phi_y$, and

$$Z = \int [d\phi] e^{-S} = (\mathrm{Det}\, M/2\pi)^{-1/2}$$

Fourier transforms:

$$\tilde{f}(k) = \sum_x a^d e^{-ikx} f(x)$$

Since $x = an$, $\tilde{f}(k + 2\pi n) = \tilde{f}(k)$, and we restrict k to *Brillouin zone*: $-\pi/2 < k_\mu \leq \pi/2$
Inverse transform:

$$f(x) = \int_{-\pi/a}^{\pi/a} \frac{d^d k}{(2\pi)^d} e^{ikx} \tilde{f}(k)$$

Note: often it is convenient to use dimensionless natural lattice units $x_\mu \in Z$, $-\pi < k_\mu \leq \pi$.

Lattice propagator:

The lattice propagator $G(x,y)$ is defined to be the inverse of operator $a^{-d}M = (\Box + m^2)$:

$$\sum_y a^d(\Box_{x,y} + m^2)G(y,z) = \delta_{x,z}$$

Take Fourier transform ($G(x,y) = G(x-y)$):

$$\left[\sum_\mu \frac{2}{a^2}(2 - 2\cos k_\mu a) + m^2\right]\tilde{G}(k) = 1$$

which gives the lattice propagator

$$\tilde{G}(k) = \frac{1}{\hat{k}^2 + m^2}, \qquad \text{where} \qquad \hat{k}_\mu = \frac{4}{a^2}\sin^2\frac{k_\mu a}{2}.$$

Continuum limit: when $a \to 0$, $\tilde{G}(k) = 1/(k^2 + m^2) + O(a^2)$.

Generating function for Green's functions:

$$S \to S(J) = \sum_x a^d\left[\tfrac{1}{2}\phi_x(\Box + m^2)\phi_x - J_x\phi_x\right]$$

Now

$$Z(J) = \int[d\phi]e^{-S(J)} = Z(0)\exp[\sum_{x,y} a^{2d}\tfrac{1}{2}J_x G(x,y)J_y]$$

N-point functions

$$\langle\phi_x \ldots \phi_y\rangle = Z(0)^{-1}\frac{\delta}{\delta J_x}\cdots\frac{\delta}{\delta J_y}Z(J)\Big|_{J=0}$$

Interactions: just modify (for example)

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \frac{1}{4!}\lambda\phi^4$$

Numbers from the lattice:

- *Low temperature expansion:*
  - Expansion of $Z = \int [d\phi] e^{-E/T}$ around $T = 0$
  - Perturbations around a $S = S_{\min}$ state
  - *Weak coupling expansion*, perturbation theory for continuous fields
- *High temperature expansion:*
  $e^{-E/T} \sim 1 - E/T + \frac{1}{2}(E/T)^2 + \ldots$
  - Expansion around a "random" state
  - *Strong coupling expansion*
  - *Hopping parameter expansion*
  - No direct continuum counterpart
- *Mean field approximation*
- *Numerical evaluation: Monte Carlo integration*
- *Exact results: dualities etc.*

Ising model: low + high temperature expansions

Ising model: every lattice point has a spin $s_i = \pm 1$

$$Z = \sum_{s_i} e^{-\beta H}, \qquad H = -\frac{1}{2} \sum_{<ij>} s_i s_j$$

$< ij >$: nearest neighbour pairs, $\beta = 1/T$

The model has a symmetry breaking phase transition at a Curie point β_c (in 2D: $\beta_c = \log(1+\sqrt{2}) \approx 0.8814$).

If $\beta \leq \beta_c$ ($T \geq T_c$), $\langle s \rangle = 0$, whereas if $\beta > \beta_c$, $\langle s \rangle \neq 0$.

Low temperature expansion for 2d Ising

- 2d Ising model at $\beta \gg 1$. For simplicity, redefine

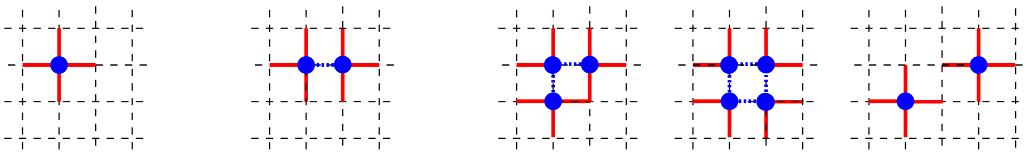
  $H \rightarrow H' = \sum_{<ij>}[1 - \delta(s_i, s_j)]$ so that a completely ordered system (all $s_i = +1$ or -1) has $H = 0$.

- Assume that $\langle s \rangle > 0$ (for example, boundaries fixed to $s = 1$)

- Classify configurations by the number of *frustrated* bonds $n_f = 0, 4, 6 \ldots$

- Partition function

$$\begin{aligned} Z &= \sum_{s_i} e^{-\beta H} = \sum_{s_i} \prod_{<ij>} e^{-\beta(1-\delta(s_i,s_j))} = \sum_{s_i} e^{-\beta n_f[s]} \\ &= 1 + V e^{-4\beta} + 2V e^{-6\beta} + (6V + V + V(V-5))e^{-8\beta} + O(e^{-10\beta}) \end{aligned}$$



- Likewise, expectation value $\langle s \rangle = \langle s_x \rangle$ (using translation invariance:

$$
\begin{aligned}
\langle s_x \rangle &= \frac{1}{Z} \sum_{s_i} s_x e^{-\beta H} \\
&= \frac{1 + (V-2)e^{-4\beta} + (2V-8)e^{-6\beta} + O(e^{-8\beta})}{1 + Ve^{-4\beta} + 2Ve^{-6\beta} + O(e^{-8\beta})} \\
&= 1 - 2e^{-4\beta} - 8e^{-6\beta} + O(e^{-8\beta})
\end{aligned}
$$

- 2d Ising: expansion to order $e^{-76\beta}$ [I. G. Enting, A, J. Guttmann, I. Jensen, J.Phys.A27 (1994)]

  3d Ising: expansion to $e^{-26\beta}$ [I. G. Enting, A, J. Guttmann, J.Phys.A26 (1993)]

- Note: expansion of $F = -\log Z$ has only connected graphs and is $\propto V$!

- Does not work for continuous d.o.f's

From [I. G. Enting, A, J. Guttmann, I. Jensen, J.Phys. A27 (1994) 6987-7006]

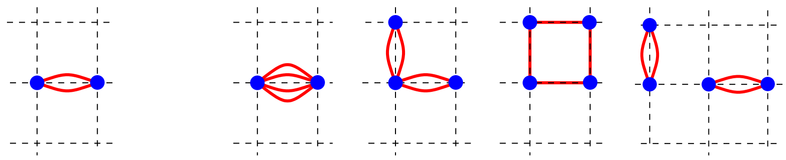
Table II: New low-temperature series for the spin-$\frac{1}{2}$ 2-dimensional Ising magnetisation ($M(u) = \sum_n m_n u^n$), susceptibility ($\chi(u) = \sum_n x_n u^n$), and specific heat ($C_v(u) = \sum_n c_n u^n$). All terms with odd n are zero.

n	m_n	x_n	c_n
0	1	0	0
2	0	0	0
4	-2	1	16
6	-8	8	72
8	-34	60	288
10	-152	416	1200
12	-714	2791	5376
14	-3472	18296	25480
16	-17318	118016	125504
18	-88048	752008	634608
20	-454378	4746341	3269680
22	-2373048	29727472	17086168
24	-12515634	185016612	90282240
26	-66551016	1145415208	481347152
28	-356345666	7059265827	2585485504
30	-1919453984	43338407712	13974825960
32	-10392792766	265168691392	75941188736
34	-56527200992	1617656173824	414593263952
36	-308691183938	9842665771649	2272626444528
38	-1691769619240	59748291677832	12502223573304
40	-9301374102034	361933688520940	68996534259040
42	-51286672777080	2188328005246304	381858968527680
44	-283527726282794	13208464812265559	2118806030647328
46	-1571151822119216	79600379336505560	11783826597027256
48	-8725364469143718	479025509574159232	65674579024955904
50	-48552769461088336	2878946431929191656	366728645195006000
52	-270670485377401738	17281629934637476365	2051443799934043632
54	-1511484024051198680	103621922312364296112	11494250259278105304
56	-8453722260102884930	620682823263814178484	64499139095733378176
58	-47350642314439048648	3714244852389988540072	362436080938852037648
60	-265579129813183372802	22206617664989885664363	2039249170926323834880
62	-1491465339550559632448	132657236460768679560864	11487673072269872540904
64	-8385872784303807639294	791843294876287279547520	64786142191741932873984
66	-47202746620874986470336	4723112509660327575046688	365754067103461706996304
68	-265975151780412455885826	28152514246598001579534217	2066925549185792626090544
70	-1500179080790296495333960	167696255471026758161692328	11691314122170272566638200
72	-8469330846027919131108866	998303936498277539688401212	66188283453887221177721568
74	-47856040705247407564621400	5939502715888619728011515904	375021938737150106426702208
76	-270636033194089067428986890	35318214476286590871820680287	2126523853550658555941372768

High temperature expansion for 2d Ising

- Use now $H = -\sum_{<ij>} s_i s_j$
- Partition function at $\beta \ll 1$:

$$
\begin{aligned}
Z &= \sum_{s_i} \prod_{<ij>} e^{\beta s_i s_j} \\
&= \sum_{s_i} \prod_{<ij>} \left(1 + \beta s_i s_j + \frac{1}{2!}\beta^2 (s_i s_j)^2 + \frac{1}{3!}\beta^3 (s_i s_j)^3 + \frac{1}{4!}\beta^4 (s_i s_j)^4 + \ldots\right) \\
&= 2^V \left[1 + \beta^2 \frac{1}{2} 2V + \beta^4 \left(\frac{1}{4!}2V + \frac{1}{2^2}6V + V + \frac{1}{2^2}2V(2V-7)\right) + O(\beta^6)\right]
\end{aligned}
$$



- $\mapsto$ Partition function as a sum of closed graphs:

$$
Z = 2^V \sum_G \beta^{L(G)} \prod_{<ij>} \frac{1}{m_{ij}(G)\,!}
$$

where $L(G)$ is the number of the links in the graph G (including links with n hops n times), and $m_{ij}(G)$ is the number of hops over link $< ij >$.

- Expectation values: for nn-pair $< ab >$

$$
\begin{aligned}
\langle s_a s_b \rangle &= \frac{1}{Z} \sum_{s_i} s_a s_b e^{-\beta H} \\
&= \frac{1}{Z} \sum_{s_i} s_a s_b \left(\beta s_a s_b + \beta^3 \left[\frac{1}{3!}(s_a s_b)^3 + \frac{1}{2} \sum_{<ij> \neq <ab>} (s_i s_j)^2 s_a s_b\right] + O(\beta^5)\right) \\
&= \beta + \beta^3 (\frac{1}{3!} - \frac{1}{2}) + O(\beta^5)
\end{aligned}
$$

- Note: again $F = -\log Z$ contains only connected graphs, and is $\propto V$.
- Generalizes to continuous spins: *hopping parameter expansion*

Hopping parameter expansion

- Scalar theory in d-dimensions:

$$S = \int d^d x \; \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda_0 \phi^4$$

- Conventional (not perhaps the best) lattice discretization: define (note: in d-dim. $[\lambda_0] = \mathrm{GeV}^{4-d}$)
- $\varphi = \sqrt{\kappa} a^{d-2}/2\phi$
- $\lambda = \dfrac{1}{4!}\lambda_0 a^{4-d}\kappa^2$
- *Hopping parameter* κ is fixed through $(d + \frac{1}{2}(ma)^2)\kappa + 2\lambda = 1$

$$S_{\rm latt} = \sum_x \left[-\kappa \sum_\mu \varphi_x \varphi_{x+\mu} + \varphi_x^2 + \lambda(\varphi_x^2 - 1)^2 \right] = \sum_x \left[-\kappa \sum_\mu \varphi_x \varphi_{x+\mu} + u(\varphi_x) \right]$$

  All quantities are dimensionless.

- $g \to \infty$: Ising model
- *Naive* continuum limit: $\kappa = \frac{1}{d} - \frac{2}{d}\lambda$.

- "High-temperature expansion": expand around $\kappa = 0$:

$$Z = \int [\prod_x d\varphi_x e^{-u(\varphi_x)}] \prod_{<xy>} e^{\kappa \varphi_x \varphi_y}$$

- Exactly as for the Ising model, we can write the last part as a sum over sets of links, *graphs* G:

$$\prod_{<xy>} e^{\kappa\varphi_x\varphi_y} \quad = \quad \prod_{<xy>} \sum_i \frac{1}{i!} \kappa^i \varphi_x^i \varphi_y^i$$

$$
\begin{aligned}
&= \sum_G \kappa^{L(G)} \prod_{<xy>\in G} \frac{1}{m_{xy}(G)!} (\varphi_x\varphi_y)^{m_{xy}(G)} \\
&= \sum_G \kappa^{L(G)} c(G) \prod_x \varphi^{N_G(x)}
\end{aligned}
$$

- $m_{xy}(G)$: the number of times link $< ij >$ is included in G
- $c(G) \equiv \prod_{<xy>} \frac{1}{m_{xy}(G)!}$
- $N_G(x)$: $\#$ of links going to point x
- Defining

  $$Z_1 = \int d\varphi e^{-u(\varphi)}, \qquad \gamma_k = \langle \varphi^k \rangle_1 = \frac{1}{Z_1} \int d\varphi\ \varphi^k e^{-u(\varphi)}$$

  we get

  $$Z = Z_1^V \sum_G \kappa^{L(G)} c(G) \prod_{x\in G} \gamma_{N_G(x)}$$

- Since $\gamma_k = 0$ for odd k, we get exactly the same closed graphs as for the Ising model.

  $$
  \begin{aligned}
  \frac{Z}{Z_1^V} &= 1 + \kappa^2 V d \frac{1}{2}\gamma_2^2 + \kappa^4 \left[V d \frac{1}{4!}\gamma_4^2 + V d(2d-1)\frac{1}{(2!)^2}\gamma_4\gamma_2^2 \right. \\
  &+ \left. \tfrac{1}{2} V d(d-1)\gamma_2^4 + \tfrac{1}{2} V d(V d - 4d + 1)\frac{1}{(2!)^2}\gamma_4^2 \right] + O(\kappa^6)
  \end{aligned}
  $$

- "Feynman rules" for Z:
  1. Draw graphs of length L
  2. Each link: $1/m!$
  3. Each point: γ_N
- Again, free energy $F = -\log Z$ contains only connected graphs.
- Various quantities calculated up to 14th order [M.Lüscher, P.Weisz, Nucl.Phys.B 295 (1988)]

Gauge fields

• Gauge field lagrangian in (Euclidean) continuum:

$$\tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} = \tfrac{1}{2}\mathrm{Tr}\,(F_{\mu\nu}F_{\mu\nu})$$

• Field tensor

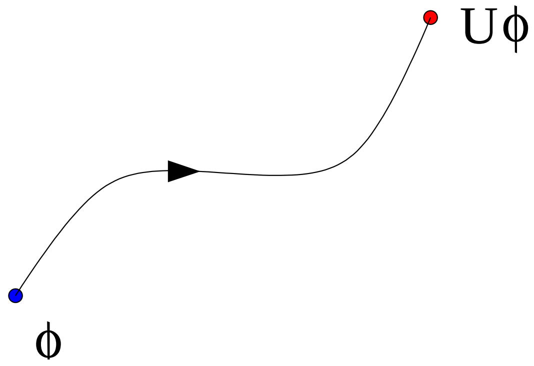
$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

where $A_\mu = A^a_\mu \lambda^a$, and λ^a are the group generators. We shall consider only unitary groups U(1) [QED] and SU(N) [QCD, Electroweak]. For SU(N), the generators are orthonormalized

$$\mathrm{Tr}\,\lambda^a\lambda^b = \tfrac{1}{2}\delta^{ab}.$$

• Put gauge potential A_μ directly on a lattice? Difficult to maintain gauge invariance.

• Good starting point for gauge field on a lattice is to consider consider gauge fields acting in a *parallel transport:* As field $\phi \in$ SU(N) is "parallel transported" along a path p, parametrized by $x_\mu(s)$, $s \in [0,1]$, gauge fields rotate it: $\phi \to U(s)\phi$



• Define $U(s)$ differentially via

$$\frac{dU(s)}{ds} = \frac{dx_\mu}{ds} igA_\mu U(s),$$

which can be formally solved:

$$U(s) = P\exp\left[ig\int_p ds \frac{dx_\mu}{ds}A_\mu\right]$$

• Here $P =$ "path ordering": in the power series expansion of the exponential one always takes $A(x)$ in the order they are encountered along the path.

• For U(1) (or any other Abelian group) path ordering is irrelevant.

• Alternatively, if we divide the path into N finite intervals of length Δs, and $x^n = x(n\Delta s)$, $n = 0 \ldots N-1$:

$$U(s) \approx P \exp\left[ig \sum_n \Delta s \frac{dx^n_\mu}{ds} A_\mu(x^n)\right] = \prod_i \exp\left[ig \Delta s \frac{dx^n_\mu}{ds} A_\mu(x^n)\right]$$

Gauge transformations:

• Gauge transformation $\Lambda(x)$ is a (SU(N)) group element defined at every point. Gauge potential transforms as

$$A_\mu \to \Lambda A_\mu \Lambda^{-1} + \frac{i}{g} \Lambda \partial_\mu \Lambda^{-1}$$

• Field ϕ:

$$\phi(x) \to \Lambda(x)\phi(x)$$

• Path p:

$$U(p) \to \Lambda(x_0) U(p) \Lambda^{-1}(x_1),$$

where x_0 and x_1 are the beginning and end of path.

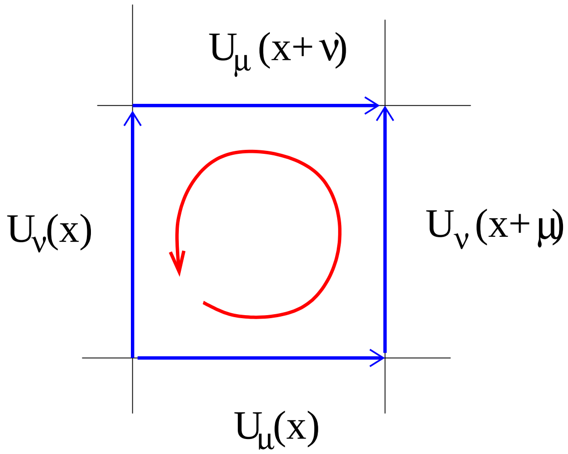
• Closed loops: $U(C) \to \Lambda(x_0) U(C) \Lambda^{-1}(x_0)$

• Trace of a closed loop: $\mathrm{Tr}\, U(C)$ is gauge invariant!

Lattice gauge fields

• Variables: parallel transporters from one lattice site to a neighbouring one, *Links:*

$$\begin{aligned} U_\mu(x) &= P\exp\left[ig\int_x^{x+\mu} dx_\mu A_\mu\right] \\ &= \exp\left[igaA_\mu(x+\tfrac{1}{2}\mu)\right] + O(ga^3) \end{aligned}$$



• The lattice action has to be gauge invariant. Only traces of closed loops are gauge invariant; the simplest one is *plaquette*

$$\mathrm{Tr}\, U_\Box = \mathrm{Tr}\; U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x)$$

• When a small (for SU(N)),

$$\mathrm{Re}\,\mathrm{Tr}\, U_\Box = N - \frac{a^4g^2}{4}F^a_{\mu\nu}F^a_{\mu\nu} + O(g^4a^6)$$

• $\to$ Wilson gauge action (K. Wilson, 1974):

$$\begin{aligned} S_g &= \frac{2N}{g^2a^{4-d}}\sum_\Box\left[1-\frac{1}{N}\mathrm{Re}\,\mathrm{Tr}\,U_\Box\right] \\ &= \int d^dx\tfrac{1}{2}\mathrm{Tr}\,[F_{\mu\nu}F_{\mu\nu}] + O(a^2g^2) \end{aligned}$$

Partition function

$$Z = \int[\prod_{x,\mu} dU_\mu(x)]e^{-S_g}$$

Here the integral is over group elements U_μ (compact), *not* over algebra A_μ
Common notation:

$$\beta = \beta_G \equiv \frac{2N}{g^2}$$

Continuum limit for U(1) (in 4d)

- $U_\mu(x) = \exp igaA_\mu(x)$
- The Wilson action for U(1) is

$$
\begin{aligned}
S &= \frac{1}{g^2}\sum_{x;\mu<\nu}\left(1 - \mathrm{Re}\, U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_n^\dagger u(x)\right) \\
&= \frac{1}{g^2}\sum_{x;\mu<\nu}\left(1 - \mathrm{Re}\,\exp\left[iga(A_\mu(x) + A_\nu(x+\mu) - A_\mu(x+\nu) - A_\nu(x))\right]\right)
\end{aligned}
$$

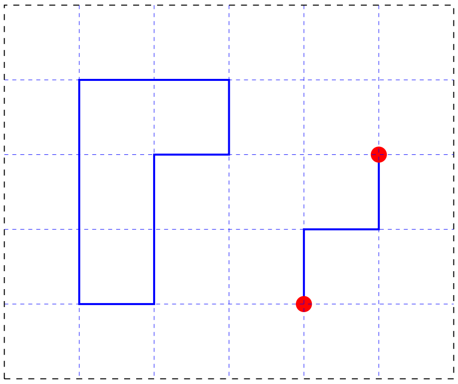
Expanding $A_\nu(x+\mu) = A_\nu(x) + a\partial_\mu A_\nu(x) + \frac{1}{2}a^2\partial_\mu^2 A_\nu(x) + \dots$

$$
\begin{aligned}
S &= \frac{1}{g^2}\sum_{x;\mu<\nu}\left(1 - \mathrm{Re}\,\exp\left[iga^2(\partial_\mu A_\nu - \partial_\nu A_\mu) + O(a^4)\right]\right) \\
&= \frac{1}{g^2}\frac{1}{2}\sum_{x;\mu\neq\nu}\left[\frac{1}{2}g^2a^2F_{\mu\nu}^2 + O(g^2a^6)\right] \\
&= \frac{1}{4}\int d^4x F_{\mu\nu}^2
\end{aligned}
$$

- Error in lagrangian $O(a^2)$
- Non-Abelian (Campbell-Baker-Hausdorff formula)

Gauge transformations:

- Gauge transform $\Lambda(x) \in$ SU(N) lives on lattice *sites*
- Link variable: $U_\mu(x) \to U'_\mu(x) = \Lambda(x) U_\mu(x) \Lambda^\dagger(x+\mu)$
  $\mapsto$
  $A_\mu(x) \to \Lambda(x) A_\mu(x) \Lambda^\dagger(x) + \frac{i}{g}\Lambda(x)\partial_m u \Lambda^\dagger(x)$
- Trace of a closed loop is gauge invariant
- Fundamental matter: $\phi(x) \to \Lambda(x)\phi(x)$, where ϕ is a N-component complex vector: operator
  $\phi^\dagger(x) U_P(x \mapsto y)\phi(y) =$
  $\phi^\dagger(x) U_\mu(x) U_\nu(x+\mu) \ldots U_\rho(y-\rho)\phi(y)$
  is gauge invariant.



- We want *only* gauge invariant animals on a lattice: lattice action + observables must consist of *closed loops* (gauge only quantities) or $\phi^\dagger U_P \phi$ - "strings" (matter).
- For example, the matter field kinetic term

$$(D_\mu\phi)^\dagger (D_\mu \phi),$$

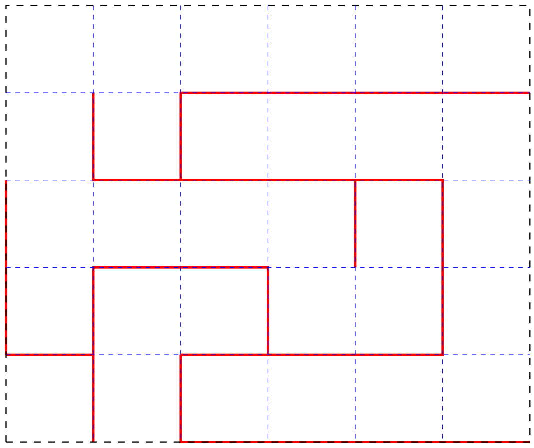
  where $D_\mu = \partial_\mu + igA_\mu$, becomes

$$\frac{1}{a^2}\left[2d\phi^\dagger\phi - 2\sum_\mu \phi^\dagger(x) U_\mu(x)\phi(x+\mu)\right]$$

- *Elitzur's theorem:* expectation values of gauge non-invariant object = 0: $\langle U_\mu(x)\rangle = 0$.

Gauge fixing on the lattice

- Something you don't want to do
- Necessary for perturbative calculations
- Most of the (Euclidean) continuum gauges go over to lattice
- Special gauge: *maximal tree*



- A tree which connects every point of the lattice, but does not have closed loops.
- Link matrices $U_\mu(x)$ are *fixed* to arbitrary values (for example all $= 0$) in the tree
- Expectation values of gauge invariant quantities do not change (proof: easy)
- (Almost) complete gauge fixing

Confinement and Wilson loop

- Potential between static charge and anticharge, separated by R:

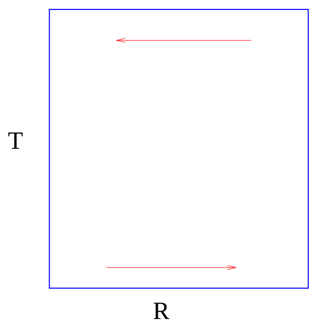
$$V(R \to \infty) \to \begin{cases} \infty: & \text{confinement} \\ \text{finite:} & \text{no confinement} \end{cases}$$

- Standard probe: Wilson loop W_{RT}. Let W be a rectangular path of size $R \times T$ along x_1 (say) and $x_0 = \tau$ directions.

$$W_{RT} = \text{Tr}\, P \exp ig \oint_W ds_\mu A_\mu = \text{Tr} \prod_{<xy>\in W} U_{xy}.$$

  Now $-\log W_{RT}$ gives the "free energy" of a static charge-anticharge ("quark-antiquark") system separated by R and which evolves for time T:

$$-\log\langle W_{RT}\rangle = V(R)T \qquad \text{valid as } T \to \infty$$



- *Perimeter law:* $-\log\langle W_{RT}\rangle \sim m(2R + 2T)$

  Free charges, m: "mass" of the charge due to gauge field

- *Area law:* $-\log\langle W_{RT}\rangle \sim \sigma RT$; $V(R) = \sigma R$

  Charges confined with linear potential, σ: string tension

- In general, $\frac{1}{T}\log\langle W\rangle \sim \sigma R + \text{const.} + c/R + \ldots$

Motivation:

- In temporal gauge $A_0 = 0 \quad \to \quad U_0 = 1$ we can define gauge field Hamiltonian $\hat{H}$ [Kogut-Susskind]
- Ψ: (trial) wave function of $q\bar{q}$ pair at $\bar{x}$, $\bar{y}$; spatial gauge transformations $\Psi_{ab} \to \Lambda_{ac}(\bar{x})\Lambda^\dagger_{bd}(\bar{y})\Psi_{cd}$
- $\hat{H}$ gauge invariant: $\langle\Psi'|e^{-T\hat{H}}|\Psi\rangle = 0$ unless Ψ, Ψ' have similar gauge transformation properties.

$$\begin{aligned}\langle\Psi|e^{-T\hat{H}}|\Psi\rangle &= \sum_n |\langle\Psi_n|\Psi\rangle|^2 e^{-E_n T} \\ &\to |\langle\Psi_0|\Psi\rangle|^2 e^{-E_0 T} \qquad \text{when } T \to \infty\end{aligned}$$

- $E_0 = V(R)$ ground state energy of static charges ($R = |\bar{x} - \bar{y}|$)
- Trial wave function: $\Psi_{ab} = U_{ab}(x \mapsto y)\Psi_{\rm vac}$, where $\Psi_{\rm vac}$ is the (gauge invariant) vacuum wave function

$$\begin{aligned}\langle\Psi|e^{-T\hat{H}}|\Psi\rangle &= \frac{1}{Z}\int [dU]\, \mathrm{Tr}\,[U^\dagger(T;\bar{x} \mapsto \bar{y})\, U(0;\bar{x} \mapsto \bar{y})]e^{-S} \\ &= \langle W_{RT}\rangle\end{aligned}$$

- W_{RT} gauge invariant: can be measured in *any* gauge:

$$-\frac{1}{T}\log\langle W_{RT}\rangle \to V(R)$$

  as $T \to \infty$.
- If we can guess/construct $\Psi \approx \Psi_0$, T need not be very large.

Strong coupling expansion and the Wilson loop

- Perturbation theory (weak coupling, $g \ll 1$) is toothless: *no confinement*
- Strong coupling: $g \gg 1$. For simplicity, consider U(1) $(beta = 1/g^2 \ll 1)$. The link variable is $U_\mu = \exp i\theta_\mu$.

$$
\begin{aligned}
S &= \beta \sum_\square (1 - \mathrm{Re}\, e^{i\theta_\square}) \\
Z &= \int_0^{2\pi} \left[\prod_{x,\mu} \frac{d\theta_\mu}{2\pi} \right] \prod_\square \exp[\beta \cos \theta_\square]
\end{aligned}
$$

Here $\theta_\square = \theta_1 + \theta_2 - \theta_3 - \theta_4$ around the plaquette $\square$.

- Expand in powers of β? OK, but more convenient is:

$$
\begin{aligned}
e^{\beta \cos\theta} &= \sum_{n=-\infty}^{\infty} I_n(\beta) e^{in\theta} \\
&= A \left[1 + \sum_{n=1}^{\infty} f_n \cos n\theta \right]
\end{aligned}
$$

where

$$
\begin{aligned}
A &= I_0(\beta) = 1 + \frac{1}{4}\beta^2 + \frac{1}{64}\beta^4 + \ldots \\
f_n &= 2I_n(\beta)/I_0(\beta) = \frac{1}{2^{n-1} n!}\beta^n + O(\beta^{n+2}) \\
f_1 &= \beta - \frac{1}{8}\beta^3 + \frac{1}{48}\beta^5 + \ldots
\end{aligned}
$$

The partition function becomes

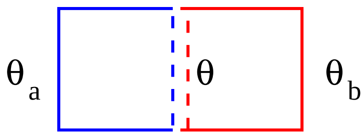
$$
Z = \int [\frac{d\theta}{2\pi}]\, A^{N_\square} \prod_\square (1 + f_1 \cos\theta_\square + f_2 \cos 2\theta_\square + \ldots)
$$

here $N_\square = 6V$ is the number of plaquettes (in 4d).

- Integration: $\int d\theta_a \cos n\theta_\square = 0$, if $\theta_a \in \square$.

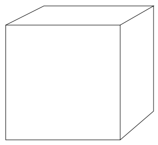
2 adjacent plaquettes:

$$\int \frac{d\phi}{2\pi} \cos(\theta + \theta_a) \cos(-\theta + \theta_b) = \tfrac{1}{2} \cos(\theta_a + \theta_b)$$



(also for $\cos n\theta$)

- Non-zero contributions only from *closed surfaces*: lowest non-trivial comes from 3-d cube:

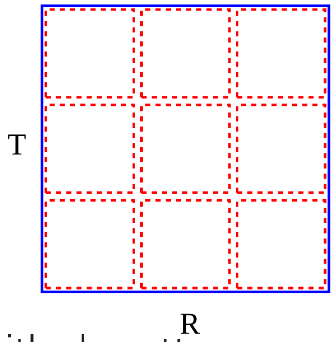


$$Z = A^{N_\Box}[1 + 4V(\tfrac{1}{2}f_1)^6 + O(\beta^{10})]$$

- Each plaquette on the surface contributes $\frac{1}{2}f_i$

- Expansion of free energy $F = -\log Z$ contains only connected graphs – *cluster expansion*

Wilson loop:

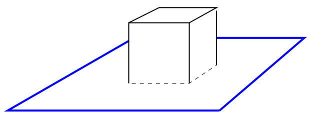


$$\langle W_{RT}\rangle = \frac{A^{N_\Box}}{Z}\int[\frac{d\theta}{2\pi}]\prod_{a\in W} e^{i\theta_a}\prod_\Box[1+\tfrac{1}{2}f_1(e^{i\theta_\Box}+e^{-i\theta_\Box})]$$

- First contribution comes when we "tile" the loop area with plaquettes:

$$\langle W_{RT}\rangle \approx (\tfrac{1}{2}f_1)^{RT} = (\tfrac{1}{2}\beta)^{RT} + O(\beta^{RT+2})$$

- We see *confinement:* $-\log\langle W\rangle/T = \log(\frac{1}{2}f_1)R = \sigma R$
- Next order: "bump"



$$\langle W_{RT}\rangle \approx (\tfrac{1}{2}f_1)^{RT}[1+4RT(\tfrac{1}{2}f_1)^4]$$

which gives

$$-a^2\sigma = -\frac{1}{RT}\log\langle W_{RT}\rangle = \log\tfrac{1}{2}f_1 + 4(\tfrac{1}{2}f_1)^4 + O([\tfrac{1}{2}f_1]^6)$$

- Order $[\frac{1}{2}f_1]^6$ gets contributions from the disconnected surfaces and Z.
- $-a^2\sigma = \log u + 4u^4 + \frac{176}{3}u^8 + \frac{10\,936}{405}u^{10} + \frac{1\,532\,044}{1\,215}u^{12} + \frac{3\,596\,102}{5\,103}u^{14}$, where $u = (\frac{1}{2}f_1)$ (see Montvay+Münster, for example)
- For U(1) this is strong coupling artefact! When g 1, there $\exists$ a phase transition. As $g \to 0$ we regain free theory.

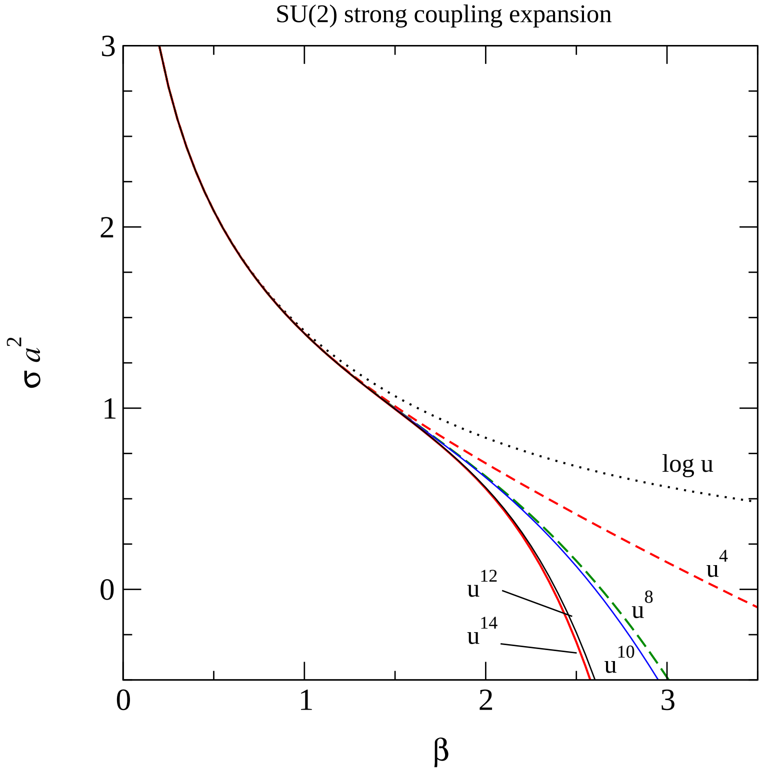
Non-Abelian gauge fields

- Same, but more difficult: use *character expansion*

$$e^{1/N\beta \,\mathrm{Re\,Tr}\,U} = A_\beta[1 + \sum_R b_R(\beta)\,\chi_R(U)]$$

  which gives "orthogonal" integration rules.

- Graphs are again surfaces – with complications!
- SU(2): $-a^2\sigma = \log u + 4u^4 + \frac{176}{3}u^8 + \ldots$,
  with $u = I_2(\beta)/I_1(\beta)$
- SU(3): $-a^2\sigma = \log u + 4u^4 + 12u^5 - 10u^6 - \ldots$
  with $u = \frac{1}{3}(x - \frac{1}{2}x^2 + \frac{2}{3}x^3 + \ldots)$, $x = \beta/6$.



- Convergence is good in *finite* range $\beta < \beta_{\max}$.

- *Mass gap:* plaquette-plaquette correlation function decays exponentially. Can be calculated using strong coupling, not in perturbation theory.
- *Roughening transition* for Wilson loops as β increases?

Realistic continuum limit

- When we change lattice spacing a, we have to adjust the *bare* coupling g in order to keep physics (at "long" distances) invariant.
- Measuring $g(a)$: choose some physical/renormalized quantity $M(a, g(a))$, (mass, string tension, Wilson loop of fixed size, vertex function ...) which can be measured/calculated. This we require to be *constant* as a is changed ("renormalization prescription")

  $\rightarrow$ *RG equation*:

$$a\frac{d}{da}M(a, g(a)) = 0 = \left(a\frac{\partial}{\partial a} - \beta_L\frac{\partial}{\partial g}\right)M$$

- β-function: $\beta_L \equiv -a\frac{\partial g}{\partial a}$

  tells how coupling g and lattice spacing a evolve.
- *Fixed point* $\beta_L(g_F) = 0$.
- If $\beta_L(g) < 0$ (> 0), g decreases (increases) as a is decreased.
- If $\partial\beta_L(g_F) < 0$, this is UV attractive fixed point and $\lim_{a\to 0} g(a) = g_F$: continuum limit
- For example: β-function at *strong coupling*, using expansion for SU(3) string tension $\sigma = -a^{-2}\log(1/(3g^2)) + O(g^{-2})$

$$\begin{aligned} 0 &= a\frac{d}{da}\sigma = -2\sigma - \beta_L\frac{2}{a^2 g} + \ldots \\ &\rightarrow \beta_L = -g\,\log(3g^2) + \ldots \end{aligned}$$

  This is < 0, so $g \searrow$ as $a \searrow$: no continuum limit at strong coupling.

Weak coupling:

- Use perturbation theory. For QCD (for example, from renormalized 3-vertex):

$$\beta_L(g) = -b_0 g^3 - b_1 g^5 + \ldots$$

Coefficients b_0 and b_1 are *universal*: independent of the regularization and prescription.

$$\begin{aligned} b_0 &= \frac{1}{16\pi^2}\left(\frac{11N}{3} - \frac{2N_F}{3}\right) \\ b_1 &= \frac{1}{(16\pi^2)^2}\left(\frac{34N^2}{3} - \frac{10NN_F}{3} - \frac{N_F(N^2-1)}{N}\right) \end{aligned}$$

- Integrate: $a = \exp[-\int^g dg'\,\beta_L^{-1}(g')] \quad \rightarrow$

$$a\Lambda_L = \exp[\frac{1}{2b_0 g^2}](b_0 g^2)^{b_1/2b_0^2}[1 + O(g^2)]$$

- UV fixed point $g \to 0$ as $a \to 0$: *asymptotic freedom*. (If $b_0 > 0$!)
- Λ_L: dimensionful integration constant, Λ-parameter; sets the scale. It is regularization dependent.
- *Dimensional transmutation:* no mass scale in the original (classical) Lagrangian!
- Thus, a physical mass m should behave as

$$am = \frac{m}{\Lambda_L}\exp[\frac{1}{2b_0 g^2}](b_0 g^2)^{b_1/2b_0^2}$$

In practice, this not well satisfied in QCD; $g \sim 1$:

Asymptotic scaling does not hold.

- Scaling holds if mass ratios are $\sim$ constant:

$$\frac{m_1}{m_2}(a) = \frac{m_1}{m_2}(0) + O(a^2)$$

This has been observed to be much better satisfied than asymptotic scaling.

Numerically:

- Principle: measure a quantity – am, $a^2\sigma$ – at several g's. Then

$$\frac{(am)(g_1)}{(am)(g_2)} = \frac{a(g_1)}{a(g_2)}$$

gives a discrete sample of $a(g_1) - a(g_2) = \exp[-\int_{g_2}^{g_1} dg' 1/\beta_L(g)]$

- Matching more than one quantity simultaneously: need more than one lattice coupling.

- *Monte Carlo renormalization group (MCRG):* direct implementation of Wilson's RG in numerical simulations.

  - Write the action $S = \sum_a \kappa_a S_a$, where S_a are a set of (local) contributions to action: plaquettes, larger loops . . .

  - With couplings κ_a^A simulate on L^d lattice, and measure a set of Wilson loops.

  - *block* the configurations by factor of 2: obtain $(L/2)^d$ lattice. Measure the Wilson loops.

  - Repeat blocking + measuring, until lattice is too small.

  - simulate directly on $(L/2)^d$ lattice, with κ^B. As before, repeat measuring Wilson loops and blocking.

  - compare Wilson loops at blocking level b from simulation B to loops at level $b+1$ from A. *Tune* κ_a^B, until the measurements match (requiring a new simulation).

  - repeat for $(L/4)^d$ etc.

  For the plaquette term, this gives function $\Delta\beta = \beta(a) - \beta(2a) = 6/g^2 - 6/g'^2$

- Violation of asymptotic scalings large at practical values of β (fig.)!

- *Perfect action:* no scaling violations; evolves along RG trajectory. Requires infinitely many coupling constants.
- *Classically perfect action:* perfect action for classical configurations. Much easier to derive, and these have been used for various models. It's also 1-loop quantum perfect action (perfect in the limit $g \to 0$).

In 3 dimensions:

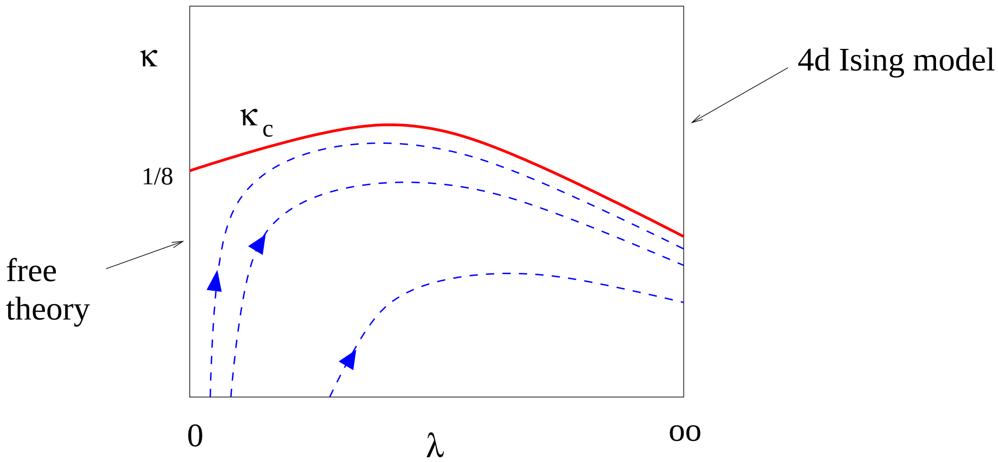
- Lattice coupling $\beta = \dfrac{2N}{g^2 a}$. Dimensions of $[g^2] =$ GeV.
- Renormalization: $g_R^2 = g^2 + O(g^4 a)$ dimensionally
- Continuum limit $\beta = \dfrac{2N}{g_R^2 a} + O[(g_R^2 a)^0]$
- Theory *superrenormalizable:* only finite number of divergent diagrams

Continuum limit in $\lambda\phi^4$ model

- Lattice lagrangian

$$\mathcal{L} = -\kappa \sum_{\mu} \varphi_x \varphi_{x+\mu} + \varphi_x^2 + \lambda(\varphi_x^2 - 1)^2$$

- In 4 dimensions, the RG trajectories of the bare couplings κ, λ have been solved [Lüscher, Weisz, NPB 295 (1988)] using hopping parameter expansion and perturbative RG equations:



Blue curves: RG trajectories of constant λ_R; λ_R decreases to $\nwarrow$; along κ_c we have $\lambda_R = 0$.

- Perturbatively $\beta_L(\lambda) = -a\dfrac{\partial\lambda}{\partial a} = \dfrac{3}{16\pi^2}\lambda^2 > 0$
- *No continuum limit!* RG trajectories hit the Ising limit $\lambda = \infty$ before the critical line $\kappa_c(\lambda)$ is reached; i.e. at finite a.
- Ising model: smallest possible a for any fixed λ_R.
- This is a generic feature of $\lambda\phi^4$ models in 4d
  $\rightarrow$ *Triviality bound* for the Standard Model Higgs mass!

- In 3 dimensions: there $\exists$ continuum limit: coupling constant is dimensionful $[\lambda] =$ GeV.
- $\lambda_R = \lambda + O(\lambda^2 a)$
- $m_R^2 = m_0^2 + C_1 \lambda a^{-1} + C_2 \lambda^2 + O(\lambda^3 a)$

  m_R^2 has linear $(1/a)$ and $(\log a/\Lambda)$ divergence. Perturbatively calculable, and can be subtracted.