Introduction to de Rham cohomology

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December 11, 2013

Preface

These are lecture notes for the course “Johdatus de Rham kohomologiaan” lectured fall 2013 at Department of Mathematics and Statistics at the University of Jyväskylä.

The main purpose of these lecture notes is to organize the topics discussed on the lectures. They are **not** meant as a comprehensive material on the topic! These lectures follow closely the book of Madsen and Tornehave “From Calculus to Cohomology” [7] and the reader is strongly encouraged to consult it for more details. There are also several alternative sources e.g. [1, 9] on differential forms and de Rham theory and [4, 5, 3] on multilinear algebra.

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Contents

Chapter 1

Alternating algebra

Let $v_1, \ldots, v_n$ be vectors in $\mathbb{R}^n$. The volume of the parallelepiped

$$P(v_1, \ldots, v_n) = \{t_1 v_1 + \cdots + t_n v_n \in \mathbb{R}^n \colon 0 \leq t_i \leq 1, i = 1, \ldots, n\}$$

is given by

$$|\det [v_1 \cdots v_n]| .$$

Whereas the absolute value of the determinant is independent on the order of vectors, the sign of $\det [v_1 \cdots v_n]$ depends on the order of $(v_1, \ldots, v_n)$. The quantity $\det [v_1 \cdots v_n]$ is therefore sometimes called "signed volume". The role of the sign is to detect the so-called "orientation" of vectors $(v_1, \ldots, v_n)$.

The notion of volume and "signed volume" based on the determinant allows us to consider n-dimensional objects in n-dimensional space. In this section, we discuss multilinear algebra which allows us to consider volumes (and "signed volumes") of k-dimensional (linear) objects in n-dimensional space when $k < n$. We discuss these geometric ideas in later sections and develop first the necessary linear theory.

1.1 Some linear algebra

We begin by recalling some facts from linear algebra; see e.g. [2, Chapter I & II] for a detail treatment.

Let V be a (real) vector space, that is, we have operations

$$\begin{aligned} +\colon V \times V \to V, &\qquad (u, v) \mapsto u + v, \\ \cdot\colon \mathbb{R} \times V \to V, &\qquad (a, v) \mapsto av, \end{aligned}$$

for which the triple $(V, +, \cdot)$ satisfies the axioms of a vector space.

Linear maps and dual spaces

Given vector spaces V and V', a function $f\colon V \to V'$ is a called a *linear map* if

$$f(u + av) = f(u) + af(v)$$

for all $u, v \in V$ and $a \in \mathbb{R}$. A bijective linear map is called a *(linear) isomorphism.*

Given linear maps $f, g\colon V \to V'$ and $a \in \mathbb{R}$, the mapping

$$f + ag\colon V \to V', \quad v \mapsto f(v) + ag(v),$$

is a linear map $V \to V'$. Thus the set $\mathrm{Hom}(V, V')$ of all linear maps $V \to V'$ is also a vector space under operations

$$\begin{aligned} +\colon \mathrm{Hom}(V,V') \times \mathrm{Hom}(V,V') &\to \mathrm{Hom}(V,V'), & (f,g) &\mapsto f+g, \\ \cdot\colon \mathbb{R} \times \mathrm{Hom}(V,V') &\to \mathrm{Hom}(V,V'), & (a,f) &\mapsto af. \end{aligned}$$

An important special case is the *dual space* $V^* = \mathrm{Hom}(V, \mathbb{R})$ *of* V.

Theorem 1.1.1 (Dual basis)**.** *Let V be a finite dimensional vector space. Then $V^* \cong V$. Furthermore, if $(e_i)_{i=1}^n$ is a basis of V, then $(\varepsilon_i)_{i=1}^n$, where $\varepsilon_i\colon V \to \mathbb{R}$ is the map*

$$\varepsilon_i(e_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j, \end{cases}$$

is a basis of V^*.

Proof. Exercise. $\square$

The basis $(\varepsilon_i)_i$ of V^* in Theorem 1.1.1 is called *dual basis (of $(e_i)_i$)*. Note that bases of V and V^* do not have the same cardinality (i.e V and V^* do not have the same "dimension") if V is infinite dimensional!

Induced maps

Given $f \in \mathrm{Hom}(V, W)$ and a linear map $\varphi\colon U \to V$, we have a composition $f \circ \varphi \in \mathrm{Hom}(U, W)$:

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[dr, dashed, "f\circ\varphi"'] & V \arrow[d, "f"] \\ & W \end{tikzcd}$$

The composition with fixed map φ induces a linear map as formalized in the following lemma.

Lemma 1.1.2. *Let U, V, W be vector spaces. Let $\varphi\colon U \to V$ be a linear map. Then*

$$\varphi^*\colon \mathrm{Hom}(V, W) \to \mathrm{Hom}(U, W),\ f \mapsto f \circ \varphi$$

is a linear map. Moreover, if φ is an isomorphism then φ^* is an isomorphism.

Similarly, we may also consider $f \in \mathrm{Hom}(U, V)$ and a fixed linear map $\psi\colon V \to W$:

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr, dashed, "f\circ\psi"'] & V \arrow[d, "\psi"] \\ & W \end{tikzcd}$$

Lemma 1.1.3. *Let U, V, W be vector spaces. Let $\psi\colon V \to W$ be a linear map. Then*

$$\varphi_*\colon \mathrm{Hom}(U, V) \to \mathrm{Hom}(U, W),\ f \mapsto \psi \circ f$$

is a linear map. Moreover, if ψ is an isomorphism then ψ_* is an isomorphism.

The mapping φ^* in Lemma 1.1.2 is so-called *pull-back (under φ)*. The mapping ψ_* is called as *push-forward*. As an immediate corollary of Lemma 1.1.2 we have the following observation.

Corollary 1.1.4. *Let U and V be vector spaces and $\varphi\colon U \to V$ a linear map. Then*

$$\varphi^*\colon V^* \to U^*, f \mapsto f \circ \varphi$$

is a linear map. Moreover, φ^* is an if φ is an isomorphism.

Subspaces and quotients

A subset $W \subset V$ of a vector space V is a *subspace of* V if $u + aw \in W$ for all $u, v \in W$ and $a \in \mathbb{R}$. A *coset of an element $v \in V$ with respect to W* is the subset

$$v + W = \{v + w \in V : w \in W\}.$$

The relation $\sim_W$ on V, given by the formula $u \sim_W v \Leftrightarrow u - v \in W$, is an equivalence relation with equivalence classes $\{v + W : v \in V\}$.

The set V/W of these equivalence classes has a natural structure of a vector space given by operations

$$\begin{aligned} +\colon V/W \times V/W \to V/W, &\qquad ((u + W), (v + W)) \mapsto (u + v) + W, \\ \cdot\colon \mathbb{R} \times V/W \to V/W, &\qquad (a, v + V/W) \mapsto (av) + W, \end{aligned}$$

that is,

$$(u + W) + a(v + W) = (u + av) + W$$

for all $u, v \in W$ and $a \in \mathbb{R}$. The space V/W is called the *quotient space of V (with respect to W).* Note that the mapping

$$\pi\colon V \to V/W, v \mapsto v + W,$$

is linear. The mapping p is called *quotient (or canonical) map.*

A fundamental fact of on linear mappings is that the kernel and the image are vector spaces, that is, let $f\colon V \to V'$ be a linear map between vector spaces, then

$$\begin{aligned} \ker f &= f^{-1}(0) = \{v \in V \colon f(v) = 0\} \\ \mathrm{Im} f &= f[V] = \{f(v) \in V' \colon v \in V\} \end{aligned}$$

are subspaces of V and V', respectively.

Theorem 1.1.5 (Isomorphism theorem)**.** *Let $f\colon V \to V'$ be a linear map between vector spaces V and V'. Then the mapping $\varphi\colon V/\ker f \to \mathrm{Im} f$, $v + W \mapsto f(v) + W$, is a linear isomorphism satisfying*

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathrm{Im} f \\ & V/\ker f \arrow[ur, "\varphi"', "\cong"] & \end{tikzcd}$$

where $p\colon V \to V/\ker f$ is the canonical map.

The proof is left as a voluntary exercise.

Products and sums

Given a set I (finite or infinite) and a vector space V_i for each $i \in I$, the product space $\prod_{i\in I} V_i$ has a natural linear structure given by

$$(v_i)_i + a(v_i')_i = (v_i + av_i')_i$$

where $v_i, v_i' \in V_i$ and $a \in \mathbb{R}$. For $I = \emptyset$, we declare $\prod_{i\in I} V_i = \{0\}$.

If $V_i = V_j =: V$ for all $i, j \in I$, denote $V^I = \prod_{i\in I} V$. Note that, V^I is, in fact, the space of all functions $I \to V$.

Note also that, for $n \in \mathbb{N}$, we have

$$V^n = V \times \cdots \times V \cong V^{\{1,\ldots,n\}} = \{\text{all functions } \{1, \ldots, n\} \to V\}.$$

The sum of two vector spaces $V \oplus W$ has (at least) three different meanings in the literature. Abstractly, $V \oplus W$ is the product space $V \times W$.

More concretely, if V and W are subspaces of a vector space U, then $V + W$ is the subspace of U spanned by V and W, that is,

$$V + W = \{v + w \in U \colon v \in V, w \in W\}.$$

If $V \cap W = \{0\}$, notation $V \oplus W$ for $V + W$ is commonly used. If U, however, is an inner product space, then notation $V \oplus W$ is commonly reserved for the case that $v \perp w$ for all $v \in V$ and $w \in W$. (Be warned!)

1.2 Multilinear maps

Definition 1.2.1. Let $V_1, \ldots, V_k$, and W be vector spaces. A function $f\colon V_1 \times \cdots \times V_k \to W$ is *k-linear* if, for all $v_j \in V_j$ $(j = 1, \ldots, k)$,

$$\begin{aligned} f(v_1, \ldots, v_{i-1}, v + aw, v_{i+1}, v_k) &= f(v_1, \cdots, v_{i-1}, v, v_{i+1}, v_k) \\ &\quad + af(v_1, \cdots, v_{i-1}, w, v_{i+1}, v_k) \end{aligned}$$

for all $i \in \{1, \ldots, k\}$, $v, w \in V_i$, and $a \in \mathbb{R}$. A mapping is *multilinear* if it is k-linear for some $k \geq 1$.

Example 1.2.2. *Let V be a (real) vector space. Then the* duality pairing *$V^* \times V \to \mathbb{R}$ defined by $(\varphi, v) \mapsto \varphi(v)$ is bilinear.*

Remark 1.2.3. *A function $f\colon V_1 \times \cdots \times V_k \to W$ is k-linear if and only if for every sequence $(v_1, \ldots, v_k) \in V_1 \times \cdots \times V_k$ and each $1 \leq i \leq k$ functions*

$$v \mapsto f(v_1, \ldots, v_{i-1}, v, v_{i+1}, v_k)$$

are linear mappings $V_i \to W$.

Example 1.2.4. *Let V be a (real) vector space. Then the* duality pairing *$V^* \times V \to \mathbb{R}$ defined by $(\varphi, v) \mapsto \varphi(v)$ is bilinear (i.e. 2-linear).*

Example 1.2.5. *An inner product $\langle \cdot, \cdot \rangle\colon V \times V \to \mathbb{R}$ is bilinear.*

Values of multilinear maps $V^k \to \mathbb{R}$ depend only on values on basis elements. This can be formalized as follows.

Lemma 1.2.6. *Let V be an n-dimensional vector space and $\{e_1, \ldots, e_n\}$ a basis of V. Let $f\colon V^k \to \mathbb{R}$ be k-linear, and $v_i = \sum_{j=1}^n v_{ij} e_j \in V$ for $i = 1, \ldots, k$. Then*

$$f(v_1, \ldots, v_k) = \sum_{(j_1, \ldots, j_k) \in \{1, \ldots, n\}^k} f(e_{j_1}, \ldots, e_{j_k}) v_{1j_i} \cdots v_{kj_k}$$

Proof. Since f is k-linear,

$$\begin{aligned} f(v_1, \ldots, v_k) &= f\left(\sum_{j_1=1}^n v_{1j_1} e_{j_i}, \ldots, \sum_{j_k=1}^n v_{kj_k} e_{j_k} \right) \\ &= \sum_{j_1} \cdots \sum_{j_k} f\left(v_{1j_1} e_{j_i}, \ldots, v_{kj_k} e_{j_k} \right) \\ &= \sum_{j_1} \cdots \sum_{j_k} (v_{1j_1} \cdots v_{kj_k}) f(e_{j_i}, \ldots, e_{j_k}) \end{aligned}$$

□

Corollary 1.2.7. *Let V be an n-dimensional vector space and $(e_1, \ldots, e_n)$ a basis of V. Suppose $f, g\colon V^k \to W$ satisfy*

$$f(e_{j_1}, \ldots, e_{j_k}) = g(e_{j_1}, \ldots e_{j_k})$$

for all multi-indices $(j_1, \ldots, j_k) \in \{1, \ldots n\}^k$. Then $f = g$.

1.2.1 Vector space of multilinear maps

Definition 1.2.8. Let $V_1, \ldots, V_k, V, W$ be vector spaces. We denote by $\mathrm{Mult}(V_1 \times \cdots \times V_k, W)$ the set of all k-linear maps $V_1 \times \cdots \times V_k \to W$. We denote $\mathrm{Mult}^k(V, W) = \mathrm{Mult}(\prod^k, W)$.

Lemma 1.2.9. *Let $V_1, \ldots, V_k, W$ be vector spaces. Then $\mathrm{Mult}(V_1 \times \cdots \times V_k, W)$ is a (vector) subspace of $\mathrm{Hom}(V_1 \times \cdots V_k, W)$.*

Proof. Clearly $f + ag \colon V_1 \times \cdots \times V_k \to W$ is k-linear if $f, g \in \mathrm{Mult}(V_1 \times \cdots \times V_k, W)$ and $a \in \mathbb{R}$. □

Definition 1.2.10. Let U, V and W be vector spaces and $\varphi \colon U \to V$ a linear map. Given a k-linear map $f \colon V^k \to W$, the map $\varphi^* f \colon U^k \to W$ is defined by formula

$$(\varphi^* f)(u_1, \ldots, u_k) = f(\varphi(u_1), \ldots, \varphi(u_k)) \tag{1.2.1}$$

for $u_1, \ldots, u_k \in U$.

Lemma 1.2.11. *Let U, V and W be vector spaces and $\varphi \colon U \to V$ a linear map. Then $\varphi^* f \in \mathrm{Mult}^k(U, W)$ for every $f \in \mathrm{Mult}^k(V, W)$. Furthermore, the map*

$$\varphi^* \colon \mathrm{Mult}^k(V, W) \to \mathrm{Mult}^k(U, W)$$

defined by formula (1.2.1)*, is linear.*

1.2.2 Tensor product

This section is added for completeness. We do not use tensor products in the following sections.

Let $f \colon V^k \to \mathbb{R}$ and $g \colon U^\ell \to \mathbb{R}$ be k- and ℓ-linear maps respectively. Define

$$\begin{aligned}
&f \otimes g \colon V^k \times U^\ell \to \mathbb{R}, \\
&(f \otimes g)(v_1, \ldots, v_k, u_1, \ldots, u_\ell) = f(v_1, \ldots, v_k) g(u_1, \ldots, v_k)
\end{aligned}$$

Lemma 1.2.12. *Let $f \colon V^k \to \mathbb{R}$ and $g \colon U^\ell \to \mathbb{R}$ be multilinear. Then $f \otimes g$ is multilinear. Moreover, if $h \colon W^m \to \mathbb{R}$ is multilinear, then $f \otimes (g \otimes h) = (f \otimes g) \otimes h$.*

Proof. Exercise. □

Lemma 1.2.13. *Let U, V be vector space, $f \in \mathrm{Mult}^k(V)$, $g \in \mathrm{Mult}^\ell(V)$, and $\varphi \colon U \to V$ a linear map. Then*

$$\varphi^*(f \otimes g) = \varphi^* f \otimes \varphi^* g.$$

Proof. Exercise. □

Lemma 1.2.14. *Let V be an n-dimensional vector space, $(e_1, \dots, e_n)$ a basis of V and $(\varepsilon_1, \dots, \varepsilon_n)$ the corresponding dual basis. Let $f\colon V^k \to \mathbb{R}$. Then there exists coefficients $a_J \in \mathbb{R}$, $J = (j_1, \dots j_k) \in \{1, \dots, n\}^k$, for which*

$$f = \sum_{J=(j_1,\dots,j_k)\in\{1,\dots n\}^k} a_J \varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_k}$$

Moreover, $a_J = f(e_{j_1}, \dots, e_{j_k})$ for $J = (j_1, \dots, j_k) \in \{1, \dots, n\}^k$.

Proof. Let $J = (j_1, \dots j_k)$ and $v_i = \sum_{j=1}^k v_{ij} e_j$ for $i = 1, \dots, k$. Then

$$\begin{aligned}
\varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_k}(v_1, \dots, v_k) &= \sum_{(i_1,\dots,i_k)\in\{1,\dots,n\}^k} (\varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_k})(e_{i_1}, \dots, e_{i_k}) v_{1i_i} \cdots v_{ki_k} \\
&= \sum_{(i_1,\dots,i_k)\in\{1,\dots,n\}^k} \varepsilon_{j_1}(e_{i_1}) \cdots \varepsilon_{j_k}(e_{i_k}) v_{1i_i} \cdots v_{ki_k} \\
&= v_{1j_i} \cdots v_{kj_k}
\end{aligned}$$

Thus, by Lemma 1.2.6,

$$\begin{aligned}
f(v_1, \dots, v_k) &= \sum_{(j_1,\dots,v_k)} f(e_{j_1}, \dots, e_{j_k}) v_{1j_1} \cdots v_{kj_k} \\
&= \left(\sum_{(j_1,\dots,v_k)} f(e_{j_1}, \dots, e_{j_k}) \varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_k} \right) (v_{1j_1}, \dots, v_{kj_k}).
\end{aligned}$$

□

Lemma 1.2.15. *Let V be an n-dimensional vector space, $(e_1, \dots, e_n)$ a basis of V and $(\varepsilon_1, \dots, \varepsilon_n)$ the corresponding dual basis. Then*

$$(\varepsilon_{j_1} \otimes \cdots \otimes \varepsilon_{j_k})_{(j_1,\dots,j_k)\in\{1,\dots,n\}^k}$$

is a basis of $\mathrm{Mult}^k(V, \mathbb{R})$.

Proof. Exercise. □

Remark 1.2.16. *Multilinear maps are discussed in many sources, see e.g. [4, 5] or [3]. Note, however, that for example in [3], the point of view is more abstract and the tensor product refers to a vector space which linearizes multilinear maps. This more abstarct approach can be viewed as the next step in this theory (which we do not take in these notes).*

1.3 Alternating multilinear maps

The material in this section is gathered from [7, Chapter 2]; alternatively see [3].

Throughout this section V and W are (real) vector spaces.

Definition 1.3.1. A k-linear map $f\colon V \times \cdots V \to W$ is *alternating* if, for $1 \le i < j \le k$,

$$f(v_1, \dots, v_{i-1}, v_j, v_{i+1}, \dots, v_{j-1}, v_i, v_{j+1}, \dots, v_k) = -f(v_1, \dots, v_k) \tag{1.3.1}$$

for all $v_1, \dots, v_k \in V$.

Example 1.3.2. *Let $\varphi\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ be the mapping*

$$(u, v) \mapsto \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix},$$

where $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Then φ is a linear map and the mapping $f\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, $(u, v) \mapsto \det \varphi(u, v)$, is an alternating 2-linear map. In fact, in this sense, a determinant $(\mathbb{R}^n)^n \to \mathbb{R}$ is an alternating n-linear map for all n.

Lemma 1.3.3. *A k-linear mapping $f\colon V^k \to W$ is alternating if and only if, for all $(v_1, \dots, v_k) \in V^k$,*

$$f(v_1, \dots, v_k) = 0 \tag{1.3.2}$$

if $v_i = v_j$ for $i \ne j$.

Example 1.3.4. *Let $f, g \in V^*$ and define $\omega\colon V \times V \to \mathbb{R}$ by formula*

$$\omega(v, v') = f(v)g(v') - f(v')g(v)$$

for $v, v' \in V$. Then ω is 2-linear and alternating, and hence $\omega \in \mathrm{Alt}^2(V)$. It will turn out that ω is an alternating product (or wedge*) of f and g.*

Example 1.3.5. *Let $(\varepsilon_1, \varepsilon_2)$ be the dual basis of the standard basis (e_1, e_2) of $\mathbb{R}^2$. If $f = \varepsilon_1$, $g = \varepsilon_2$, and ω are as in the previous example, then*

$$\omega((x_1, y_1), (x_2, y_2)) = x_1 y_2 - x_2 y_1 = \det \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.

1.3.1 Space $\mathrm{Alt}^k(V)$

Definition 1.3.6. Let V be a vector space. We denote

$$\mathrm{Alt}^k(V) = \{f\colon V^k \to \mathbb{R}\colon f \text{ is an alternating } k\text{−linear}\}.$$

Here $V^0 \cong \mathbb{R}$ and we identify $\mathrm{Alt}^0(V) = \mathbb{R}^* = \mathbb{R}$.

Lemma 1.3.7. *Let V be a vector space. Then $\mathrm{Alt}^k(V)$ is a subspace of $\mathrm{Mult}^k(V,\mathbb{R})$, and hence of $\mathrm{Hom}(\prod^k V,\mathbb{R})$.*

Proof. Since $\mathrm{Alt}^k(V) \subset \mathrm{Mult}^k(V,\mathbb{R})$, it suffices to observe that (clearly) $f + ag\colon V^k \to \mathbb{R}$ is alternating for every $f, g \in \mathrm{Alt}^k(V)$ and $a \in \mathbb{R}$. □

Remark 1.3.8.

$$\mathrm{Alt}^1(V) = \{f\colon V \to \mathbb{R}\colon f \text{ is } k\text{−linear}\} = \mathrm{Hom}(V,\mathbb{R}) = V^*.$$

Lemma 1.3.9. *Let V, W be vector spaces and $\varphi\colon V \to W$ a linear map. Then*

$$\varphi^*\colon \mathrm{Alt}^k(W) \to \mathrm{Alt}^k(V)$$

defined by

$$(v_1, \ldots, v_k) \mapsto \omega(\varphi(v_1), \ldots, \varphi(v_k))$$

(as in Definition 1.2.1), is well-defined and linear.

Proof. Since $\varphi^*\colon \mathrm{Mult}^k(W,\mathbb{R}) \to \mathrm{Mult}^k(V,\mathbb{R})$ is well-defined and linear, it suffices to observe that (clearly) $\varphi^*\omega$ is alternating for every $\omega \in \mathrm{Alt}^k(V)$. □

1.3.2 Intermission: permutations

Let $k \geq 1$. A bijection $\{1,\ldots,k\} \to \{1,\ldots,k\}$ is called a *permutation.* We denote by S_k the set of all permutations $\{1,\ldots k\} \to \{1,\ldots,k\}$. A permutation $\tau \in S_k$ is a *transposition* if there exists $1 \leq i < j \leq k$ for which $\tau(m) = m$ for $m \neq i, j$ and $\tau(i) = j$ (and $\tau(j) = i$); we denote by $\tau_{ij} = \tau_{ij}^{(k)}$ the transposition satisfying $\tau_{ij}(i) = j$.

A permutation is *even* if it can be written as an even number of transpositions. Otherwise, a permutation is called *odd.* A permutation is either even or odd. We formalize this as follows.

Proposition 1.3.10. *A permutation has a sign, that is, there exists a function* $\mathrm{sign}\colon S_k \to \{\pm 1\}$ *satisfying*

$$\mathrm{sign}(\sigma) = \begin{cases} +1, & \sigma \text{ is an even permutation,} \\ -1, & \sigma \text{ is an odd permutation.} \end{cases}$$

Proof. Induction on k. (Exercise.) □

Corollary 1.3.11. *The* sign: $S_k \to \{\pm 1\}$ *satisfies*

$$\mathrm{sign}(\sigma \circ \sigma') = \mathrm{sign}(\sigma)\mathrm{sign}(\sigma').$$

In particular,

$$\mathrm{sign}(\tau_1 \circ \cdots \circ \tau_m) = (-1)^m$$

for transpositions $\tau_1, \ldots, \tau_m$.

Remark 1.3.12. *The set S_k is a* group *under composition, that is, the product $\sigma\sigma'$ is the composition $\sigma \circ \sigma'$. In Proposition 1.3.10 and Corollary 1.3.10 we implicitely state that S_k is generated by transpostions and the function* sign *is, in fact, a group homomorphism $(S_k, \circ) \to (\{\pm 1\}, \cdot)$. We do not pursue these details here.*

The main observation on permutions and alternating maps is the following lemma.

Lemma 1.3.13. *Let $f\colon V^k \to W$ be an alternating k-linear map and $\sigma \in S_k$. Then, for $(v_1, \ldots, v_k) \in V^k$,*

$$f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \mathrm{sign}(\sigma) f(v_1, \ldots, v_k).$$

Proof. Let $\sigma = \tau_1 \circ \cdots \circ \tau_m \in S_k$ and denote $\sigma' = \tau_1^{-1} \circ \sigma$. Then $\sigma = \tau_1 \circ \sigma'$ and, by alternating,

$$\begin{aligned} f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) &= f(v_{\tau_1(\sigma'(1))}, \ldots, v_{\tau_1(\sigma'(k))}) \\ &= -f(v_{\sigma'(1)}, \ldots, v_{\sigma'(k)}). \end{aligned}$$

Thus, by induction,

$$f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = (-1)^m f(v_1, \ldots, v_k) = \mathrm{sign}(\sigma) f(v_1, \ldots, v_k).$$

□

Lemma 1.3.13 gives an easy proof for an alternative characterization of alternate multilinear maps.

Lemma 1.3.14. *Let V and W be vector spaces and $f\colon V^k \to W$ be a k-linear map. Then f is alternating if and only if $f(v_1, \ldots, v_k) = 0$ whenever $v_i = v_{i+1}$ for some $1 \le i < k$.*

Proof. Exercise. □

Exercise 1.3.15. *Let $f\colon V^k \to W$ be a map and $\sigma \in S_k$. Define the map $\sigma_\# \omega\colon V^k \to W$ by*

$$(\sigma_\# f)(v_1, \ldots, v_k) = f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

Show that $(\sigma_\# f)$ is an alternating k-linear map if and only if f is an alternating k-linear map.

1.3.3 Exterior product

Definition 1.3.16. Let $k, \ell \geq 1$. A permutation $\sigma \in S_{k+\ell}$, is a (k,ℓ)-*shuffle* if

$$\sigma(1) < \sigma(2) < \cdots < \sigma(k) \text{ and } \sigma(k+1) < \cdots < \sigma(k+\ell).$$

We denote by $S(k,\ell)$ the set of all (k,ℓ)-shuffles.

Definition 1.3.17. Let $\omega \in \mathrm{Alt}^k(V)$ and $\tau \in \mathrm{Alt}^\ell(V)$. The *exterior (or the wedge) product* $\omega \wedge \tau \in \mathrm{Alt}^{k+\ell}(V)$ is defined by

$$\omega \wedge \tau(v_1, \ldots, v_{k+\ell}) = \sum_{\sigma \in S(k,\ell} \mathrm{sign}(\sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\tau(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)})$$

for $v_1, \ldots, v_{k+\ell} \in V$.

Lemma 1.3.18. *Let* $\omega \in \mathrm{Alt}^k(V)$ *and* $\tau \in \mathrm{Alt}^\ell(V)$. *Then* $\omega \wedge \tau \in \mathrm{Alt}^{k+\ell}(V)$.

Proof. (See also [7, Lemma 2.6].)

By Lemma 1.3.14 it suffices to show that

$$\omega \wedge \tau(v_1, \ldots, v_{k+1}) = 0$$

if $v_i = v_{i+1}$ for some $1 \leq i < k + \ell$.

Let

$$\begin{aligned}
S_0 &= \{\sigma \in S(k,\ell) \colon \{i, i+1\} \in \sigma(\{1, \ldots, k\}) \text{ or } \{i, i+1\} \in \sigma(\{k+1, \ldots, k+\ell\})\} \\
S_- &= \{\sigma \in S(k,\ell) \colon i \in \sigma(\{1, \ldots, k\}) \text{ and } i+1 \in \sigma(\{k+1, \ldots, k+\ell\})\} \\
S_+ &= \{\sigma \in S(k,\ell) \colon i+1 \in \sigma(\{1, \ldots, k\}) \text{ and } i \in \sigma(\{k+1, \ldots, k+\ell\})\}
\end{aligned}$$

Since $v_i = v_{i+1}$, we have, for every $\sigma \in S_0$, either $\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = 0$ or $\tau(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}) = 0$. Thus

$$\begin{aligned}
&\omega \wedge \tau(v_1, \ldots, v_{k+\ell}) \\
&\quad = \sum_{\sigma \in S(k,\ell)} \mathrm{sign}(\sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\tau(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}) \\
&\quad = \sum_{\sigma \in S_+} \mathrm{sign}(\sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\tau(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}) \\
&\qquad + \sum_{\sigma \in S_-} \mathrm{sign}(\sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\tau(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}).
\end{aligned}$$

It suffices to show that

$$\begin{aligned}
&\sum_{\sigma \in S_+} \mathrm{sign}(\sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\tau(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}) \\
&\quad = -\sum_{\sigma \in S_-} \mathrm{sign}(\sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\tau(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}).
\end{aligned}$$

Let $\alpha = \tau_{i,i+1}$ be the transposition interchanging i and $i+1$.
Step 1: We show first that $S_- \to S_+$, $\sigma \mapsto \alpha \circ \sigma$, is a well-defined bijection.

The inverse of the map is $\sigma' \mapsto \alpha \circ \sigma'$, so it a bijection. It suffices to show that the target is S_+.

Let $\sigma \in S_-$. Then there exists $1 \leq j \leq k$ for which $\sigma(j) = i$ and $k+1 \leq j' \leq k+\ell$ for which $\sigma(j') = i+1$.

We observe that, for $1 \leq j < k$,

$$i+1 < \sigma(j+1)$$

since $i+1 \notin \sigma(\{1,\ldots,k\})$ and $\sigma(j+1) > \sigma(j) = i$. Similarly,

$$\sigma(j'-1) < i.$$

Since α is a transposition, we have

$$\begin{aligned}
&(\alpha(\sigma(1)),\ldots,\alpha(\sigma(k))) \\
&\quad = (\sigma(1),\ldots,\sigma(j-1),\alpha(\sigma(j)),\sigma(j+1),\ldots,\sigma(k)) \\
&\quad = (\sigma(1),\ldots,\sigma(j-1),i+1,\sigma(j+1),\ldots,\sigma(k))
\end{aligned}$$

and

$$\begin{aligned}
&(\alpha(\sigma(k+1)),\ldots,\alpha(\sigma(k+\ell))) \\
&\quad = \big(\sigma(k+\ell),\ldots,\sigma(j'-1),\alpha(\sigma(j')),\sigma(j'+1),\ldots,\sigma(k+\ell)\big) \\
&\quad = \big(\sigma(k+1),\ldots,\sigma(j'-1),i,\sigma(j'+1),\ldots,\sigma(k+\ell)\big).
\end{aligned}$$

Thus

$$\alpha(\sigma(1)) < \cdots < \alpha(\sigma(k)) \text{ and } \alpha(\sigma(k+1)) < \cdots < \alpha(\sigma(k+\ell)).$$

Hence $\alpha \circ \sigma \in S(k,\ell)$ and $\alpha \circ \sigma \in S_+$. This ends Step 1.
Step 2: We show now that, for $\sigma \in S_-$,

$$\begin{aligned}
&\omega\big(v_{\alpha(\sigma(1))},\ldots,v_{\alpha(\sigma(k))}\big)\tau\big(v_{\alpha(\sigma(k+1))},\ldots,v_{\alpha(\sigma(k+\ell))}\big) \\
&\quad = \omega\big(v_{\sigma(1)},\ldots,v_{\sigma(k)}\big)\tau\big(v_{\sigma(k+1)},\ldots,v_{\sigma(k+\ell)}\big).
\end{aligned}$$

Let $1 \leq j \leq k$ and $k+1 \leq j' \leq k+\ell$ be as in the proof of Subclaim 1. Since $v_i = v_{i+1}$, we have

$$\begin{aligned}
&\omega\big(v_{\alpha(\sigma(1))},\ldots,v_{\alpha(\sigma(k))}\big)\tau\big(v_{\alpha(\sigma(k+1))},\ldots,v_{\alpha(\sigma(k+\ell))}\big) \\
&\quad = \omega\big(v_{\sigma(1)},\ldots,v_{\sigma(j-1)},v_{i+1},v_{\sigma(j+1)},\ldots,v_{\sigma(k)}\big) \\
&\qquad \cdot\tau\big(v_{\sigma(k+1)},\ldots,v_{\sigma(j'-1)},v_i,v_{\sigma(j'+1)},\ldots,v_{\sigma(k+\ell)}\big) \\
&\quad = \omega\big(v_{\sigma(1)},\ldots,v_{\sigma(j-1)},v_i,v_{\sigma(j+1)},\ldots,v_{\sigma(k)}\big) \\
&\qquad \cdot\tau\big(v_{\sigma(k+1)},\ldots,v_{\sigma(j'-1)},v_{i+1},v_{\sigma(j'+1)},\ldots,v_{\sigma(k+\ell)}\big) \\
&\quad = \omega\big(v_{\sigma(1)},\ldots,v_{\sigma(k)}\big)\tau\big(v_{\sigma(k+1)},\ldots,v_{\sigma(k+\ell)}\big).
\end{aligned}$$

This ends Step 2.
Final step: Combining previous steps, we have

$$\begin{aligned}
&\omega \wedge \tau(v_1, \ldots, v_{k+\ell}) \\
&\quad = \sum_{\sigma' \in S_-} \operatorname{sign}(\alpha \circ \sigma')\omega(v_{\alpha\circ\sigma'(1)}, \ldots, v_{\alpha\circ\sigma'(k)})\tau(v_{\alpha\circ\sigma'(k+1)}, \ldots, v_{\alpha\circ\sigma'(k+\ell)}) \\
&\qquad + \sum_{\sigma \in S_-} \operatorname{sign}(\sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\tau(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}) \\
&\quad = (-1) \sum_{\sigma' \in S_-} \operatorname{sign}(\sigma')\omega(v_{\sigma'(1)}, \ldots, v_{\sigma'(k)})\tau(v_{\sigma'(k+1)}, \ldots, v_{\sigma'(k+\ell)}) \\
&\qquad + \sum_{\sigma \in S_-} \operatorname{sign}(\sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\tau(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}) \\
&\quad = 0.
\end{aligned}$$

□

The exterior product is bilinear in the following sense.

Lemma 1.3.19. *Let $\omega, \omega' \in \operatorname{Alt}^k(V)$, $\tau, \tau' \in \operatorname{Alt}^\ell(V)$, and $a \in \mathbb{R}$. Then*

$$\begin{aligned}
(\omega + a\omega') \wedge \tau &= \omega \wedge \tau + a\omega' \wedge \tau \\
\omega \wedge (\tau + a\tau') &= \omega \wedge \tau + a\omega \wedge \tau'.
\end{aligned}$$

Proof. Exercise. □

Lemma 1.3.20. *Let $\omega \in \operatorname{Alt}^k(V)$ and $\tau \in \operatorname{Alt}^\ell(V)$. Then*

$$\omega \wedge \tau = (-1)^{k\ell}\tau \wedge \omega.$$

The proof of Lemma 1.3.20 ([7, Lemma 2.8]) is based on the following observation on shuffles.

Lemma 1.3.21. *Let $k, \ell \geq 1$ and $\alpha \in S_{k+\ell}$ the permutation*

$$\alpha(i) = \begin{cases} k + i, & i \leq \ell \\ i - \ell, & i \geq \ell + 1 \end{cases}$$

Then $S(\ell, k) \to S(k, \ell)$, $\sigma \mapsto \sigma \circ \alpha$, is a bijection. Furthermore, $\operatorname{sign}(\alpha) = (-1)^{k\ell}$.

Proof. Let $1 \leq j < \ell$. Then $k < \alpha(j) < \alpha(j+1) \leq k + \ell$. Thus, for $\alpha \in S(\ell, k)$,

$$\sigma(\alpha(j)) < \sigma(\alpha(j+1)).$$

Hence $\sigma \circ \alpha \in S(k, \ell)$. Thus $\sigma \mapsto \sigma \circ \alpha$ is well-defined. It is clearly a bijection (the inverse map is given by $\sigma \mapsto \sigma \circ \alpha^{-1}$).

We show now that $\mathrm{sign}(\alpha) = (-1)^{kl}$. For $1 \leq i \leq k$, let $\beta_i \in S_{k+\ell}$ be the permutation

$$\beta_i = \tau_{i,i+1} \circ \tau_{i+1,i+2} \circ \cdots \circ \tau_{\ell+i-1,\ell+i}.$$

Then

$$\beta_i(j) = \begin{cases} i, & j = \ell + i \\ j+1, & \text{for } i \leq j < \ell + i \\ j, & \text{otherwise.} \end{cases}$$

Moreover, $\mathrm{sign}(\beta_i) = (-1)^\ell$ for every $1 \leq i \leq k$.

By induction,

$$\alpha = \beta_k \circ \cdots \circ \beta_1.$$

Thus

$$\mathrm{sign}(\alpha) = \mathrm{sign}(\beta_k \circ \cdots \circ \beta_1) = \mathrm{sign}(\beta_k) \cdots \mathrm{sign}(\beta_1) = (-1)^{k\ell}.$$

$\square$

Proof of Lemma 1.3.20. Let $\alpha \in S_{k+\ell}$ be the permutation in Lemma 1.3.21. Then

$$\tau(v_{\sigma(\alpha(1))}, \ldots, v_{\sigma(\alpha(\ell))}) = \tau(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)})$$

and

$$\omega(v_{\sigma(\alpha(\ell+1)}, \ldots, v_{\sigma(\alpha(k+\ell)}) = \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

Hence

$$\begin{aligned} &\tau \wedge \omega(v_1, \ldots, v_{k+\ell}) \\ &\quad = \sum_{\sigma \in S(\ell,k)} \mathrm{sign}(\sigma)\tau(v_{\sigma(1)}, \ldots, v_{\sigma(\ell)})\omega(v_{\sigma(\ell+1)}, \ldots, v_{\sigma(\ell+k)}) \\ &\quad = \sum_{\sigma' \in S(k,\ell)} \mathrm{sign}(\sigma' \circ \alpha)\tau(v_{\sigma'(\alpha(1))}, \ldots, v_{\sigma'(\alpha(\ell)})\omega(v_{\sigma'(\alpha(\ell+1))}, \ldots, v_{\sigma'(\alpha(\ell+k))}) \\ &\quad = \mathrm{sign}(\alpha) \sum_{\sigma' \in S(k,\alpha)} \mathrm{sign}(\sigma')\tau(v_{\sigma'(k+1))}, \ldots, v_{\sigma'(\ell+k)})\omega(v_{\sigma'(1)}, \ldots, v_{\sigma'(k)}) \\ &\quad = (-1)^{k\ell}\omega \wedge \tau(v_1, \ldots, v_{k+\ell}). \end{aligned}$$

$\square$

For completeness, we remark that the exterior product is *associative.*

Lemma 1.3.22. *Let* $\omega \in \mathrm{Alt}^k(V)$, $\tau \in \mathrm{Alt}^\ell(V)$, *and* $\xi \in \mathrm{Alt}^m(V)$. *Then*

$$(\omega \wedge \tau) \wedge \xi = \omega \wedge (\tau \wedge \xi) \in \mathrm{Alt}^{k+\ell+m}(V).$$

Proof. [7, Lemma 2.9]. $\square$

1.3.4 Alternating multilinear maps and determinants

Lemma 1.3.23. *Let V be a vector space and $\omega_1, \ldots, \omega_k \in \mathrm{Alt}^1(V)$. Then*

$$\omega_1 \wedge \cdots \wedge \omega_k(v_1, \ldots, v_k) = \det \begin{bmatrix} \omega_1(v_1) & \cdots & \omega_1(v_k) \\ \vdots & \ddots & \vdots \\ \omega_k(v_1) & \cdots & \omega_k(v_k) \end{bmatrix}$$

for all $v_1, \ldots, v_k \in V$.

Proof. The proof is by induction. The claim clearly holds for $k = 1$. Suppose it holds for $k \geq 1$. Then, by the definition of the exterior product, we have, for $k+1$,

$$\begin{aligned} &\omega_1 \wedge \cdots \wedge \omega_{k+1}(v_1, \ldots, v_{k+1}) \\ &\quad = \omega_1 \wedge (\omega_2 \wedge \cdots \wedge \omega_{k+1})(v_1, \ldots, v_{k+1}) \\ &\quad = \sum_{\sigma \in S(1,k)} \mathrm{sign}(\sigma) \omega_1(v_{\sigma(1)}) \omega_2 \wedge \cdots \wedge \omega_{k+1}(v_{\sigma(2)}, \ldots, v_{\sigma(k+1)}) \\ &\quad = \sum_{i=1}^{k+1} (-1)^{i+1} \omega_1(v_i) \omega_2 \wedge \cdots \wedge \omega_{k+1}(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{k+1}). \end{aligned}$$

For $i = 1, \ldots, k+1$, let A_{i1} be the matrix

$$A_{i1} = \begin{bmatrix} \omega_2(v_1) & \cdots & \omega_2(v_{i-1}) & \omega_2(v_{i+1}) & \cdots & \omega_{k+1}(v_{\sigma(k+1)}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \omega_{k+1}(v_1) & \cdots & \omega_{k+1}(v_{i-1}) & \omega_{k+1}(v_{i+1}) & \cdots & \omega_{k+1}(v_{\sigma(k+1)}) \end{bmatrix}.$$

Then, by the induction assumption and the expansion of the determinant along the first row

$$\begin{aligned} \omega_1 \wedge \cdots \wedge \omega_{k+1}(v_1, \ldots, v_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i+1} \omega_1(v_i) \det A_{i1} \\ &= \det \begin{bmatrix} \omega_1(v_1) & \cdots & \omega_1(v_{k+1}) \\ \vdots & \ddots & \vdots \\ \omega_{k+1}(v_1) & \cdots & \omega_{k+1}(v_{k+1}) \end{bmatrix} \end{aligned}$$

□

Corollary 1.3.24. *Let $\omega_1, \ldots, \omega_k \in \mathrm{Alt}^1(V)$. Then*

$$\omega_1 \wedge \cdots \wedge \omega_k \neq 0$$

if and only if $\omega_1, \ldots, \omega_k$ are linearly independent in V^*.

Proof. Suppose $\omega_1 \wedge \cdots \wedge \omega_k \neq 0$. Suppose towards contradiction that $\omega_1, \ldots, \omega_k$ are linearly dependent, that is, there exists $a_1, \ldots, a_k \in \mathbb{R}$ not all zeroes, so that

$$a_1\omega_1 + \cdots + a_k\omega_k = 0.$$

We may assume that $a_1 \neq 0$ and solve ω_1 to obtain

$$\omega_1 = -\frac{1}{a_1}\left(a_2\omega_2 + \cdots + a_k\omega_k\right).$$

Thus

$$\omega_1 \wedge \cdots \wedge \omega_k = -\sum_{i=2}^{k} \frac{a_i}{a_1}\omega_i \wedge \omega_2 \wedge \cdots \wedge \omega_k = 0,$$

since $\omega_i \wedge \omega_2 \wedge \cdots \wedge \omega_k = 0$ for all $i = 1, \ldots, k$ by Lemma 1.3.23. This is a contradiction. Thus linear maps $\omega_1, \ldots, \omega_k$ are linearly independent.

Suppose now that $\omega_1 \wedge \cdots \wedge \omega_k = 0$ and suppose towards contradiction that $(\omega_1, \ldots, \omega_k)$ is linearly independent. Then there exists vectors $e_1, \ldots, e_k \in V$ for which $\omega_i(e_j) = \delta_{ij}$.

Then

$$\begin{aligned} \det \begin{bmatrix} \omega_1(e_1) & \cdots & \omega_k(e_1) \\ \vdots & \ddots & \vdots \\ \omega_1(e_k) & \cdots & \omega_k(e_k) \end{bmatrix} &= \det \begin{bmatrix} \omega_1(e_1) & \cdots & \omega_1(e_k) \\ \vdots & \ddots & \vdots \\ \omega_k(e_1) & \cdots & \omega_k(e_k) \end{bmatrix} \\ &= \omega_1 \wedge \cdots \wedge \omega_k(e_1, \ldots, e_k) = 0. \end{aligned}$$

Thus columns of the determinant are linearly dependent and there exists coefficients $a_1, \ldots, a_k \in \mathbb{R}$ (not all zero) so that

$$a_1 \begin{bmatrix} \omega_1(e_1) \\ \vdots \\ \omega_1(e_k) \end{bmatrix} + \cdots + a_k \begin{bmatrix} \omega_1(e_1) \\ \vdots \\ \omega_k(e_k) \end{bmatrix} = 0.$$

We may assume that $a_1 \neq 0$ and we obtain

$$\omega_1(e_i) = \frac{1}{a_1}\sum_{j=1}^{k} \omega_j(e_i)$$

for all $i = 1, \ldots, k$. But this is contradiction, since $\omega_1(e_1) = 1$ and

$$\frac{1}{a_1}\sum_{j=1}^{k} \omega_j(e_1) = 0.$$

Thus $(\omega_1, \omega_2, \ldots, \omega_k)$ is linearly dependent. $\square$

1.3.5 Basis of $\mathrm{Alt}^k(V)$

Let $n \geq k \geq 1$. Recall that $\sigma \in S(k, n-k)$ is a bijection $\{1, \ldots, n\} \to \{1, \ldots, n\}$ satisfying

$$1 \leq \sigma(1) < \cdots < \sigma(k) \leq n \text{ and } \sigma(k+1) < \cdots < \sigma(n).$$

Theorem 1.3.25. *Let V be n-dimensional (real) vector space, $(e_1, \ldots, e_n)$ a basis of V and $(\varepsilon_1, \ldots, \varepsilon_n)$ the corresponding dual basis. Then*

$$(\varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(k)})_{\sigma \in S(k,n-k)} \tag{1.3.3}$$

is a basis of $\mathrm{Alt}^k(V)$. Moreover, for every $\omega \in \mathrm{Alt}^k(V)$,

$$\omega = \sum_{\sigma \in S(k,n-k)} \omega(e_{\sigma(1)}, \ldots, e_{\sigma(k)}) \varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(k)}. \tag{1.3.4}$$

Corollary 1.3.26. *Let V be an n-dimensional vector space. For $0 \leq k \leq n$,*

$$\dim \mathrm{Alt}^k(V) = \# S(k, n-k) = \binom{n}{k}.$$

In particular, $\mathrm{Alt}^n(V) \cong \mathbb{R}$. Moreover, for $k > n$, $\mathrm{Alt}^k(V) = \{0\}$.

Proof of Theorem 1.3.25. We show first the linear independence. Suppose $a_\sigma \in \mathbb{R}$ $(\sigma \in S(k, n-k))$ are such coefficients that

$$\sum_\sigma a_\sigma \varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(k)} = 0.$$

Let $\tau \in S(k, n-k)$. By Lemma 1.3.23,

$$\varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(k)}(e_{\tau(1)}, \ldots, e_{\tau(k)}) = \delta_{\sigma,\tau}$$

for every $\sigma \in S(k, n-k)$. Thus

$$\begin{aligned}
a_\tau &= a_\tau \varepsilon_{\tau(1)} \wedge \cdots \wedge \varepsilon_{\tau(k)}(e_{\tau(1)}, \ldots, e_{\tau(k)}) \\
&= \sum_\sigma a_\sigma \varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(k)}(e_{\tau(1)}, \ldots, e_{\tau(k)}) \\
&= 0.
\end{aligned}$$

for every $\tau \in S(k, n-k)$. Hence (1.3.3) is linearly independent.

We show now (1.3.4). This proves that the sequence in (1.3.3) spans $\mathrm{Alt}^k(V)$. Let $v_i = \sum_{j=1}^k v_{ij} e_j \in V$ for $i = 1, \ldots, k$. Then

$$\begin{aligned}
\omega(v_1, \ldots, v_k) &= \sum_{(j_1, \ldots, j_k)} v_{1j_1} \cdots v_{kj_k} \omega(e_{j_1}, \ldots e_{j_k}) \\
&= \sum_{\sigma \in S(n,k)} \sum_{\tau \in S_k} v_{1,\tau(\sigma(1))} \cdots v_{k,\tau(\sigma(k))} \omega(e_{\tau(\sigma(1))}, \ldots e_{\tau(\sigma(k))}) \\
&= \sum_{\sigma \in S(n,k)} \sum_{\tau \in S_k} \mathrm{sign}(\tau) v_{1,\tau(\sigma(1))} \cdots v_{k,\tau(\sigma(k))} \omega(e_{\sigma(1)}, \ldots e_{\sigma(k)})
\end{aligned}$$

On the other hand,

$$\varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(k)}(e_{\alpha(1)}, \ldots e_{\alpha(k)}) = \begin{cases} 1, & \alpha = \sigma \\ 0, & \text{otherwise} \end{cases}$$

for all $\sigma, \alpha \in S(k, n-k)$ (by Lemma 1.3.23 or directly from definition).

Thus

(1.3.5)

$$\begin{aligned} &\varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(k)}(v_1, \ldots, v_k) \\ &\quad = \sum_{\alpha \in S(k,n-k)} \left(\sum_{\tau \in S_k} \operatorname{sign}(\tau) v_{1,\tau(\alpha(1))} \cdots v_{k,\tau(\alpha(k))} \right) \varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(k)}(e_{\alpha(1)}, \ldots e_{\alpha(k)}) \\ &\quad = \sum_{\tau \in S_k} \operatorname{sign}(\tau) v_{1,\tau(\sigma(1))} \cdots v_{k,\tau(\sigma(k))}. \end{aligned}$$

Hence

$$\begin{aligned} \omega(v_1, \ldots, v_k) &= \sum_{\sigma \in S(k,n-k)} \sum_{\tau \in S_k} \operatorname{sign}(\tau) v_{1,\tau(\sigma(1))} \cdots v_{k,\tau(\sigma(k))} \omega(e_{\sigma(1)}, \ldots e_{\sigma(k)}) \\ &= \sum_{\sigma \in S(k,n-k)} \varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(k)}(v_1, \ldots, v_k) \omega(e_{\sigma(1)}, \ldots e_{\sigma(k)}) \\ &= \left(\sum_{\sigma \in S(k,n-k)} \omega(e_{\sigma(1)}, \ldots e_{\sigma(k)}) \varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(k)} \right) (v_1, \ldots, v_k). \end{aligned}$$

□

Remark 1.3.27. *Note that, for $n = k$ and $\sigma = \mathrm{id}$, we may combine Lemma 1.3.23 and* (1.3.5) *to obtain the formula*

$$\det [v_1 \cdots v_n] = \varepsilon_1 \wedge \cdots \wedge \varepsilon_n(v_1, \ldots, v_n) = \sum_{\tau \in S_n} \operatorname{sign}(\tau) v_{1\tau(1)} \cdots v_{n\tau(n)}.$$

This representation formula is usually taken as a definition of the determinant.

Taking the formula for the determinant in Remark 1.3.27 and combining it with Lemma 1.3.23, we also get a (useful) formula which could be taken as the definition of exterior product of dual vectors.

Lemma 1.3.28. *Let V be a vector space and let $\omega_1, \ldots, \omega_k \in \mathrm{Alt}^k(V)$. Then*

$$\omega_1 \wedge \cdots \wedge \omega_k(v_1, \ldots, v_k) = \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) \omega_1(v_{\sigma(1)}) \cdots \omega_n(v_{\sigma(k)}).$$

1.3.6 Exterior ring $\mathrm{Alt}^*(V)$

Let V be an n-dimensional vector space. Having the exterior product at our disposal, we may define the *exterior ring* $\mathrm{Alt}^*(V)$ of the vector space V by setting

$$\mathrm{Alt}^*(V) = \bigoplus_{k\geq 0} \mathrm{Alt}^k(V) = \mathrm{Alt}^0(V) \times \mathrm{Alt}^1(V) \times \cdots \times \mathrm{Alt}^n(V).$$

Thus the elements of $\mathrm{Alt}^*(V)$ are sequences

$$\omega = (\omega_0, \ldots, \omega_n).$$

Since there is no confusion between k- and ℓ-linear maps, we may also unambiguously write

$$\omega = \sum_{k=0}^{n} (0, \ldots, 0, \omega_k, 0, \ldots, 0) = \omega_0 + \omega_1 + \cdots + \omega_n,$$

where $\omega_k \in \mathrm{Alt}^k(V)$; note that components in this sum are uniquely determined.

The exterior product $\wedge\colon \mathrm{Alt}^k(V) \times \mathrm{Alt}^\ell(V) \to \mathrm{Alt}^{k+\ell}(V)$ induces a multiplication $\wedge\colon \mathrm{Alt}^*(V) \times \mathrm{Alt}^*(V) \to \mathrm{Alt}^*(V)$ with

$$\left(\sum_{k=0}^{n} \omega_k\right) \wedge \left(\sum_{k=0}^{n} \tau_k\right) = \sum_{k,\ell=0}^{n} \omega_k \wedge \tau_\ell.$$

Definition 1.3.29. Vector space $\mathrm{Alt}^*(V)$ with exterior product $\wedge$ is called an exterior ring of V.

Given a linear map $\varphi\colon V \to W$, there exists a ring homomorphism $\varphi^*\colon \mathrm{Alt}^*(W) \to \mathrm{Alt}^*(V)$ given by the formula

$$\varphi^*(\sum_{k=0}^{n} \omega_k) = \sum_{k=0}^{n} \varphi^* \omega_k.$$

The map φ^* is clearly linear and, by Lemma 1.3.30 (below), it satisfies

$$\varphi^*(\omega \wedge \tau) = \varphi^*\omega \wedge \varphi^*\tau$$

for all $\omega, \tau \in \mathrm{Alt}^*(W)$.

Lemma 1.3.30. *Let V, W be vector spaces, $\omega \in \mathrm{Alt}^k(W)$, $\tau \in \mathrm{Alt}^\ell(W)$, and $\varphi\colon V \to W$ a linear map. Then*

$$\varphi^*(\omega \wedge \tau) = \varphi^*\omega \wedge \varphi^*\tau.$$

Proof. Exercise. $\square$

Chapter 2

Differential forms

Let $n, k \geq 0$. Let $(e_1, \ldots, e_n)$ be a standard basis of $\mathbb{R}^n$ and $(\varepsilon_1, \ldots, \varepsilon_n)$ the corresponding dual basis.

By results in the previous section, we have that $\mathrm{Alt}^k(\mathbb{R}^n)$ is a vector space with basis $(\varepsilon_\sigma)_{\sigma \in S(k,n-k)}$, where

$$\varepsilon_\sigma = \varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(k)}.$$

Thus we have linear isomorphism

$$\mathbb{R}^{\binom{n}{k}} \xrightarrow{\cong} \mathbb{R}^{S(k,n-k)} \xrightarrow{\cong} \mathrm{Alt}^k(\mathbb{R}^n) ,$$

where $\mathbb{R}^{S(k,n-k)}$ is the vector space of all functions $S(k, n-k) \to \mathbb{R}$.

We give $\mathbb{R}^{S(k,n-k)}$ and $\mathrm{Alt}^k(\mathbb{R}^n)$ topologies which are induced by these linear isomorphisms. (A careful reader checks that the given topologies are independent on the chosen linear isomorphisms. We leave this to the careful reader, though.)

Let $U \subset \mathbb{R}^n$ be an open set. Let $\omega \colon U \to \mathrm{Alt}^k(\mathbb{R}^n)$ be a function. Then $\omega(x) \in \mathrm{Alt}^k(\mathbb{R}^n)$ for every $x \in U$. Thus, for every $x \in U$, there exists coeffiecients $f_\sigma(x) \in \mathbb{R}$ for which

$$\omega(x) = \sum_{\sigma \in S(k,n-k)} f_\sigma(x)\varepsilon_\sigma.$$

In particular, for every $\sigma \in S(k, n-k)$, we have a function $f_\sigma \colon U \to \mathbb{R}$. For every $\alpha \in S(k, n-k)$ and $x \in U$, we also have the formula

$$(\omega(x))\,(e_{\alpha(1)}, \ldots, e_{\alpha(k)}) = f_\alpha(x).$$

Based on this observation, we give the following definition for smoothness. (A careful reader again verifies that the smoothness does not depend on chosen basis.)

Definition 2.0.31. Let $U \subset \mathbb{R}^n$ be an open set. A function $\omega\colon U \to \mathrm{Alt}^k(\mathbb{R}^n)$ is C^∞-smooth if the function $U \to \mathbb{R}$,

$$x \mapsto (\omega(x))\,(e_{\alpha(1)}, \ldots, e_{\alpha(k)}),$$

is smooth for every $\alpha \in S(k, n-k)$.

Convention 2.0.32. *To avoid unneccessary parenthesis, it is common to write*

$$\omega_x = \omega(x) \in \mathrm{Alt}^k(\mathbb{R}^n)$$

and

$$\omega_x(v_1, \ldots, v_k) = (\omega(x))\,(v_1, \ldots, v_k)$$

for $x \in \Omega$ and $v_1, \ldots, v_k \in \mathbb{R}^n$.

Definition 2.0.33. Let $U \subset \mathbb{R}^n$ be an open set. A *differential k-form* is a C^∞-smooth function $U \to \mathrm{Alt}^k(\mathbb{R}^n)$. We denote by $\Omega^k(U)$ the vector space of all differential k-forms $U \to \mathrm{Alt}^k(\mathbb{R}^n)$.

Remark 2.0.34. *We also call elements of $\Omega^k(U)$* differential forms, *k-*forms, *or just* forms, *for short.*

Example 2.0.35. *(1) Let $U \subset \mathbb{R}^n$ be an open set. Then $x \mapsto \varepsilon_1 \wedge \cdots \wedge \varepsilon_n$ is a differential form $\Omega^k(U)$. We call this usually a volume form and denote it by* vol_U.

(2) Let $U = \mathbb{R}^n \setminus \{0\}$. Then the map $\rho\colon U \to \mathrm{Alt}^{n-1}(U)$,

$$(\rho_x)(v_1, \ldots, v_{n-1}) = \varepsilon_1 \wedge \cdots \wedge \varepsilon_n \left(\frac{x}{|x|}, v_1, \ldots, v_{n-1}\right)$$

is an $(n-1)$-form in $\Omega^{n-1}(U)$ (Exercise).

The exterior product $\mathrm{Alt}^*(\mathbb{R}^n)$ extends to differential forms as follows.

Definition 2.0.36. Let $U \subset \mathbb{R}^n$ be an open set. Let $\omega \in \Omega^k(U)$ and $\tau \in \Omega^\ell(U)$. The map $\omega \wedge \tau\colon U \to \mathrm{Alt}^{k+\ell}(\mathbb{R}^n)$ defined by

$$(\omega \wedge \tau)_x = \omega_x \wedge \tau_x \tag{2.0.1}$$

for $x \in \Omega$ is the *exterior product $\omega \wedge \tau \in \Omega^{k+\ell}(U)$ of ω and τ.*

Abstractly, we have have obtained a bilinear operator

$$\wedge\colon \Omega^k(U) \times \Omega^\ell(U) \to \Omega^{k+\ell}(U).$$

2.1 Exterior derivative

Let U be an open set in $\mathbb{R}^n$ and $f\colon U \to \mathbb{R}$ a smooth function.

Definition 2.1.1. The *differential* df *of* f is the 1-form $df\colon U \to \operatorname{Alt}^1(\mathbb{R}^n)$ defined by

$$(df)_x(v) = \lim_{t\to 0} \frac{f(x+tv)-f(x)}{t}.$$

Remark 2.1.2. *The differential df of f is nothing but the standard linear map given by the directional derivatives. By the multivariable calculus,*

$$df(v) = \langle v, \nabla f\rangle,$$

where $\nabla f\colon U \to \mathbb{R}^n$ is the gradient of f

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{bmatrix}.$$

Example 2.1.3. *Let $x = (x_1, \ldots, x_n)\colon \mathbb{R}^n \to \mathbb{R}^n$ be the indentity map with coordinate functions $x_i\colon \mathbb{R}^n \to \mathbb{R}$ (i.e. $x_i(v_1, \ldots, v_n) = v_i$). Then*

$$dx_i = \varepsilon_i,$$

where ε_i is the element of the (standard) dual basis.(Exercise.)

Corollary 2.1.4. *Let $U \subset \mathbb{R}^n$ be an open set and $\omega \in \Omega^k(U)$. Then*

$$\omega = \sum_{\sigma\in S(k,n-k)} \omega_\sigma dx_{\sigma(1)} \wedge \cdots \wedge dx_{\sigma(k)},$$

where ω_σ is the function $\omega_\sigma(x) = \omega_x(e_{\sigma(1)}, \ldots, e_{\sigma(k)})$.

Proof. Exercise. □

2.1.1 Exterior derivitive of a k-form

Observation 2.1.5. *Let U be an open set and $\omega \in \Omega^k(U)$. Let $v = (v_1, \ldots, v_k) \in (\mathbb{R}^n)^k$ and consider the function $\omega^v\colon U \to \mathbb{R}$ defined by*

$$\omega^v(x) = \omega_x(v_1, \ldots, v_k) = \sum_{\sigma\in S(k,n-k)} \omega_\sigma(x)\varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(k)}(v_1, \ldots, v_k).$$

Then, for $w \in \mathbb{R}^n$,

$$\begin{aligned}
(d\omega^v)_x(w) &= \lim_{t\to 0} \frac{\omega^v(x+tw)-\omega^v(x)}{t} \\
&= \lim_{t\to 0} \frac{1}{t}\sum_\sigma (\omega_\sigma(x+tw)-\omega_\sigma(x))\varepsilon_\sigma(v_1, \ldots, v_k) \\
&= \sum_\sigma (d\omega_\sigma)_x(w)\varepsilon_\sigma(v_1, \ldots, v_k)
\end{aligned}$$

Remark 2.1.6 (Alternative approach)**.** *Since* $\mathrm{Alt}^k(V)$ *is a vector space, we may also consider* $\omega\colon U \to \mathrm{Alt}^k(U)$ *as a vector valued function (or a map!) and define the differential* $(D\omega)_x$ *at* $x \in U$ *as a linear map* $\mathbb{R}^n \to \mathrm{Alt}^k(\mathbb{R}^n)$ *by formula*

$$(D\omega)_x(w) = \lim_{t\to 0} \frac{1}{t}\left(\omega_{x+tw} - \omega_x\right).$$

Note that $(D\omega)_x(w)$ *is an alternating* k*-linear map and*

$$\begin{aligned}
\left((D\omega)_x(w)\right)(v_1,\dots,v_k) &= \lim_{t\to 0} \frac{1}{t}\left(\omega_{x+tw} - \omega_x\right)(v_1,\dots,v_k) \\
&= \lim_{t\to 0} \frac{1}{t}\left(\omega_{x+tw}(v_1,\dots,v_k) - \omega_x(v_1,\dots,v_k)\right)
\end{aligned}$$

for $v_1,\dots,v_k \in \mathbb{R}^n$*. Thus, for* $v = (v_1,\dots,v_k) \in (\mathbb{R}^n)^k$ *and* $w \in \mathbb{R}^n$,

$$\left((D\omega)_x(w)\right)(v_1,\dots,v_k) = (d\omega^v)_x(w).$$

Convention 2.1.7. *Given a* k*-form* $\omega \in \Omega^k(U)$*. In what follows we use the notation*

$$\omega_x(v_1,\dots,\widehat{v_i},\dots,v_k) = \omega_x(v_1,\dots,v_{i-1},v_{i+1},\dots,v_k)$$

to indicate that we have removed the i*th entry.*

Observation 2.1.8. *We make now the final observation before the definition of the exterior product. Let* $\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \Omega^k(U)$*. Then, for* $v_1,\dots,v_{k+1} \in \mathbb{R}^n$,

$$\begin{aligned}
&df \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}(v_1,\dots,v_{k+1}) \\
&\quad= \sum_{\sigma\in S(1,k)} \mathrm{sign}(\sigma) df(v_{\sigma(1)}) dx_{i_1} \wedge \cdots \wedge dx_{i_k}(v_{\sigma(2)},\dots,v_{\sigma(k+1)}) \\
&\quad= \sum_{i=1}^{k+1} (-1)^{i-1} df(v_i) dx_{i_1} \wedge \cdots \wedge dx_{i_k}(v_1,\dots,\widehat{v_i},\dots,v_{k+1})
\end{aligned} \tag{2.1.1}$$

Definition 2.1.9. Let $\omega \in \Omega^k(U)$. The *exterior derivative* $d\omega \in \Omega^{k+1}(U)$ *of* ω is the $(k+1)$-form

$$(d\omega)_x(v_1,\dots,v_{k+1}) = \sum_{\sigma\in S(k,n-k)} \sum_{i=1}^{k+1} (-1)^{i-1} (d\omega_\sigma)_x(v_i) \varepsilon_\sigma(v_1,\dots,\widehat{v_i},\dots,v_k).$$

Remark 2.1.10. *(1) By* (2.1.1)*, we have that*

$$d\omega = \sum_{\sigma\in S(k,n-k)} d\omega_\sigma \wedge dx_{\sigma(1)} \wedge \cdots \wedge dx_{\sigma(k)}.$$

(2) Clearly

$$d(\omega + a\tau) = d\omega + ad\tau$$

for $\omega, \tau \in \Omega^k(U)$ and $a \in \mathbb{R}$.

(3) Since $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ has constant coefficients, we have

$$d\left(dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) = 0$$

for every $i_1, \ldots, i_k \in \{1, \ldots, n\}$.

The two most important properties of the exterior product are the Leibniz rule and the fact that the composition of exterior derivatives is zero. We begin with the latter.

Lemma 2.1.11. *Let $\omega \in \Omega^k(U)$. Then $d^2\omega = d(d\omega) = 0$.*

Proof. Let $f \in C^\infty(U)$ and $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, where $I = (i_1, \ldots, i_k)$. Then

$$\begin{aligned}
d^2(f dx_I) &= d\left(\sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i\right) \wedge dx_I = \sum_{j=1}^n \sum_{i=1}^n \left(\frac{\partial^2 f}{\partial x_j x_i} dx_j\right) \wedge dx_i \wedge dx_I \\
&= \sum_{j,i=1}^k \frac{\partial^2 f}{\partial x_j x_i} (dx_j \wedge dx_i) \wedge dx_I \\
&= \sum_{j<i} \left(\frac{\partial^2 f}{\partial x_j x_i} - \frac{\partial^2 f}{\partial x_i x_j}\right) (dx_j \wedge dx_i) \wedge dx_I = 0.
\end{aligned}$$

The claim now follows by linearity. □

The Leibniz rule for the exterior product has the following form.

Lemma 2.1.12. *Let $\omega \in \Omega^k(U)$ and $\tau \in \Omega^\ell(U)$. Then*

$$d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^k \omega \wedge d\tau.$$

Proof. It suffices to consider cases $\omega = f dx_I$ and $\tau = g dx_J$, where $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ and $dx_J = dx_{j_1} \wedge \cdots \wedge dx_{j_\ell}$ for $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_\ell)$. Since

$$\begin{aligned}
d\left(\omega \wedge \tau\right) &= d\left((fg) dx_I \wedge dx_J\right) \\
&= d(fg) \wedge dx_I \wedge dx_J \\
&= (gdf + fdg) \wedge dx_I \wedge dx_J \\
&= (df \wedge dx_I) \wedge g dx_J + f dg \wedge dx_I \wedge dx_J \\
&= d\omega \wedge \tau + (-1)^k f dx_I \wedge dg \wedge dx_J \\
&= d\omega \wedge \tau + (-1)^k \omega \wedge d\tau
\end{aligned}$$

□

We record as an observation that the exterior product for functions, Leibniz rule and the composition property uniquely determine the exterior product.

Corollary 2.1.13. *Suppose that, for every $k \geq 0$, $\hat{d}_k \colon \Omega^k(U) \to \Omega^{k+1}(U)$ is a linear operator satisfying*

(1) $\hat{d}_0 f = df$ *for every* $f \in \Omega^0(U)$*,*

(2) $\hat{d}_{k+1} \circ \hat{d}_k = 0$*,*

(3) $\hat{d}_{k+\ell}(\omega \wedge \tau) = d_k\omega \wedge \tau + (-1)^k \omega \wedge \hat{d}_\ell \tau$ *for* $\omega \in \Omega^k(U)$ *and* $\tau \in \Omega^\ell(U)$*.*

Then $\hat{d}_k = d$ *for every* $k \geq 0$*.*

Proof. See [7, Theorem 3.7]. □

2.2 Pull-back of forms

We begin with an observation, which we record as a lemma.

Lemma 2.2.1. *Let $f \colon U \to U'$ be a C^∞-smooth map. Then $f^* \colon \Omega^k(U') \to \Omega^k(U)$ defined by formula*

$$(f^*\omega)_x(v_1, \ldots, v_k) = \omega_{f(x)}((Df)v_1, \ldots, (Df)v_k) \tag{2.2.1}$$

is a well-defined linear operator.

Remark 2.2.2. *Note that* (2.2.1) *is equivalent to*

$$(f^*\omega)_x = ((Df)_x)^* \omega_{f(x)}$$

for every $x \in U$.

Definition 2.2.3. Let $f \colon U \to U'$ be a C^∞-smooth map. The mapping $f^* \colon \Omega^k(U') \to \Omega^k(U)$ is the *pull-back (of differential forms) defined by f.*

Having Lemma 1.3.30 at our disposal, we immediately obtain the fact that pull-back and the wedge product commute.

Lemma 2.2.4. *Let $U \subset \mathbb{R}^m$ and $U' \subset \mathbb{R}^n$ be open sets, and let $f \colon U \to U'$ be a C^∞-smooth map. Then, for $\omega \in \Omega^k(U')$ and $\tau \in \Omega^\ell(U')$, we have*

$$f^*(\omega \wedge \tau) = (f^*\omega) \wedge (f^*\tau).$$

In particular, for $u \in \Omega^0(U) = C^\infty(U)$,

$$f^*(u) = u \circ f$$

We also have the following observation.

Lemma 2.2.5. *Let $U \subset \mathbb{R}^n$ and $U' \subset \mathbb{R}^m$ be open sets, and let $f = (f_1, \ldots, f_n) \colon U \to U'$ be a C^∞-smooth map. Then*

$$f^*(dx_i) = df_i$$

for every $i = 1, \ldots, n$.

Proof. Exercise. $\square$

The exterior derivative and the pull-back commute.

Lemma 2.2.6. *Let $U \subset \mathbb{R}^m$ and $U' \subset \mathbb{R}^n$ be open sets, and let $f \colon U \to U'$ be a C^∞-smooth map. Then*

$$df^*\omega = f^*d\omega$$

for every $\omega \in \Omega^k(U')$.

Proof. By linearity, it suffices to consider $\omega = udx_{i_1} \wedge \cdots \wedge dx_{i_k}$. By the chain rule,

$$\begin{aligned}
f^*(du) &= f^*\left(\sum_{i=1}^m \frac{\partial u}{\partial x_i} dx_i\right) = \sum_{i=1}^m \left(\frac{\partial u}{\partial x_i} \circ f\right) f^*(dx_i) \\
&= \sum_{i=1}^m \left(\frac{\partial u}{\partial x_i} \circ f\right) \left(\sum_{j=1}^n \frac{\partial f_i}{\partial x_j} dx_j\right) = \sum_{j=1}^n \sum_{i=1}^m \left(\frac{\partial u}{\partial x_i} \circ f\right) \frac{\partial f_i}{\partial x_j} dx_j \\
&= \sum_{j=1}^n \frac{\partial (u \circ f)}{\partial x_j} dx_j = d(u \circ f).
\end{aligned}$$

The second observation is that

$$d\left(df_{i_1} \wedge \cdots \wedge df_{i_k}\right) = 0$$

for all $I = (i_1, \ldots, i_k)$. Indeed, this follows from the Leibniz rule by indeuction in k as follows:

$$\begin{aligned}
d\left(df_{i_1} \wedge \cdots \wedge df_{i_k}\right) &= d(df_{i-1}) \wedge df_{i_2} \wedge \cdots \wedge df_{i_k} + (-1) df_{i_1} \wedge d\left(df_{i_2} \wedge \cdots \wedge df_{i_k}\right) \\
&= 0,
\end{aligned}$$

where we used the induction assumption on $df_{i_2} \wedge \cdots \wedge df_{i_k}$ and the fact that $d^2 = 0$.

Having both of these observation at our disposal, we get

$$\begin{aligned}
df^*\omega &= d\left((u \circ f) df_{i_1} \wedge \cdots \wedge df_{i_k}\right) \\
&= d(u \circ f) \wedge \left(df_{i_1} \wedge \cdots \wedge df_{i_k}\right) + (u \circ f) \wedge d\left(df_{i_1} \wedge \cdots \wedge df_{i_k}\right) \\
&= f^*du \wedge f^*dx_{i_1} \wedge \cdots \wedge f^*dx_{i_k} \\
&= f^*\left(du \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) \\
&= f^*(d\omega).
\end{aligned}$$

$\square$

Chapter 3

De Rham cohomology

Let $U \subset \mathbb{R}^n$ be an open set. Since $d^2 = d \circ d \colon \Omega^{k-1}(U) \to \Omega^{k+1}(U)$ is zero opeator for every $k \geq 1$, we have

$$\text{(3.0.1)} \qquad \operatorname{im}\left(d \colon \Omega^{k-1}(U) \to \Omega^k(U)\right) \subset \ker\left(d \colon \Omega^k(U) \to \Omega^{k+1}(U)\right).$$

Thus Imd is a subspace of $\ker d$ for all $k \geq 1$.

For convenience, we set $\Omega^k(U) = \{0\}$ for all $k < 0$ and set $d = 0 \colon \Omega^k(U) \to \Omega^{k+1}(U)$ for all $k < 0$. Then (3.0.1) holds for all $k \in \mathbb{Z}$.

Definition 3.0.7. Let $U \subset \mathbb{R}^n$ be an open set. The quotient vector space

$$H^k(U) = \frac{\ker(d \colon \Omega^k(U) \to \Omega^{k+1}(U))}{\operatorname{im}(d \colon \Omega^{k-1}(U) \to \Omega^k(U))} = \frac{\{\omega \in \Omega^k(U) \colon d\omega = 0\}}{\{d\tau \colon \tau \in \Omega^{k-1}(U)\}}$$

is the *kth de Rham cohomology (group) of U.*

Recall that elemtents of the quotient space $H^k(U)$ are equivalence classes of k-forms. Given $\omega \in \ker(d \colon \Omega^k(U) \to \Omega^{k+1}(U))$, we denote the equivalence class by

$$[\omega] = \{\omega + d\tau \in \Omega^k(U) \colon \tau \in \Omega^{k-1}(U)\}.$$

Definition 3.0.8. Let $U \subset \mathbb{R}^n$ be an open set. A form $\omega \in \Omega^k(U)$ is *closed* if $d\omega = 0$ and *exact* if there exists a $(k-1)$-form $\tau \in \Omega^{k-1}(U)$ for which $d\tau = \omega$.

Thus

$$H^k(U) = \frac{\{\text{closed } k\text{−forms in } U\}}{\{\text{exact } k\text{−forms in } U\}}$$

and *de Rham cohomology $H^k(U)$ is a vector space which classifies the closed k-forms in U upto exact forms.*

The 0th cohomology $H^0(U)$ counts the number of components of the set U. This is typically formalized as follows.

Lemma 3.0.9. *Let U be an open set. Then $H^0(U)$. is the vector space of maps $U \to \mathbb{R}$ which are locally constant.*

Proof. By definition

$$H^0(U) = \frac{\ker(d\colon \Omega^0(U) \to \Omega^1(U))}{\operatorname{Im}(d\colon \Omega^{-1}(U) \to \Omega^0(U))}.$$

Since $d\colon \Omega^{-1}(U) \to \Omega^0(U)$ is the zero map, we have

$$H^0(U) \cong \ker(d\colon \Omega^0(U) \to \Omega^1(U))$$

On the other hand, given $f \in C^\infty(U)$, $df = 0$ if and only if f is locally constant. The claim follows. $\square$

Again there is a natural pull-back.

Lemma 3.0.10. *Let $U \subset \mathbb{R}^n$ and $U' \subset \mathbb{R}^m$ be open sets, and let $f\colon U \to U'$ be a smooth mapping. Then the mapping $H^k(f)\colon H^k(U') \to H^k(U)$ defined by the formula*

$$[\omega] \mapsto [f^*\omega]$$

is well-defined and linear.

Proof. To show that the formula is well-defined, we have to show that it is independent on the representative. Let $\omega, \omega' \in \Omega^k(U')$ be closed forms so that $[\omega] = [\omega']$. Then, by definition of the quotient, $\omega - \omega' = d\tau$ where $\tau \in \Omega^{k-1}(U')$. Since

$$f^*\omega = f^*(\omega' + d\tau) = f^*\omega' + df^*\tau,$$

we have $f^*\omega - f^*\omega' = df^*\tau$. Thus $[f^*\omega] = [f^*\omega']$ and $H^k(f)$ is well-defined.

To show the linearity, let again $\omega, \omega' \in \Omega^k(U')$ and $a \in \mathbb{R}$. Then $f^*(\omega + a\omega') = f^*\omega + af^*\omega'$. Since $[f^*\omega + af^*\omega'] = [f^*\omega] + a[f^*\omega']$ by the definition of operations in the quotient space, linearity of $H^k(f)$ follows. $\square$

Remark 3.0.11. *It is typical to denote the mapping $H^k(f)$ by f^*. Thus, on the cohomological level, the pull-back is defined by*

$$f^*[\omega] = [f^*\omega].$$

Lemma 3.0.12. *Let U, V, W be open sets in $\mathbb{R}^n$, $\mathbb{R}^m$, and $\mathbb{R}^\ell$, respectively, and let $f\colon U \to V$ and $g\colon V \to W$ be C^∞-smooth maps. Then*

$$f^* \circ g^* = (g \circ f)^*\colon H^k(W) \to H^k(U)$$

and

$$\mathrm{id}^* = \mathrm{id}\colon H^k(U) \to H^k(U)$$

for each k. In paricular, $f^*\colon H^k(V) \to H^k(U)$ if f is a diffeomorphism (i.e. f is a homeomorphism so that f and f^{-1} are C^∞-smooth).

Proof. Given $\omega \in \Omega^k(W)$ and $x \in U$, we have by the chain rule

$$\begin{aligned}
(f^* \circ g*)_x\omega &= (f^*)_x(g^*\omega) = (Df_x)^*(g^*\omega)_{f(x)} = (Df)_x(Dg)^*_{f(x)}\omega_{g(f(x))} \\
&= ((Dg)_{f(x)}(Df)_x)^*\omega_{(g\circ f)(x)} = D(g \circ f)_x\omega_{(g\circ f)(x)} \\
&= (g \circ f)^*_x\omega.
\end{aligned}$$

Thus $(f^* \circ g^*)[\omega] = [(f^* \circ g^*)\omega] = [(g \circ f)^*(\omega)] = (g \circ f)^*[\omega]$.

Since $\mathrm{id}^*\omega = \omega$, the second claim follows immediately.

If f is a diffeomorphism, we observe that

$$f^* \circ (f^{-1})^* = \mathrm{id}^* = \mathrm{id} \quad \text{and} \quad (f^{-1})^* \circ f^* = \mathrm{id}.$$

Thus f^* is an isomorphism. □

The exterior product is naturally defined on the cohomological level.

Lemma 3.0.13. *Let $U \subset \mathbb{R}^n$ be an open set. Then the mapping $H^k(U) \times H^\ell(U) \to H^{k+\ell}(U)$ defined by*

$$[\xi] \wedge [\zeta] = [\xi \wedge \zeta]$$

is well-defined and bilinear.

Proof. Let $\xi, \xi' \in \Omega^k(U)$ and $\zeta, \zeta' \in \Omega^k(U)$ be closed forms for which $\xi - \xi' = d\alpha$ and $\zeta - \zeta' = d\beta$ for some $\alpha \in \Omega^{k-1}(U)$ and $\beta \in \Omega^{k-1}(U)$. Since ζ is closed,

$$d(\alpha \wedge \zeta) = d\alpha \wedge \zeta + (-1)^{\ell-1}\alpha \wedge d\zeta = d\alpha \wedge \zeta.$$

Similarly,

$$d(\xi \wedge \beta) = d\xi \wedge \beta + (-1)^k\xi \wedge d\beta = (-1)^k\xi \wedge d\beta.$$

Thus

$$\begin{aligned}
\xi' \wedge \zeta' &= (\xi + d\alpha) \wedge (\zeta + d\beta) \\
&= \xi \wedge \zeta + \xi \wedge d\beta + d\alpha \wedge \zeta + d\alpha \wedge d\beta \\
&= \xi \wedge \zeta + (-1)^k d(\xi \wedge \beta) + d(\alpha \wedge \zeta) + d(\alpha \wedge d\beta) \\
&= \xi \wedge \zeta + d\left((-1)^k\xi \wedge \beta + \alpha \wedge \zeta + \alpha \wedge \beta\right).
\end{aligned}$$

Thus

$$[\xi' \wedge \zeta'] = [\xi \wedge \zeta']$$

and the mapping is well-defined. Cheking the bilinearity is left to the interested reader. □

3.1 Poincaré lemma

A fundamental fact in de Rham theory is that

$$H^k(B^n) \cong \begin{cases} \mathbb{R}, & k = 0 \\ 0, & k > 0. \end{cases}$$

We obtain this fact from slightly more general result.

Recall that a set $U \subset \mathbb{R}^n$ is *star-like* if there exists $x_0 \in U$ so that, $\{(1-t)x_0 + tx \in \mathbb{R}^n \colon 0 \le t \le 1\} \subset U$ for every $x \in U$. The point x_0 is a *center of* U.

Theorem 3.1.1 (Poincaré lemma)**.** *Let $U \subset \mathbb{R}^n$ be an open star-like set. Then*

$$\dim H^k(U) = \begin{cases} 1, & k = 0 \\ 0, & k > 0. \end{cases}$$

Proof. Every star-like set is connected. Thus every locally constant function on U is constant. Hence $H^0(U) \cong \mathbb{R}$ by Lemma 3.0.9.

Let $k > 0$. We need to show that

$$\ker(d \colon \Omega^k(U) \to \Omega^{k+1}(U)) = \mathrm{Im}(d \colon \Omega^{k-1}(U) \to \Omega^k(U)).$$

Idea: Suppose we find, for every $k > 0$, a linear map $S_k \colon \Omega^k(U) \to \Omega^{k-1}(U)$ satisfying

$$\omega = dS_k\omega + S_{k+1}d\omega$$

for every $\omega \in \Omega^k(U)$. Then every closed k-form is exact. Indeed, suppose ω is a closed k-form. Then

$$\omega = dS_k\omega + S_{k+1}d\omega = dS_k\omega.$$

(This operator S_k is a "chain homotopy operator".)
Execution of the idea.

We abuse the notation and denote by $t \colon U \times \mathbb{R} \to \mathbb{R}$ the projection $(x,s) \mapsto s$. We also denote by $x_k \colon U \times \mathbb{R} \to \mathbb{R}$ the projection $x_k(y_1, \ldots, y_n, s) = y_k$ (as usual). Given $\omega \in \Omega^k(U \times \mathbb{R})$, we have

(3.1.1)
$$\begin{aligned}\omega &= \sum_{\substack{I=(i_1,\ldots,i_k)\\ 1\le i_1<\cdots<i_k\le n}} \omega_I dx_{i_1} \wedge \cdots \wedge d_{i_k} + \sum_{\substack{J=(j_1,\ldots,j_{k-1})\\ 1\le j_1<\cdots<j_{k-1}\le n}} \omega_J dt \wedge dx_{j_1} \wedge \cdots \wedge d_{j_{k-1}} \\ &= \sum_I \omega_I dx_I + \sum_J \omega_J dt \wedge dx_J,\end{aligned}$$

where functions $\omega_I \in C^\infty(U)$ and $\omega_J \in C^\infty(U)$ are uniquely determined. Thus the linear map $\hat{S}_k \colon \Omega^k(U \times \mathbb{R}) \to \Omega^{k-1}(U)$, defined by formula

$$(\hat{S}_k\omega)_y = \sum_{\substack{J=(j_1,\ldots,j_{k-1})\\ 1\le j_1<\cdots<j_{k-1}\le n}} \left(\int_0^1 \omega_J(y,s)\, \mathrm{d}s\right) dx_J$$

for $y \in U$, is well-defined; here we use the notation in (3.1.1).

Let ω as in (3.1.1). Then

$$\begin{aligned}
\hat{S}_{k+1} d\omega &= \hat{S}_{k+1}\left(\sum_I d\omega_I \wedge dx_I + \sum_J d\omega_J \wedge dt \wedge dx_J\right) \\
&= \hat{S}_{k+1}\left(\sum_I \frac{\partial \omega_I}{\partial t} dt \wedge dx_I + \sum_I \sum_{\ell=1}^n \frac{\partial \omega_I}{\partial x_\ell} dx_\ell \wedge dx_I + \sum_J \sum_{\ell=1}^n \frac{\partial \omega_J}{\partial x_\ell} dx_\ell \wedge dt \wedge dx_J\right) \\
&= \hat{S}_{k+1}\left(\sum_I \frac{\partial \omega_I}{\partial t} dt \wedge dx_I - \sum_{J,\ell} \frac{\partial \omega_J}{\partial x_\ell} dt \wedge dx_\ell \wedge dx_J\right)
\end{aligned}$$

Thus

$$\begin{aligned}
(d\hat{S}_k + \hat{S}_{k+1} d)\omega &= d\sum_J \left(\int_0^1 \omega_J(\cdot, s)\, \mathrm{d}s\right) dx_J + \hat{S}_{k+1}\left(\sum_I d\omega_I \wedge dx_I + \sum_J d\omega_J \wedge dt \wedge dx_J\right) \\
&= \sum_{J,\ell} \left(\int_0^1 \frac{\omega_J}{\partial x_\ell}(\cdot, s)\, \mathrm{d}s\right) dx_\ell \wedge dx_J \\
&\quad + \sum_I \left(\int_0^1 \frac{\partial \omega_I}{\partial t}(\cdot, s)\, \mathrm{d}s\right) dx_I - \sum_{J,\ell} \left(\int_0^1 \frac{\partial \omega_J}{\partial x_\ell}(\cdot, s)\, \mathrm{d}s\right) dx_\ell \wedge dx_J \\
&= \sum_I \left(\omega_I(\cdot, 1) - \omega_I(\cdot, 0)\right) dx_I
\end{aligned}$$

Use of star-likeness. Since U is star-like, we may fix a center $x_0 \in U$ of U. Define $F\colon U \times \mathbb{R} \to U$ by

$$(x, t) \mapsto x_0 + \lambda(t)(x - x_0),$$

where $\lambda \in C^\infty(\mathbb{R})$ is a function so that $\lambda(t) = 0$ for $t \le 0$, $0 \le \lambda(t) \le 1$ for $0 \le t \le 1$, and $\lambda(t) = 1$ for $t \ge 1$. (Existence: exercise.) Then F is a smooth map satisfying $F(x, 0) = x$, and $F(x, 1) = x_0$ for every $x \in U$. Note that $F^*\colon \Omega^k(U) \to \Omega^k(U \times \mathbb{R})$.

For every $\omega \in \Omega^k(U)$, we have (Exercise!)

$$(F^*\omega)_{(\cdot,1)} = \omega \quad \text{and} \quad (F^*\omega)_{(\cdot,0)} = 0.$$

Let $S_k = \hat{S}_k \circ F^*\colon \Omega^k(U) \to \Omega^{k-1}(U)$. Then

$$\begin{aligned}
d \circ S_k + S_{k+1} \circ d &= d \circ \hat{S}_k \circ F^* + \hat{S}_{k+1} \circ F^* \circ d \\
&= d \circ \hat{S}_k \circ F^* + \hat{S}_{k+1} \circ d \circ F^* = \left(d \circ \hat{S}_k + \hat{S}_{k+1} \circ d\right) \circ F^*.
\end{aligned}$$

Thus

$$\left((\hat{S}_{k+1} d + d\hat{S}_k) \circ F^*\right)\omega = (F^*\omega)_{(\cdot,1)} - (F^*\omega)_{(\cdot,0)} = \omega.$$

The proof is complete. $\square$

3.2 Chain complexes

Let $U \subset \mathbb{R}^n$ open. Then $\big(\Omega^k(U)\big)_{k\in\mathbb{Z}}$ is a sequence of vector spaces. Exterior derivatives $(d\colon \Omega^k(U) \to \Omega^{k+1}(U)$ give a sequence of linear maps with the property that the composition

$$\Omega^k(U) \xrightarrow{d} \Omega^{k+1}(U) \xrightarrow{d} \Omega^{k+2}(U)$$

is always a zero map. Thus the sequence of pairs $\big((\Omega^k(U), d)\big)$ is a chain complex. In this section we consider abstract chain complexes and then apply the algebraic observations to de Rham cohomology.

Definition 3.2.1 (Chain complex)**.** The sequence $A_* = (A_k, d_k)_{k\in\mathbb{Z}}$ vector spaces and linear maps $d_k\colon A_k \to A_{k+1}$ is a *chain complex* if $d_{k+1} \circ d_k = 0$.

Remark 3.2.2. *Note that we have chosen here so-called* $+1$*-grading (i.e.* d_k *increases the index by* 1*). It would be also possible to choose* -1*-grading, which is used e.g. in singular homology (or in homological algebra in general). Both gradings lead to the same theory.*

Definition 3.2.3. The kth homology the chain complex $A_* = (A_k, d_k)$ is

$$H^k(A_*) = \frac{\ker d_k}{\mathrm{Im} d_{k-1}}.$$

The elements of $\ker d_k$ are called k-cycles and the elements of $\mathrm{Im} d_{k-1}$ k-boundaries. The elements of $H^k(A_*)$ are *homology classes.*

Example 3.2.4. *Let* $U \subset \mathbb{R}^n$ *open. Denote by* $\Omega^k_c(U)$ *the compactly supported* k*-forms in* $\mathbb{R}^n$*, that is, forms* $\omega \in \Omega^k(U)$ *for which the closure of* $\{x \in U\colon \omega_x \neq 0\}$ *is compact.*

Since $d\omega \in \Omega^{k+1}_c(U)$ *for every* $\omega \in \Omega^k_c(U)$*, we obtain a chain complex* $\Omega^*_c(U) = (\Omega^k_c(U), d)_{k\in\mathbb{Z}}$*. Its homology*

$$H^k(\Omega^*_c(U)) = \frac{\ker(d\colon \Omega^k_c(U) \to \Omega^{k+1}_c(U))}{\mathrm{Im}(d\colon \Omega^{k-1}_c(U) \to \Omega^k_c(U))}$$

is called the compactly supported cohomology $H^k_c(U)$ of U

3.2.1 Exact sequences of vector spaces

Definition 3.2.5. Let A, B, C be vector spaces and $f\colon A \to B$ and $g\colon B \to C$ linear maps. The sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is *exact* if $\ker g = \mathrm{Im} f$.

Similarly, a sequence

$$\cdots \xrightarrow{f_{k-2}} A_{k-1} \xrightarrow{f_{k-1}} A_k \xrightarrow{f_k} A_{k+1} \xrightarrow{f_{k+1}} \cdots$$

of vector spaces and linear maps is *exact* if

$$A_{k-1} \xrightarrow{f_{k-1}} A_k \xrightarrow{f_k} A_{k+1}$$

is exact whenever defined.

Exactness of a sequence encodes familiar properties of linear maps.

Lemma 3.2.6. *(1) If the sequence*

$$B \xrightarrow{g} C \longrightarrow 0$$

is exact, then g is surjective.

(2) If the sequence

$$0 \longrightarrow A \xrightarrow{f} B$$

is exact, then f is injective.

Proof. Exercise. □

Definition 3.2.7. An exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called a *short exact sequence.*

Example 3.2.8. *Let $f\colon A \to B$ be an injective linear map. Then*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{\pi} B/\ker f \longrightarrow 0$$

is an exact sequence; here π is the (canonical) linear map $v \mapsto v + \ker f$.

A fundamental observation on short exact sequences of vector spaces is the following lemma on splitting.

Lemma 3.2.9. *Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence. Then $B \cong A \oplus C$. In particularly, if A and C are finite dimensional then so is B.*

Proof. (We give a (slightly) abstract proof which does not refer to a basis. See [7, Lemma 4.1] for another proof.)

Since g is surjective, we have the diagram

$$\begin{tikzcd} B \arrow[rr, "g"] \arrow[dr, "\pi"'] & & C \\ & B/\ker g \arrow[ur, "\hat{g}"'] & \end{tikzcd}$$

where π is the (canonical) quotient map and $\hat{g}$ a linear isomorphism. Let $V \subset B$ be a subspace so that $V \oplus \ker g = B$ (i.e $V + \ker g = B$ and $V \cap \ker g = \{0\}$). Since $\mathrm{Im}(\pi|V) = B/\ker g$ and $\ker(\pi|V) = \{0\}$, we have that $\pi|V \colon V \to B/\ker g$ is an isomorphism.

Since f is injective, we have that $f\colon A \to \mathrm{Im} f$ is an isomorphism. Since $\mathrm{Im} f = \ker g$, we conclude that

$$B = \ker g \oplus V \cong \mathrm{Im} f \oplus (B/\ker g) \cong A \oplus C.$$

(Note that first $\oplus$ is understood in the sense of subspaces and the latter abstractly.) $\square$

3.2.2 Exact sequences of chain complexes

Definition 3.2.10. Let $A_* = (A_k, d_k^A)$ and $B_* = (B_k, d_k^B)$ be chain complexes. A chain map $f\colon A_* \to B_*$ is a sequence $(f_k \colon A_k \to B_k)$ of linear maps satisfying

$$\begin{tikzcd} A_k \arrow[r, "d_k^A"] \arrow[d, "f_k"'] & A_{k+1} \arrow[d, "f_{k+1}"] \\ B_k \arrow[r, "d_k^B"] & B_{k+1} \end{tikzcd}$$

for each k.

Lemma 3.2.11. *Let $f\colon A_* \to B_*$ be a chain map. Then $f_* = H_k(f)\colon H_k(A_*) \to H_k(B_*)$, $[c] \mapsto [f(c)]$, is well-defined and linear.*

Proof. To show that f_* is well-defined, let c and c' be k-cycles so that $c - c' = dc''$ for $c'' \in A_{k-1}$, i.e. $[c] = [c']$. Then

$$f(c) - f(c') = f(dc'') = d'f(c'').$$

Thus $[f(c)] = [f(c')]$ and f_* is well-defined. Linearity is left to the interested reader. $\square$

Definition 3.2.12. Let $A_* = (A_k, d_k^A)$ and $B_* = (B_k, d_k^B)$ are chain complexes. Chain maps $f\colon A_* \to B_*$ and $g\colon A_* \to B_*$ are *chain homotopic* if there exists a linear maps $s_k \colon A_k \to B_{k-1}$ for which

$$f - g = d_{k-1}^B s_k - s_{k+1} d_k^A.$$

Lemma 3.2.13. *For chain homotopic maps $f, g\colon A_* \to B_*$, $f^* = g^*\colon H_k(A^*) \to H_k(B^*)$.*

Proof. Exercise. (Compare to the operator S_k in the proof of Poincaré lemma.) $\square$

Exercise 3.2.14. *Let $A_* = (A_k, d_k^A)$ and $B_* = (B_k, d_k^B)$ be chain complexes. Define $A_* \oplus B_* = (A_k \oplus B_k, d_k^A \oplus d_k^B)$; here $d_k^A \oplus d_k^B\colon A_k \oplus B_k \to A_{k+1} \oplus B_{k+1}$ is the map $(c, c') \mapsto (d_k^A c, d_k^B c')$. Show that*

$$H^k(A_* \oplus B_*) \cong H^k(A_*) \oplus H^k(B_*).$$

Terminology introduced for vector spaces can easily be extended to chain complexes.

A sequence $A_* \xrightarrow{f} B_* \xrightarrow{g} C_*$ of chain complexes and chain maps is *exact* if $A_k \xrightarrow{f_k} B_k \xrightarrow{g_k} C_k$ is exact for each k. Similarly, an exact sequence $0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$ (of chain complexes) is called a short exact sequence (of chain complexes).

A non-trivial example

Let U_1 and U_2 be open sets in $\mathbb{R}^n$. We have four inclusions:

$$\begin{tikzcd} & U_1 \cap U_2 \arrow[dl, "j_1"'] \arrow[dr, "j_2"] & \\ U_1 \arrow[dr, "i_1"'] & & U_2 \arrow[dl, "i_2"] \\ & U_1 \cup U_2 & \end{tikzcd}$$

It is clear that

$$I_k\colon \Omega^k(U_1 \cup U_2) \to \Omega^k(U_1) \oplus \Omega^k(U_2), \quad \omega \mapsto (i_1^*\omega, i_2^*\omega),$$

and

$$J_k\colon \Omega^k(U_1) \oplus \Omega^k(U_2) \to \Omega^k(U_1 \cap U_2), \quad (\omega_1, \omega_2) \mapsto j_1^*\omega_1 - j_2^*\omega_2,$$

are chain maps.

Theorem 3.2.15. *The sequence*

$$0 \longrightarrow \Omega^k(U_1 \cup U_2) \xrightarrow{I_k} \Omega^k(U_1) \oplus \Omega^k(U_2) \xrightarrow{J_k} \Omega^k(U_1 \cap U_2) \longrightarrow 0$$

is a short exact sequence of chain complexes.

We use the following version on the partition of unity in the proof.

Lemma 3.2.16. *Let U and V be open sets in $\mathbb{R}^n$. Then there exist C^∞-smooth functions $\lambda\colon U\cup V\colon [0,1]$ and $\mu\colon U\cup V\to[0,1]$ for which* $\operatorname{spt}\lambda\subset U$, $\operatorname{spt}\mu\subset V$, *and $\lambda+\mu=1$; here* $\operatorname{spt} f=\operatorname{cl}_{U\cup V}\{x\in U\cup V\colon f(x)\neq 0\}$.

Proof. Exercise. $\square$

Proof of Theorem 3.2.15. We begin with a general remark. Let $V\subset W\subset\mathbb{R}^n$ be open sets and let $\iota\colon V\to W$ be the inclusion. Then, for all $\omega=\sum_I\omega_I dx_I\in\Omega^k(W)$,

$$\iota^*(\omega)=\sum_I(\omega_I\circ\iota)\iota^*(dx_I)=\sum_I(\omega_I|V)dx_I.$$

We begin now the actual proof. Let $\iota\colon U_1\cap U_2\to U_1\cup U_2$ be the inclusion. Then $\iota=i_1\circ j_1=i_2\circ j_2$. In particular, $j_1^*i_1^*=(i_1\circ j_1)^*=(i_2\circ j_2)^*=j_2^*i_2^*$.

We have to show the exactness at three places.

Case I: To show that I_k is injective, suppose $\omega=\sum_I\omega_I dx_I\in\Omega^k(U_1\cup U_2)$ is in the kernel of I_k. Then $i_1^*\omega=0$ and $i_2^*\omega=0$. Thus $\omega_I|U_1=0$ and $\omega_I|U_2=0$ for each I. Hence $\omega_I=0$ for every I. Hence $\omega=0$ and I_k is injective.

Case II: Let $\omega=\sum_I\omega_I dx_I\in\Omega^k(U_1\cup U_2)$. Then

$$J_k\circ I_k(\omega)=J_k(i_1^*\omega,i_2^*\omega)=j_1^*i_1^*\omega-j_2^*i_2^*\omega=0.$$

Thus $\mathrm{Im} I_k\subset\ker J_k$. To show the converse, let $(\omega_1,\omega_2)\in\ker J_k$, where $\omega_1=\sum_I\omega_{1,I}dx_I$ and $\omega_2=\sum_I\omega_{2,I}dx_I$. Then

$$0=j_1^*\omega_1-j_2^*\omega_2=\sum_I\left(\omega_{1,I}|U_1\cap U_2-\omega_{2,I}|U_1\cap U_2\right)dx_I$$

and, in particular, $\omega_{1,I}|U_1\cap U_2=\omega_{2,I}|U_1\cap U_2$. We extend $\omega_{1,I}$ and $\omega_{2,I}$ by zero to functions of $U_1\cup U_2$.

Let (λ_1,λ_2) be a smooth partition of unity for (U_1,U_2) so that $\lambda_1|U_1\setminus U_2=1$. Then

$$\lambda_2(x)\omega_{1,I}(x)+\lambda_1(x)\omega_{2,I}(x)=\begin{cases}\omega_{1,I}(x), & x\in U_1\setminus U_2\\ \omega_{1,I}(x)=\omega_{2,I}(x), & x\in U_1\cap U_2\\ \omega_{2,I}(x), & x\in U_2\setminus U_1\end{cases}\tag{3.2.1}$$

Let

$$\omega=\sum_I\left(\lambda_2\omega_{1,I}+\lambda_1\omega_{2,I}\right)dx_I\in\Omega^k(U_1\cup U_2).$$

By (3.2.1),

$$\begin{aligned}
I_k(\omega) &= (i_1^*\omega, i_2^*\omega) \\
&= \left(\sum_I ((\lambda_2\omega_{1,I} + \lambda_1\omega_{2,I})|U_1)\, dx_I, \sum_I ((\lambda_2\omega_{1,I} + \lambda_1\omega_{2,I})|U_2)\, dx_I\right) \\
&= \left(\sum_I \omega_{1,I} dx_I, \sum_I \omega_{2,I} dx_I\right) = (\omega_1, \omega_2).
\end{aligned}$$

Case III: Finally, we have show the surjectivity of J_k. Let $\omega \in \Omega^k(U_1 \cap U_2)$. Let again (λ_1, λ_2) be a partition of unity for (U_1, U_2). Let $\omega_1 = \lambda_2\omega \in \Omega^k(U_1)$ and $\omega_2 = -\lambda_1\omega \in \Omega^k(U_2)$. Then

$$\begin{aligned}
J_k(\omega_1, \omega_2) &= j_1^*\omega_1 - j_2^*\omega_2 = (\lambda_2|U_1 \cap U_2)\omega - (-\lambda_1|U_1 \cap U_2)\omega \\
&= ((\lambda_1 + \lambda_2)|U_1 \cap U_2)\,\omega = \omega.
\end{aligned}$$

□

3.3 Long exact sequence

A beautiful algebraic fact is that a short exact sequence of chain complexes gives rise to a long exact sequence of the homological level.

Theorem 3.3.1 (Long exact sequence)**.** *Let*

$$0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$$

be a short exact sequence of chain complexes. Then there exist linear maps $\partial_k \colon H^k(C_*) \to H^{k+1}(A_*)$ $(k \in \mathbb{Z})$ for which the sequence

$$\cdots \xrightarrow{\partial_{k-1}} H^k(A_*) \xrightarrow{f_*} H^k(B_*) \xrightarrow{g_*} H^k(C_*) \xrightarrow{\partial_k} H^{k+1}(A_*) \xrightarrow{f_*} \cdots$$

is exact.

An particular example of a long exact sequence is the Meyer–Vietoris sequence for de Rham cohomology. We state it as a theorem for importance, but, in fact, is an immediate corollary of Theorems 3.2.15 and 3.3.1.

Theorem 3.3.2. *Let U_1 and U_2 be open sets in $\mathbb{R}^n$, $U = U_1 \cup U_2$, and let $i_m \colon U_m \to U_1 \cup U_2$ and $j_m \colon U_1 \cap U_2 \to U_m$ be inclusions. Let*

(3.3.1)
$$0 \longrightarrow \Omega^k(U_1 \cup U_2) \xrightarrow{I_k} \Omega^k(U_1) \oplus \Omega^k(U_2) \xrightarrow{J_k} \Omega^k(U_1 \cap U_2) \longrightarrow 0$$

be a short exact sequence as in Theorem 3.2.15; $I(\omega) = (i_1^*(\omega), i_2^*(\omega))$ and $J(\omega_1, \omega_2) = j_1^*\omega_1 - j_2^*(\omega_2)$.

Then the sequence

$$\cdots \longrightarrow H^k(U) \xrightarrow{I_*} H^k(U_1) \oplus H^k(U_2) \xrightarrow{J_*} H^k(U_1 \cap U_2) \xrightarrow{\partial_k} H^{k+1}(U) \longrightarrow \cdots$$

where ∂_k is the boundary operator for the short exact sequence (3.3.1), *is a long exact sequence.*

Example 3.3.3. *Using this sequence we show that*

$$H^k(\mathbb{R}^2 \setminus \{0\}) \cong \begin{cases} \mathbb{R}, & k = 0, 1 \\ 0, & k \geq 2 \end{cases}$$

Let $L_+ = \{(x,0) \in \mathbb{R}^2 \colon x \geq 0\}$ and $L_- = \{(x,0) \in \mathbb{R}^2 \colon x \leq 0\}$, and set $U_+ = \mathbb{R}^2 \setminus L_+$ and $U_- = \mathbb{R}^2 \setminus L_-$. Then $U_+ \cup U_- = \mathbb{R}^2 \setminus \{0\}$ and $U_+ \cap U_- = (\mathbb{R} \setminus \{0\}) \times \mathbb{R}$.

Since $H^k(W) = 0$ for $k > 2$ and $k < 0$ all open sets W in $\mathbb{R}^2$, the Meyer–Vietoris theorem implies that the sequence
(3.3.2)

$$\begin{aligned} &0 \longrightarrow H^0(\mathbb{R}^2 \setminus \{0\}) \xrightarrow{(I_0)_*} H^0(U_+) \oplus H^0(U_-) \xrightarrow{(J_0)_*} H^0(U_+ \cup U_-) \xrightarrow{\partial_0} \\ &\xrightarrow{\partial_0} H^1(\mathbb{R}^2 \setminus \{0\}) \xrightarrow{(I_1)_*} H^1(U_+) \oplus H^1(U_-) \xrightarrow{(J_1)_*} H^1(U_+ \cup U_-) \xrightarrow{\partial_1} \\ &\xrightarrow{\partial_1} H^2(\mathbb{R}^2 \setminus \{0\}) \longrightarrow 0 \end{aligned}$$

is exact.

Since U_+ and U_- are starlike,

$$H^k(U_+) \cong H^k(U_-) \cong \begin{cases} \mathbb{R}, & k = 0, \\ 0, & k > 0. \end{cases}$$

On the other hand, $U_+ \cap U_-$ is a pair-wise disjoint union of two star-like sets. Thus

$$H^k(U_+ \cap U_-) = H^k(U_+) \oplus H^k(U_-) = \begin{cases} \mathbb{R}^2, & k = 0, \\ 0. & k > 0. \end{cases}$$

Furthermore, $\mathbb{R}^2 \setminus \{0\}$ is clearly connected and hence $H^0(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{R}$. Thus the Meyer-Vietoris sequence (3.3.2) *actually takes the form*

$$\begin{aligned} &0 \longrightarrow \mathbb{R} \xrightarrow{f_0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{g_0} \mathbb{R}^2 \xrightarrow{\partial_0} \\ &\xrightarrow{\partial_0} H^1(\mathbb{R}^2 \setminus \{0\}) \longrightarrow 0 \longrightarrow 0 \longrightarrow \\ &\longrightarrow H^2(\mathbb{R}^2 \setminus \{0\}) \longrightarrow 0 \end{aligned}$$

where f_0, g_0 are the linear maps obtained from maps $(I_0)_*$ and $(J_0)_*$ after identifications observed above, e.g. f_0 is the map

$$\begin{array}{ccc} H^0(\mathbb{R}^2 \setminus \{0\}) & \xrightarrow{(I_0)_*} & H^1(U_+) \oplus H^1(U_-) \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{R} & \xrightarrow{f_0} & \mathbb{R} \oplus \mathbb{R} \end{array}$$

Since the sequence is exact, we see that f_0 is injective and $\dim(\ker g_0) = 1$*. Thus* $\dim\ker\partial_0 = 1$*. Since* $H^1(\mathbb{R}^2 \setminus \{0\}) = \ker f_1 = \mathrm{Im}\partial_0$*, we have that* $H^1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{R}$*. On the other hand,* $H^2(\mathbb{R}^2 \setminus \{0\}) = 0$*. Thus*

$$H^k(\mathbb{R}^2 \setminus \{0\}) = \begin{cases} \mathbb{R}, & k = 0, 1, \\ 0, & k > 1. \end{cases}$$

3.3.1 Proof of Theorem 3.3.1

The proof of Theorem 3.3.1 consists of four parts: definition of ∂_k and three verifications of exactness. We begin with the definition of ∂_k rising from the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_{k-1} & \longrightarrow & B_{k-1} & \longrightarrow & C_{k-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_k & \longrightarrow & B_k & \xrightarrow{g_k} & C_k & \longrightarrow & 0 \\ & & \downarrow & & \downarrow d_k^B & & \downarrow & & \\ 0 & \longrightarrow & A_{k+1} & \xrightarrow{f_{k+1}} & B_{k+1} & \longrightarrow & C_{k+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_{k+2} & \longrightarrow & B_{k+2} & \longrightarrow & C_{k+2} & \longrightarrow & 0 \end{array}$$

Lemma 3.3.4. *Let* $0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$ *be a short exact sequence of chain complexes.*

(1) Let $c \in C_k$ be a cycle. Then $(f_{k+1})^{-1}(d_k^B(g_k^{-1}(c))) \neq \emptyset$ and all elements in $(f_{k+1})^{-1}(d_k^B(g_k^{-1}(c)))$ are cycles.

(2) Suppose k-cycles $c, c' \in C_k$ are in the same cohomology class. Then

$$a \in (f_{k+1})^{-1}(d_k^B(g_k^{-1}(c))) \quad \text{and} \quad a' \in (f_{k+1})^{-1}(d_k^B(g_k^{-1}(c')))$$

are in the same cohomology class in $H^{k+1}(A_*)$.

Proof. To show (1), we observe first that g_k is surjective. Let $b \in g_k^{-1}(c)$. Since c is a cycle, we have by

$$\begin{tikzcd} B_k \arrow[r, "g_k"] \arrow[d, "d_k^B"] & C_k \arrow[d, "d_k^C"] \\ B_{k+1} \arrow[r, "g_{k+1}"] & C_{k+1} \end{tikzcd}$$

that $g_{k+1}(d_{k+1}(c)) = 0$. Since $\ker g_{k+1} = \mathrm{Im} f_{k+1}$, there exists $a \in A_{k+1}$ so that $f_{k+1}(a) = d_k^B(b)$. Furthermore, since

$$\begin{tikzcd} A_{k+1} \arrow[r, "f_{k+1}"] \arrow[d, "d_{k+1}^A"] & B_{k+1} \arrow[d, "d_{k+1}^B"] \\ A_{k+2} \arrow[r, "f_{k+2}"] & B_{k+2} \end{tikzcd}$$

we have that $f_{k+2}(d_{k+1}^A(a)) = d_{k+1}^B(f_{k+1}(a)) = d_{k+1}^B d_k^B(b) = 0$. Since $\ker f_{k+2} = 0$ we have that b is a cycle.

To show (2), let $c'' \in C_{k-1}$ be such that $c - c' = d_k^C c''$. We have to show that there exists $a'' \in A_k$ for which $a - a' = d_k^A(a'')$.

Let $b, b' \in B_k$ be elements as in the proof of (1) for c and c', respectively. Then

$$f_{k+1}(a) - f_{k+1}(a') = d_k^B(b) - d_k^B(b') = d_k^B(b - b').$$

On the other hand,

$$g_k(b - b') = c - c' = d_k^C c''.$$

By surjectivity of g_{k-1}, there exists $b'' \in B_{k-1}$ for which $g_{k+1}(b'') = c''$. Then $g_k(d_{k-1}^B(b'')) = d_{k-1}^C(g_{k-1}(b'')) = d_{k-1}^C(c'') = c - c'$ and

$$g_k((b - b') - d_{k-1}^B(b'')) = 0.$$

Thus there exists $a'' \in A_k$ so that $f_k(a'') = (b - b') - d_{k-1}^B(b'')$. Since

$$f_{k+1}(a - a' - d_k^A(a'')) = f_{k+1}(a - a') - d_k^B f_k(a'') = d_k^B d_{k-1}^B(b'') = 0$$

and f_{k+1} is injective, the claim follows. $\square$

By Lemma 3.3.4, the following operator is well-defined; linearity we leave to the interested reader.

Definition 3.3.5. Let $0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$ be a short exact sequence of chain complexes. The linear operator $\partial_k \colon H^k(C_*) \to H^{k+1}(A_*)$ given by formula

$$\partial_k[c] = [(f_{k+1})^{-1}(d_k^B(g_k^{-1}(c)))]$$

is called the *boundary operator (for this short exact sequence)*.

To prove Theorem 3.3.1 it suffices now to verify the following lemma.

Lemma 3.3.6. *Let* $0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$ *be a short exact sequence of chain complexes. Then sequences*

$$H^k(B_*) \xrightarrow{g_*} H^k(C_*) \xrightarrow{\partial_k} H^{k+1}(A_*), \tag{3.3.3}$$

$$H^k(C_*) \xrightarrow{\partial_k} H^{k+1}(A_*) \xrightarrow{f_*} H^k(A_*) \tag{3.3.4}$$

and

$$H^k(A_*) \xrightarrow{(f_k)_*} H^k(B_*) \xrightarrow{(g_k)_*} H^k(B_*) \tag{3.3.5}$$

are exact for every $k \in \mathbb{Z}$.

Proof. We consider first (3.3.3). Let $[b] \in H^k(B_*)$. Thus

$$\partial_k \circ g_*([b]) = \partial_k[g_k(b)] = [f_{k+1}^{-1} d_k^B(b)] = 0.$$

and $\mathrm{Im} g_* \subset \ker \partial_k$ for every k.

To show the converse, let $[c] \in \ker \partial_k$, i.e. $[f_{k+1})^{-1}(d_k^B(g_k^{-1}(c)))] = 0$. Fix $b \in B_k$ and $a_{k+1} \in A_{k+1}$ so that $g_k(b) = c$ and $f_{k+1}(a) = d_k^B(b)$. Since $[a] = 0$, there exists $a' \in A_k$ for which $a = d_k^A a'$. Since $d_k^B(b - f_k(a')) = 0$ and $g_k(b - f_k(a')) = c - g_k(f_k(a')) = c$, we have that $[c] \in \mathrm{Im}(g_*)$. This proves the exactness of (3.3.3).

We consider now (3.3.4). Let $[c] \in H^k(C_*)$. Then

$$f_* \circ \partial_k[c] = [d_k^B(g_k^{-1}(c)] = 0$$

and $\mathrm{Im}\partial_k \subset \ker f_*$ for every k.

To show the converse, let $[a] \in \ker f_*$. Then there exists $b \in B_k$ for which $f_{k+1}(a) = d_k^B(b)$. Let $c = g_k(b)$. Since $d_k^C(c) = g_{k+1}(d_k^B(b)) = g_{k+1} \circ f_{k+1}(a)) = 0$, c is a cycle. Thus, by definition of ∂_k, $\partial_k[c] = [a]$ and $\ker f_* \subset \mathrm{Im}\partial_k$.

Finally, we consider the exactness of (3.3.4). Let $[a] \in H^k(A_*)$. Then $g_* f_*[a] = [g_k(f_k(a))] = 0$. Thus $\mathrm{im} f_* \subset \ker g_*$. Suppose now that $[b] \in \ker g_*$. Then $g_k(b) = d_{k+1}^C(c')$ for some $c' \in C_{k-1}$. Thus, by surjectivity of g_{k-1}, there exists $b' \in B_{k-1}$ so that $g_{k-1}(b') = c'$. Thus $b - d_{k-1}^B b' \in \ker g_k = \mathrm{Im} f_k$. Hence there exists $a \in A_k$ so that $f_k(a) = b - d_{k-1}^B b'$. Thus $f_*[a] = [f_k(a)] = [b]$. Hence $\ker g_* \subset \mathrm{Im} f_*$. $\square$

3.4 Homotopy

The kth de Rham cohomology $H^k(U)$ is defined using smooth forms in $\Omega^k(U)$. Furtheremore, we know that smooth mappings $f\colon U \to V$ induce linear mappings $f^*\colon H^k(V) \to H^k(U)$. These linear maps are isomorphisms if f is a diffeomorphism. It is an interesing fact that it is possible to define the pull-back $f^*\colon H^k(V) \to H^k(U)$ also when f is merely a continuous map. This leads to a fundamental observation that *de Rham cohomology is actually a topological invariant and not only a smooth invariant.* We discuss these (and other) questions in this section.

Convention 3.4.1. *In this section, we assume that all mappings we consider are (at least!) continuous.*

Definition 3.4.2. Let X and Y be topological spaces. Mappings $f_i\colon X \to Y$ (for $i = 0, 1$) are *homotopic* if there exists a map $F\colon X \times [0,1] \to Y$ so that $F(\cdot, 0) = f_0$ and $F(\cdot, 1) = f_1$. We denote $f_0 \simeq f_1$.

We list two fundamental facts on homotopy. Proofs are left to interested readers.

Lemma 3.4.3. *The relation $\simeq$ is an equivalence relation.*

Lemma 3.4.4. *Let $f_i\colon X \to Y$ and $g_i\colon Y \to Z$ be mappings between topological spaces for $i = 0, 1$. If $f_0 \simeq f_1$ and $g_0 \simeq g_!$ then $g_1 \circ f_1 \simeq g_0 \circ f_0$.*

Definition 3.4.5. A mapping $f\colon X \to Y$ is a *homotopy equivalence* if there exists a mapping $g\colon Y \to X$ for which $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$; here g is called a *homotopy inverse of f*.

If there exists a homotopy equivalence $X \to Y$ then X and Y are said to be homotopy equivalent and we denote $X \simeq Y$. The space X is said to be *contractible* if it is homotopy equivalent to a point.

Example 3.4.6. *(i) Star-like sets are contractible. In particular, $\mathbb{R}^n$ is contractible for all $n \geq 1$.*

(ii) $\mathbb{R}^2 \setminus \{0\} \simeq \mathbb{S}^1$.

(iii) $\mathbb{R}^2 \setminus ([-1,1] \times \{0\}) \simeq \mathbb{R}^2 \setminus \{0\}$.

Theorem 3.4.7. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets and let $f_0, f_1\colon U \to V$ be C^∞-smooth maps. Suppose f_0 and f_1 are smoothly homotopic, that is, there exists a C^∞-smooth map $F\colon U \times \mathbb{R} \to V$ so that $F(\cdot, 0) = f_0$ and $F(\cdot, 1) = f_1$. Then $f_0^*, f_1^*\colon \Omega^k(V) \to \Omega^k(U)$ are chain homotopic. In particularly,*

$$f_0^* = f_1^*\colon H^k(V) \to H^k(U).$$

Proof. Let $\phi_i\colon U \to U \times \mathbb{R}$ be the inclusion maps $x \mapsto (x,i)$ for $i = 0,1$. Then
$$F \circ \phi_i = f_i$$
for $i = 0,1$.

Let $\hat{S}_k\colon \Omega^k(U\times\mathbb{R}) \to \Omega^{k-1}(U)$ be the operators in the proof of Poincaré lemma. Then (Problem # 4 in Exercise set 3.)
$$d\hat{S}_k(\omega) - \hat{S}_{k+1}(d\omega) = \phi_1^*(\omega) - \phi_0^*(\omega)$$
for every $\omega \in \Omega^k(U\times\mathbb{R})$.

We define $S_k\colon \Omega^k(V) \to \Omega^{k-1}(U)$ by
$$S_k = \hat{S}_k \circ F^*$$
Then
$$\begin{aligned} dS_k - S_{k+1}d &= d\circ \hat{S}_k \circ F^* - \hat{S}_{k+1}\circ F^* \circ d \\ &= (d\circ \hat{S}_k - \hat{S}_{k+1}\circ d)\circ F^* \\ &= (\phi_1^* - \phi_0^*)\circ F^* = \phi_1^*\circ F^* - \phi_0 \circ F^* \\ &= (F\circ\phi_1)^* - (F\circ\phi_0)^* = f_1^* - f_0^*. \end{aligned}$$
Thus linear maps $f_i^*\colon \Omega^k(V) \to \Omega^k(U)$ are chain homotopic. We conclude that $f_1^* = f_0^*\colon H^k(V) \to H^k(U)$ by Lemma 3.2.13. □

3.4.1 Topological invariance of de Rham cohomology

In this subsection we prove (among other things) the following theorem.

Theorem 3.4.8 (Topological invariance of de Rham cohomology)**.** *Let $f\colon U \to V$ be a homeomorphism. Then $H^k(U) \cong H^k(V)$ for each $k \geq 0$.*

We begin with the following approximation theorem.

Theorem 3.4.9. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets and $f\colon U \to V$ a continuous map. Then there exists a C^∞-smooth map $g\colon U \to V$ homotopic to f. Furthermore, if C^∞-smooth maps $g_0, g_1\colon U \to V$ are homotopic then there exists a smooth homotopy $G\colon U\times\mathbb{R} \to V$ so that $G(\cdot,0) \simeq g_0$ and $G(\cdot,1) \simeq g_1$.*

The proof of Theorem 3.4.9 is based on partition of unity.

Theorem 3.4.10. *Let $U \subset \mathbb{R}^n$ be an open set and $\mathcal{V} = (V_i)_{i\in I}$ a cover of U by open sets. Then there exists C^∞-smooth functions $\phi_i\colon U \to [0,1]$ satisfying*

(1) $\mathrm{spt}\phi_i \subset V_i$ *for each $i \in I$,*

(2) (local finiteness) every $x \in U$ has a neighborhood W so that $\#\{i \in I : \phi_i | W \neq 0\} < \infty$,

(3) $\sum_{i \in I} \phi_i = 1$.

Proof. See [7, Theorem A.1]. □

As an application of the partition of unity, we have the following approximation result.

Lemma 3.4.11. *Suppose $A \subset U' \subset U \subset \mathbb{R}^n$ where U' and U are open sets and A is closed in U. Suppose $W \subset \mathbb{R}^m$ is an open set and $f : U \to W$ be a continuous map so that $f|U'$ is C^∞-smooth. Given a function $\epsilon : U \to (0, \infty)$ there exists a C^∞-smooth map $g : U \to W$ so that*

(1) $|f(x) - g(x)| \leq \epsilon(x)$ for all $x \in U$ and

(2) $f(x) = g(x)$ for $x \in A$.

Proof. See [7, Lemma A.9]. □

Proof of Theorem 3.4.9. We begin by fixing a function $\epsilon : U \to (0, \infty)$ satisfying $B^n(f(x), \epsilon(x)) \subset V$ for every $x \in U$. Then, by Lemma 3.4.11 ($A = U = \emptyset$), there exists a C^∞-smooth map $g : U \to V$ satisfying $|f(x) - g(x)| < \epsilon(x)$ for every $x \in U$. Then $f \simeq g$ by homotopy $(x, t) \mapsto (1 - t)f(x) + tg(x)$.

Suppose now that $g_0, g_1 : U \to V$ are C^∞-smooth functions and $F : U \times [0, 1] \to V$ a homotopy satisfying $F(\cdot, 0) = g_0$ and $F(\cdot, 1) = g_1$. Let $\psi : \mathbb{R} \to [0, 1]$ be a continuous function satisfying $\psi(t) = 0$ for $t \leq 1/3$ and $\psi(t) = 1$ for $t \geq 2/3$. Define $H : U \times \mathbb{R} \to V$ by $H = F \circ (\mathrm{id} \times \psi)$. Now, by Lemma 3.4.11, there exists a C^∞-smooth map $G : U \times \mathbb{R} \to V$ satisfying $G|U \times \{0, 1\} = H|U \times \{0, 1\}$. □

Combining Theorems 3.4.7 and 3.4.9, we obtain the following corollary.

Lemma 3.4.12. *Let $f : U \to V$ be a continuous map. Then there exists a C^∞-smooth map $g : U \to V$ homotopic to f. Furthermore, if $h : U \to V$ another C^∞-smooth map homotopic to f, then*

$$g^* = h^* : H^k(V) \to H^k(U)$$

for every $k \geq 0$.

Definition 3.4.13. Let $f : U \to V$ be a continuous map between open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$. We define

$$f^* = H^k(f) : H^k(V) \to H^k(U)$$

by $f^* = g^*$, where $g : U \to V$ is any C^∞-smooth map homotopic to f.

Corollary 3.4.14. *Let $f\colon U \to V$ be a homotopy equivalence. Then the induced map $f^*\colon H^k(V) \to H^k(U)$ is an isomorphism for each $k \geq 0$.*

Sketch of a proof. Let $g\colon V \to U$ be the homotopy inverse of f. Then $f^* \circ g^* = (g \circ f)^* = \mathrm{id}_U^* = \mathrm{id}$ and $g^* \circ f^* = (f \circ g)^* = \mathrm{id}_V^* = id$. (Why this is only a sketch?) □

Theorem 3.4.8 is now a corollary of this corollary.

3.4.2 First applications

We combine now methods of calculation to obtain the following result.

Theorem 3.4.15. *For $n \geq 2$,*

$$H^k(\mathbb{R}^n \setminus \{0\}) \cong \begin{cases} \mathbb{R}, & k = 0, n-1, \\ 0, & \text{otherwise.} \end{cases}$$

Combining Theorems 3.4.8 and 3.4.15, we have the following corollary which we state as a theorem (for its importance).

Theorem 3.4.16. *Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic if and only if $n = m$.*

Proof. Suppose $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic. Then $\mathbb{R}^n \setminus \{0\}$ and $\mathbb{R}^m \setminus \{0\}$ are homeomorphic. Thus $m = n$ by Theorem 3.4.15. The other direction (really) is trivial. □

Theorem 3.4.15 is a consequence of following theorem which is based on homotopy invariance and the Meyer–Vietoris sequence. Recall that we may identify $H^0(U)$ with the space of locally constant functions, and that this space contains a 1-dimensional space $\mathbb{R} \cdot 1$ spanned by the constant function 1.

Theorem 3.4.17. *Let $A \subset \mathbb{R}^n = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ be a closed set such that $A \neq \mathbb{R}^n$. Then*

$$H^k(\mathbb{R}^{n+1} \setminus A) \cong \begin{cases} H^{k-1}(\mathbb{R}^n \setminus A), & k > 1 \\ H^0(\mathbb{R}^n \setminus A)/\mathbb{R} \cdot 1, & k = 1 \\ \mathbb{R}, & k = 0. \end{cases}$$

Proof. Let $U_1 = \mathbb{R}^n \times (0, \infty) \cup \mathbb{R}^n \setminus A \times (-1, \infty)$ and $U_2 = \mathbb{R}^n \times (-\infty, 0) \cup \mathbb{R}^n \setminus A \times (-\infty, 1)$. Then

$$U_1 \cup U_2 = \mathbb{R}^n \setminus A$$

and

$$U_1 \cap U_2 = \big(\mathbb{R}^{n-1} \setminus A\big) \times (-1, 1).$$

The sets U_1 and U_2 are contractible. (Exercise!) Furthermore, the projection $\mathrm{pr}\colon (\mathbb{R}^{n-1} \setminus A) \times (-1,1) \to \mathbb{R}^{n-1} \setminus A$, $(x,t) \mapsto x$, is a homotopy equivalence. Indeed, the inclusion $\mathbb{R}^{n-1} \setminus A \to (\mathbb{R}^{n-1} \setminus A) \times (-1,1)$, $x \mapsto (x,0)$, is the homotopy inverse. (Exercise!)

We conclude that $\mathrm{pr}^*\colon H^k(\mathbb{R}^n \setminus A) \to H^k(U_1 \cap U_2)$ is an isomorphism for every $k \geq 0$.

On the other hand, by Meyer–Vietoris, we have an exact sequence

$$\cdots \longrightarrow H^k(U_1) \oplus H^k(U_2) \longrightarrow H^k(U_1 \cap U_2) \xrightarrow{\partial_k}$$

$$\xrightarrow{\partial_k} H^{k+1}(U_1 \cup U_2) \longrightarrow H^{k+1}(U_1) \oplus H^k k+1(U_2) \longrightarrow \cdots$$

For $k \geq 1$, we have

$$0 \longrightarrow H^k(U_1 \cap U_2) \xrightarrow{\partial_k} H^{k+1}(U_1 \cup U_2) \longrightarrow 0$$

Thus ∂_k is an isomorphism for $k \geq 1$.

Since $U_1 \cup U_2$ is connected (Exercise!), we have $H^0(U_1 \cup U_2) \cong \mathbb{R}$. Thus, we have an exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow H^0(U_1 \cap U_2) \xrightarrow{\partial_0} H^1(U_1 \cup U_2) \longrightarrow 0 \longrightarrow \cdots$$

We conclude that $\dim H^1(U_1 \cup U_2) = \dim H^0(U_1 \cap U_2) - 1$. (Exercise!) $\square$

Proof of Theorem 3.4.15. We know that

$$H^k(\mathbb{R}^2 \setminus \{0\}) \cong \begin{cases} \mathbb{R}, & k = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, by Theorem 3.4.17, we have, for $k > 1$,

$$H^k(\mathbb{R}^n \setminus \{0\}) \cong H^{k-(n-2)}(\mathbb{R}^2 \setminus \{0\}) \cong \begin{cases} \mathbb{R}, & k = 0, n-1 \\ 0, & \text{otherwise} \end{cases}$$

Similarly, $H^1(\mathbb{R}^n \setminus \{0\}) \cong H^0(\mathbb{R}^{n-1} \setminus \{0\})/(\mathbb{R} \cdot 1) = 0$ and $H^0(\mathbb{R}^n \setminus \{0\}) \cong \mathbb{R}$, since $\mathbb{R}^{n-1} \setminus \{0\}$ and $\mathbb{R}^n \setminus \{0\}$ are connected for $n \geq 2$. $\square$

As an another application of Theorem 3.4.17 we prove the following result which statates that the cohomology of the complement does not depend on the embedding of the set.

Theorem 3.4.18. *Let A and B be homeomorphic closed sets in $\mathbb{R}^n$ so that $A \neq \mathbb{R}^n \neq B$. Then*

$$H^k(\mathbb{R}^n \setminus A) \cong H^k(\mathbb{R}^n \setminus B)$$

for all $k \geq 0$.

Corollary 3.4.19. *Suppose A and B are closed homoeomorphic subsets $\mathbb{R}^n$. Then $\mathbb{R}^n \setminus A$ and $\mathbb{R}^n \setminus B$ have the same number of components.*

The proof is based on so-called *Klee trick*, which is based on the *Urysohn-Tietze extension theorem* (see e.g. [7, Lemma 7.4].

Lemma 3.4.20. *Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ be homeomorphic closed sets and $f\colon A \to B$ a homeomorphism. Then there exists a homeomorphism $\phi\colon \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ so that $\phi(x,0) = (0, f(x))$ for every $x \in A$.*

Proof. Let $f_1\colon \mathbb{R}^n \to \mathbb{R}^m$ be a continuous extension of f and define $h_1\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ by

$$(x,y) \mapsto (x, f_1(x) + y).$$

Then h_1 is a homeomorphism with the inverse $(x,y) \mapsto (x, y - f_1(x))$.

Let $f_2\colon \mathbb{R}^m \to \mathbb{R}^n$ be a continuous extension of f^{-1} and define a homeomorphism $h_2 \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ by

$$(x,y) \mapsto (x + f_2(y), y).$$

Now the mapping $\phi = h_2^{-1} \circ h_1$ is the required homeomorphism.(Check!) $\square$

Corollary 3.4.21 (Corollary of the Klee trick)**.** *Every homeomorphism $\phi\colon A \to B$ of closed sets in $\mathbb{R}^n$ can be extended to a homeomorphism $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$. (Here $\mathbb{R}^n = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$.)*

Proof of Theorem 3.4.18. For $k \geq 1$, we have by Theorem 3.4.17 and Corollary 3.4.21, we have

$$H^k(\mathbb{R}^n \setminus A) \cong H^{k+n}(\mathbb{R}^{2n} \setminus (A \times \{0\})) \cong H^{k+n}(\mathbb{R}^{2n} \setminus (B \times \{0\})) \cong H^k(\mathbb{R}^n \setminus B).$$

Similarly,

$$H^0(\mathbb{R}^n \setminus A)/(\mathbb{R} \cdot 1) \cong H^0(\mathbb{R}^n \setminus B)/(\mathbb{R} \cdot 1).$$

The claim is proven. $\square$

Chapter 4

Applications

In this chapter we list some classical applications of the de Rham cohomology. In this section we use the notations $B^n = \{x \in \mathbb{R}^n \colon |x| < 1\}$ and $\mathbb{S}^{n-1} = \partial B^n$.

4.0.3 Brower's fixed point theorem

Theorem 4.0.22 (Brower's fixed point theorem)**.** *Let $f \colon \bar{B}^n \to \bar{B}^n$ be a continuous map. Then f has a fixed point, that is, there exists $x \in \bar{B}^n$ for which $f(x) = x$.*

The proof is an easy application of the following lemma. (We leave the actual details to the interested reader; or see [7, Theorem 7.1].)

Lemma 4.0.23. *There is no continuous map $f \colon \bar{B}^n \to \mathbb{S}^{n-1}$ extending the identity, that is, satisfying $f|\partial\bar{B}^n = \mathrm{id}$.*

Proof. Suppose such map f exists. Let $F \colon \mathbb{S}^{n-1} \times [0,1] \to \mathbb{S}^{n-1}$ be the map $F(x,t) = f(tx)$. Then F is a homotopy from the constant map $x \mapsto f(0)$ to the identity $\mathrm{id}_{\mathbb{S}^{n-1}} = f|\mathbb{S}^{n-1}$. Thus $\mathbb{S}^{n-1}$ is contractible. This is a contradiction, since $\mathbb{S}^{n-1}$ is not contractible (Exercise!). □

4.0.4 Hairy ball theorem

For the purposes of the next statement, recall from Multivariable calculus the notion of a tangent space. For simplicity, we say that the tangent space of the sphere $\mathbb{S}^n$ at point $x \in \mathbb{S}^n$ is the linear subspace $T_x\mathbb{S}^n \subset \mathbb{R}^{n+1}$ which is orthogonal to x, that is, $v \in T_x\mathbb{S}^n$ satisfy $\langle v, x\rangle = 0$. Furthermore, we say that a vector field $X \colon \mathbb{S}^n \to \mathbb{R}^{n+1}$ is *tangential* if $X(x) \in T_x\mathbb{S}^n$ for every x.

Theorem 4.0.24. *The sphere $\mathbb{S}^n$ has a tangent vector field X with $X(x) \neq 0$ for all $x \in \mathbb{S}^n$ if and only if n is odd.*

Proof. For $n = 2m-1$ define

$$X(x_1, \ldots, x_{2m}) = (-x_2, x_1, -x_3, x_4, \ldots, -x_{2m}, x_{2m-1}).$$

Then $|X(x)| = |x| = 1$ for every x.

Suppose now that $X \colon \mathbb{S}^n \to \mathbb{R}^{n+1}$ is a tangent vector field which does not vanish ($X(x) \neq 0$ for all x). We show that n is odd.

Let $Y \colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{n+1} \setminus \{0\}$ be the vector field $Y(x) = X(x/|x|)$. Then $\langle Y(x), x\rangle = 0$ for all $x \in \mathbb{R}^{n+1} \setminus \{0\}$. Let $F \colon (\mathbb{R}^{n+1} \setminus \{0\}) \times [0,1] \to \mathbb{R}^{n+1} \setminus \{0\}$ be the map

$$(x,t) \mapsto \cos(\pi t)x + \sin(\pi t)Y(x).$$

Then F is a homotopy from id to $-$id. (Check that F is really well-defined! Draw a picture.)

Thus $\mathrm{id}^* = (-\mathrm{id})^* = (-1)^{n+1}\mathrm{id}^* \colon H^k(\mathbb{R}^{n+1} \setminus \{0\}) \to H^k(\mathbb{R}^{n+1} \setminus \{0\})$ for all $k > 0$. (Exercise!) Thus $n+1$ is even and hence n is odd. $\square$

4.0.5 Jordan–Brouwer separation theorem

Theorem 4.0.25. *Let $n \geq 2$ and let $\Sigma \subset \mathbb{R}^n$ be a subset homeomorphic to $\mathbb{S}^{n-1}$. Then*

(a) $\mathbb{R}^n \setminus \Sigma$ has exactly two components U_1 and U_2, where U_1 is bounded and U_2 is unbounded.

(b) the set Σ is the boundary of both U_1 and U_2.

Proof. To show (a), it suffices, by Corollary 3.4.19, find the number of components of $\mathbb{R}^n \setminus \Sigma$, it suffices to observe that $\mathbb{R}^n \setminus \mathbb{S}^{n-1}$ has two components, B^n and $\mathbb{R}^n \setminus \bar{B}^n$. To check that one of the components of $\mathbb{R}^n \setminus \Sigma$ is bounded and other unbounded, let $r = \max_{x\in\Sigma} |x|$. Then $W = \mathbb{R}^n \setminus \bar{B}^n(0,r)$ is a connected set and $W \cap \Sigma = \emptyset$. Thus W is contained in one of components, say U_2, of $\mathbb{R}^n \setminus \Sigma$. Thus U_2 is unbounded. By connectedness, the other component U_1 of $\mathbb{R}^n \setminus \Sigma$ is contained in $\mathbb{R}^n \setminus W$. Thus $U_1 \subset \bar{B}^n(0,r)$ and U_1 is bounded.

The proof of (b) is harder. Let $x \in \Sigma$. We need to show that $x \in \partial U_1$ and $x \in \partial U_2$. This follows if we show that $W \cap U_1 \neq \emptyset$ and $W \cap U_2 \neq \emptyset$ for all neighborhoods W of x in $\mathbb{R}^n$.

Let W be a neighborhood of x in $\mathbb{R}^n$. By passing to a smaller neighborhood if necessary we may assume that $\Sigma \not\subset W$. Then $A = \Sigma \setminus W$ is a closed subset of $\mathbb{R}^n$. We show first that $\mathbb{R}^n \setminus A = (\mathbb{R}^n \setminus \Sigma) \cup W$ is connected.

Let $\psi \colon \Sigma \to \mathbb{S}^{n-1}$ be a homeomorphism and denote $B = \psi(A)$. Then B is a closed subset of $\mathbb{S}^{n-1}$ in the relative topology. Since $\mathbb{S}^{n-1}$ is compact, B is a closed subset of $\mathbb{R}^n$. By Corollary 3.4.19 (again), $\mathbb{R}^n \setminus A$ and $\mathbb{R}^n \setminus B$ have the same number of components. However, it is an easy (geometric) exercise to show that $\mathbb{R}^n \setminus B$ is connected. Thus $\mathbb{R}^n \setminus A$ is connected.

Let $y_1 \in W \cap U_1$ and $y_2 \in W \cap U_2$. Since $\mathbb{R}^n \setminus A$ is connected there exists a path $\gamma\colon [0,1] \to \mathbb{R}^n \setminus A$ so that $\gamma(0) = y_1$ and $\gamma(1) = y_2$. Then, by connectedness, $\gamma[0,1] \cap \Sigma \neq \emptyset$. Note also that $\gamma[0,1] \cap \Sigma = \gamma[0,1] \cap \Sigma \cap W$.

Let $t_1 = \min \gamma^{-1}\Sigma$ and $t_2 = \max \gamma^{-1}\Sigma$. Then $t_1 > 0$ since $\gamma(0) = y_1$ and $W \cap U_1$ is an open neighborhood of t_1. Now $\gamma[0, t_1) \subset U_1$. Indeed, if this is not the case, then there exists $0 < t < t_1$ so that $\gamma(t) \in U_2$. But then there exists $0 < t' < t$ so that $\gamma(t) \in \Sigma$ by the connectedness argument above, and this contradicts the minimality of t_1. By similar arguments, $t_2 < 1$ and $\gamma(t_2, 1] \subset U_2$.

It follows now from the continuity of γ that $U_1 \cap W \neq \emptyset$ and $U_2 \cap W \neq \emptyset$. $\square$

Remark 4.0.26 (Warning!). *It is true for $n = 2$ that the components of $\mathbb{R}^2 \setminus \Sigma$ are homeomorphic to B^2 and $\mathbb{R}^2 \setminus \bar{B}^2$. This is* **not** *true for $n > 2$. Google Alexander's horned sphere.*

4.0.6 Invariance of domain

Theorem 4.0.27 (Brouwer)**.** *Let $U \subset \mathbb{R}^n$ be an open set and $f\colon U \to \mathbb{R}^n$ an injective continuous map. Then the image $f(U)$ is open in $\mathbb{R}^n$ and $f|U\colon U \to f(U)$ a homeomorphism.*

Corollary 4.0.28. *(Invariance of domain) If $V \subset \mathbb{R}^n$, in the topology induced from $\mathbb{R}^n$, is homeomorphic to an open set in $\mathbb{R}^n$, then V is open in $\mathbb{R}^n$.*

As an application of invariance of domain, we have the following local version of Theorem 3.4.16.

Corollary 4.0.29. *(Dimension invariance) Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets. If U and V are homeomorphic then $n = m$.*

Proof. We may assume that $n < m$. Let $\iota\colon \mathbb{R}^n \to \mathbb{R}^m$ be the natural inclusion $x \mapsto (x, 0)$. Since $\iota(U) \approx U \approx V$, $\iota(U)$ is open in $\mathbb{R}^m$ by the invariance of domain. This is a contradiction since $\partial\iota(U) = \iota(U)$ in $\mathbb{R}^m$. $\square$

Proof of Theorem 4.0.27. We need to show that $g\colon f(U) \to U$ is continuous, where $g = f^{-1}$. We show that $f(W)$ is open for every open set $W \subset U$. Then g^{-1} is continuous (by definition!).

Let $W \subset U$ be an open set and let $x_0 \in W$. Let $B = B^n(x_0, \delta)$ be a ball so that $\bar{B} \subset W$ and denote $S = \partial B$. We show that $f(B)$ is open.

For $n = 1$, we have that $f(B)$ is an open interval and we are done. Suppose $n \geq 2$.

Since f is a continuous injection and S is compact, $f|S\colon S \to f(S)$ is a homeomorphism. Then, by Jordan–Brouwer separation theorem (Theorem 4.0.25), $\mathbb{R}^n \setminus f(S)$ has two components V_1 and V_2 so that V_1 is bounded and V_2 is unbounded, and which satisfy $\partial V_1 = \partial V_2 = f(S)$.

Since $f|\bar{B}\colon \bar{B} \to f(\bar{B})$ is a homeomorphism, we have by Theorem 3.4.18 that $H^0(\mathbb{R}^n \setminus f(\bar{B})) \cong H^0(\mathbb{R}^n \setminus \bar{B}) \cong \mathbb{R}$. Thus $\mathbb{R}^n \setminus f(\bar{B})$ is connected. Furthermore, $\mathbb{R}^n \setminus f(\bar{B})$ is unboudned.

Since f is injective, $f(S) \cap f(B) = \emptyset$. Thus, by connectedness, $\mathbb{R}^n \setminus f(\bar{B}) \subset U_2$. Thus $f(S) \cup U_1 \subset f(\bar{B})$. Thus $U_1 \subset f(B)$.

We finish by showing that $U_1 = f(B)$. Since B is connected, so is $f(B)$. Since $f(B) \subset U_1 \cup U_2$ by injectivity of f, we have $f(B) \subset U_1$. Thus $U_1 = f(B)$. Thus $f(B)$ is open, which completes the proof. $\square$

Chapter 5

Manifolds and bundles

5.1 Topological manifolds and bundles

Definition 5.1.1. A topological space M is an n-manifold, for $n \geq 0$, if

(M1) M is Hausdorff,

(M2) M has countable basis,

(M3) every point in M has a (open) neighborhood which is homeomorphic to an open set in $\mathbb{R}^n$.

Definition 5.1.2. Let M be an n-manifold. A pair (U, φ), where $U \subset M$ is an open subset and $\varphi\colon U \to V$ is a homeomorphism to an open subset of $\mathbb{R}^n$, is a *chart of M*.

Since $\varphi\colon U \to V$ carries all necessary information, also φ is called sometimes a chart. For $x \in U$, we also sometimes say that (U, φ) is *chart of M at x*.

Remark 5.1.3. *Condition (M2) is some times replaced with another condition (like paracompactness) or left out. Recall that $\mathcal{B}$ is a* basis of topology *$\mathcal{T}$ if each open set (i.e. element of $\mathcal{T}$) is a union of elements of $\mathcal{B}$ (or empty). Recall also that a topological space is* paracompact *if every open cover has a locally finite refinement.*

Remark 5.1.4. *We identify $\mathbb{R}^0 = \{0\}$. Thus components of 0-manifolds are points.*

By de Rham theory, we may define the dimension $\dim M$ of M uniquely using condition (M3). Indeed, by the invariance of dimension, we have the following corollary which we record as a lemma.

Lemma 5.1.5. *Let M be a topological space which is an n-manifold and an m-manifold. Then $n = m$.*

Lemma 5.1.6. *Let M be an n-manifold. Then every non-empty open subset of M is an n-manifold.*

Proof. Let U be an open subset of M. Then, in the relative topology, U is Hausdorff and has countable basis. To verify (M3), let $x \in U$. Since M is an n-manifold x has a neighborhood $W \subset M$ which is homeomorphic to an open set in M. Thus $U \cap W$ is an open neighborhood of x which is homeomorphic to an open subset of $\mathbb{R}^n$. □

Definition 5.1.7. A topological space M satisfying (M1) and (M2) is an *n-manifold with boundary* if

(M3') every point in M has a neighborhood homeomorphic to an open set in $\mathbb{R}^{n-1} \times [0, \infty) \subset \mathbb{R}^n$ (in the relative topology).

Let M be an n-manifold with boundary. We say that $x \in M$ is a *manifold point* if it has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$. We call

$$\mathrm{int} M = \{x \in M : x \text{ is a manifold point}\}$$

the *manifold interior of M*. The set $\partial M = M \setminus \mathrm{int} M$ we call *the manifold boundary of M*.

By (non-)contractibility results from de Rham theory, we have the following characterization; here and in what follows $B^n = B^n(0,1) = \{x \in \mathbb{R}^n : |x| < 1\}$.

Lemma 5.1.8. *Let M be an n-manifold with boundary. Then $x \in \partial M$ if and only if x has a neighborhood U so that all neigborhoods of x contained in U are homeomorphic to relatively open sets in $\mathbb{R}^{n-1} \times [0, \infty)$ but not to open sets in $\mathbb{R}^n$.*

Proof. Exercise. □

The interior $\mathrm{int} M$ of M is a non-empty n-manifold and ∂M is an $(n-1)$-manifold. In fact both are submanifolds.

Definition 5.1.9. Let M be an n-manifold with boundary. A subset $N \subset M$ is a k-submanifold if for every $x \in N$ there exists a neighborhood U of x and an embedding $\varphi \colon U \to X$, where X is $\mathbb{R}^n$ if $x \in \mathrm{int} M$ and $\mathbb{R}^{n-1} \times [0, \infty)$ otherwise, satisfying $\varphi(U \cap N) = \varphi(U) \cap \mathbb{R}^k \subset \mathbb{R}^k \times \{0\}$.

Lemma 5.1.10. *A k-submanifold of an n-manifold with boundary is a k-manifold. In addition, $k \leq n$.*

Proof. Let $N \subset M$ be a k-submanifold of an n-manifold with boundary M. Then N is Hausdorff and has countable basis in the relative topology. Thus it suffices to verify (M3). Let $x \in N$.

Suppose first that $x \in \mathrm{int}M$. We fix a neighborhood U and an embedding $\varphi\colon U \to \mathbb{R}^n$ for which $\varphi(U \cap N) \subset \mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$. Then $\varphi(U \cap N)$ is open in the relative topology of $\mathbb{R}^k \times \{0\}$. Thus $U \cap N$ is a neighborhood of x in N which is homeomorphic to an open set in $\mathbb{R}^k$. The case $x \in \partial M$ is almost verbatim.

The last claim, $k \le n$, follows from the invariance of domain. □

Lemma 5.1.11. *Let M be an n-manifold with boundary. Then* $\mathrm{int}M$ *is an n-submanifold and ∂M is an $(n-1)$-submanifold.*

Proof. We readily observe that $\mathrm{int}M$, in the relative topology, is Hausdorff and has countable basis. Since every point in ∂M has a neighborhood homeomorphic to an open set in $\mathbb{R}^n$ by definition, it is an n-manifold.

To show that ∂M is an $(n-1)$-manifold it suffices to verify (M3). Let $x \in \partial M$ and U a neighborhood of x in M. Then, by Lemma 5.1.8, there exists a homeomorphism $\varphi\colon U \to V$, where V is an relatively open set in $\mathbb{R}^{n-1} \times [0,\infty)$ which is not open in $\mathbb{R}^n$. Let $W = U \cap \partial M$. Since W is an open neighborhood of x in the relative topology of ∂M, it suffices to show that $\varphi(W)$ is an open subset of $\mathbb{R}^{n-1} \times \{0\}$.

By Lemma 5.1.8, $\varphi(W) \subset \mathbb{R}^{n-1} \times \{0\}$. Indeed, otherwise, a point in W would have a neighborhood homeomorphic to an open set in $\mathbb{R}^n$. On the other hand, none of the points in $\varphi^{-1}(U \cap \mathbb{R}^{n-1} \times \{0\})$ is in $\mathrm{int}M$. Thus $\varphi(W) = \varphi(U) \cap \mathbb{R}^{n-1} \times \{0\}$. Since $\varphi(U)$ is open in $\mathbb{R}^{n-1} \times [0,\infty)$, $\varphi(U) \cap \mathbb{R}^{n-1} \times \{0\}$ is open in the relative topology of $\mathbb{R}^{n-1} \times \{0\}$. Thus ∂U is an $(n-1)$-submanifold. □

5.1.1 Examples

First examples

We begin with concrete elementary examples. The most commonly used concrete example is the n-sphere $\mathbb{S}^n$.

Example 5.1.12. *The n-sphere $\mathbb{S}^n$ is the subset*

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \colon |x| = 1\}.$$

The n-sphere is a submanifold of $\mathbb{R}^{n+1}$, but we treat it as an n-manifold. Since $\mathbb{S}^n \subset \mathbb{R}^n$, it is Hausdorff and has countable basis in the relative topology. To verify (M3), consider projections $\pi_i\colon \mathbb{S}^n \to \mathbb{R}^n$ defined by $(x_1,\ldots,x_{n+1}) \mapsto (x_1,\ldots,\widehat{x_i},\ldots,_{n+1})$ and subsets

$$S_{i,\pm} = \{(x_1,\ldots,x_{n+1}) \in \mathbb{S}^n \colon \pm x_i > 0\}.$$

Then $\pi_i|S_{i,+}\colon S_{i,+} \to B^n$ and $\pi_i|S_{i,-}\colon S_{i,-} \to B^n$ are homeomorphisms for each i. Furthermore, $\mathbb{S}^n = \bigcup_i S_{i,+} \cup S_{i,-}$. Thus $\mathbb{S}^n$ is an n-manifold.

The second is perhaps the 2-torus.

Example 5.1.13. *Let $Q = [0,1]^2$ and let $\sim$ be the minimal equivalence relation Q which satisfies $(0,t) \sim (1,t)$ and $(t,0) \sim (t,1)$ for each $t \in [0,1]$. Then the quotient space $Q/\sim$ (with quotient topology) is the called the* 2-torus.

The 2*-torus has two (nice) geometric realizations. First is $\mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^4$ and the other*

$$T = \{f(\theta) + \frac{1}{4}(\cos\phi f(\theta) + \sin\phi(0,0,1)) : \theta, \phi \in [0,2\pi]\} \subset \mathbb{R}^3,$$

where $f\colon [0,2\pi] \to \mathbb{R}^3$, $f(\theta) = (\cos\theta, \sin\theta, 0)$.

Perhaps not the second, but third, most common example is the Möbius strip.

Example 5.1.14. *Let $Q = [-1,1]^2$ and $\sim$ the minimal equivalence relation in Q satisfying $(t,0) \sim (1-t,1)$ for $t \in [0,1]$. Then $Q/\sim$ is a* 2*-manifold with boundary. It has a nice realization in $\mathbb{R}^3$ as*

$$\Sigma = \{f(\theta) + \frac{s}{4}(\cos(\theta/2)f(\theta) + \sin(\theta/2)(0,0,1)) : \theta \in [0,2\pi], s \in [-1,1]\}.$$

Products

Almost trivial, but essential, observation is that product of manifolds is a manifold.

Lemma 5.1.15. *Let M and N be m- and n-manifolds, respectively. Then $M \times N$ is an $(m+n)$-manifold. Furthermore, if M and N are manifolds with boundary, then $M \times N$ is a manifold with boundary.*

Proof. Exercise. □

Example 5.1.16. *Since $\mathbb{S}^1$ is a manifold, so is the* 2*-torus $\mathbb{S}^1 \times \mathbb{S}^1$.*

Covering maps

Definition 5.1.17. A continuous mapping $f\colon X \to Y$ between topological spaces is a *covering map* if f is surjective and for each $x \in X$ there exists a neighborhood V of $f(x)$ so that components of $f^{-1}(V)$ are homeomorphic to V.

Since the basis of the product topology is given by products of open sets in the factors, we readily obtain the following result which we record as a lemma.

Lemma 5.1.18. *Let $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$ be covering maps. Then $f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2$ is a covering map.*

Example 5.1.19. *Let $\phi\colon \mathbb{R} \to \mathbb{S}^1$ be the map $t \mapsto (\cos 2\pi t, \sin 2\pi t)$, i.e. $t \mapsto e^{i2\pi t}$. Let $t_0 \in \mathbb{R}$ and set $Q_0 = (t_0 - 1/2, t_0 + 1/2)$. Then $\phi^{-1}(\phi(Q_0)) = \bigcup_{z\in\mathbb{Z}}(z + Q_0)$, where $z + Q_0 = \{z + w \in \mathbb{R}\colon w \in Q_0\}$. Furthermore, $\varphi|(z + Q_0)\colon z + Q_0 \to \phi(Q_0)$ is a homeomorphism for each z. Thus ϕ is a covering map.*

Lemma 5.1.20. *Let M be an manifold and let $\phi\colon X \to M$ and $\psi\colon M \to Y$ be covering maps. Then Y is an n-manifold. Moreover, X is an n-manifold if it has a countable basis.*

Proof. Exercise. □

Example 5.1.21. *The two torus $\mathbb{S}^1 \times \mathbb{S}^1$ is a 2-manifold, since $\phi \times \phi\colon \mathbb{R}^2 \to \mathbb{S}^1 \times \mathbb{S}^1$ is a covering map.*

Quotients and orbit maps

Suppose $\phi\colon M \to N$ is a covering map between two n-manifolds. Then N is homeomorphic to $M/\sim$ where $\sim$ is the equivalence relation on M given by $x \sim y$ if and only if $\phi(x) = \phi(y)$. Thus N is a *quotient manifold of M*.

A particular class of quotient manifolds is obtained by group actions.

Let M be an n-manifold and $\mathrm{Homeo}(M)$ the set of all homeomorphisms of M. Then $\mathrm{Homeo}(M)$ is a groups where the group law is given by composition.

Let $\Gamma \subset \mathrm{Homeo}(M)$ be a subgroup. Given $x \in M$, we denote by $\Gamma x = \{\gamma(x) \in M\colon \gamma \in \Gamma\}$ the *orbit of x with respect to* Γ. Then $M/\Gamma = \{\Gamma x\colon x \in M\}$ is a partition of M. Let $\pi_\Gamma\colon M \to M/\Gamma$ be the canonical map $x \mapsto \Gamma x$, called *orbit map*.

Theorem 5.1.22. *Let M be an n-manifold and let $\Gamma \subset \mathrm{Homeo}(M)$ be a subgroup. Suppose Γ satisfies the conditions*

(Γ *1)* *$\gamma(x) \neq x$ for each $x \in M$ and $\gamma \neq \mathrm{id} \in \Gamma$ and*

(Γ *2)* *every point $x \in M$ has a neighborhood U so that*

$$\#\{\gamma \in \Gamma\colon U \cap \gamma(U) \neq \emptyset\} < \infty.$$

Then M/Γ is an n-manifold and the orbit map $\pi_\Gamma\colon M \to M/\Gamma$ a covering map.

Proof. Exercise. □

Remark 5.1.23. *The subgroup $\Gamma \subset \mathrm{Homeo}(M)$ is said to* act on M freely *if it satisfies condition (Γ 1). Similarly, Γ is said to act* discontinously *if it satisfies (Γ 2). Note that this terminology sligly varies in the literature depending on the underlying setting. We work on manifold which are locally compact (σ-compact) spaces.*

Remark 5.1.24. *The notion of 'action' comes from a more general framework, where* Γ *is an abstract group and there exists a homomorphism* $A\colon \Gamma \to \mathrm{Homeo}(M)$*. Then* Γ *is said to* act on M (via A).

Example 5.1.25. *The 2-torus* $\mathbb{S}^1 \times \mathbb{S}^1$ *is homeomorphic to* $\mathbb{R}^2/\mathbb{Z}^2$*, where* $\mathbb{Z}^2$ *is identified with the group of translations* $x \mapsto x + z$, $z \in \mathbb{Z}^2$, *in* $\mathbb{R}^2$.

Example 5.1.26. *Let* $M = \mathbb{S}^n$ *and* $\Gamma = \{\mathrm{id}, -\mathrm{id}\}$*. Then* M/Γ *is a* real projective space $\mathbb{R}P^n$ *which is an* n*-manifold.*

Amalgamations

Let $A \subset X$ and $B \subset Y$ be homeomorphic subsets and $\phi\colon A \to B$ a homeomorphism. Let $\sim$ be the minimal equivalence relation so that $\phi(x) \sim x$ for every $x \in A$. We denote by $X \coprod_\phi Y$ the quotient space $(X \coprod Y)/\sim$ and call it *amalgamation of* X *and* Y *along* ϕ.

Lemma 5.1.27. *Let* M *be an* n*-manifold with boundary and let* $\psi\colon \bar{B}^{n-1}(0,2) \times \{0,1\} \to \partial M$ *be an embedding. Let* $C = \bar{B}^{n-1} \times [0,1]$*, and* $\phi = \psi|\bar{B}^{n-1} \times [0,1]$*. Then* $M \coprod_\phi C$ *is an* n*-manifold with boundary.*

Proof. Exercise. □

The operation in Lemma 5.1.27 is called an *attachment of a (1-)handle.*

Example 5.1.28. *Consider a closed 3-ball* $\bar{B}^3 = \{x \in \mathbb{R}^3 \colon |x| \le 1\}$ *and attach a handle to it, for example, as follows. Let* D_0 *and* D_1 *be closed disks* $\partial B^3 = \mathbb{S}^2$*, e.g.* $D_0 = \{x \in \mathbb{S}^2 \colon x_3 \le -1/2\}$ *and* $D_1 = \{x \in \mathbb{S}^2 \colon x_3 \ge 1/2\}$*. Let* $\phi\colon D_0 \cup D_1 \to \bar{B}^2 \times \{0,1\}$ *be the homeomorphism*

$$(x_1, x_2, x_3) \mapsto \begin{cases} 2(x_1, x_2, 0), & x_3 < 0 \\ 2(x_1, x_2, 1), & x_3 > 0 \end{cases}$$

Then $\bar{B}^3 \coprod_\phi C$ *is homeomorphic to* $\bar{B}^2 \times \mathbb{S}^1$*, i.e. it is a* solid 3-torus. *In particularly, it is a* 3*-manifold with boundary homeomorphic to* $\partial \bar{B}^2 \times \mathbb{S}^1 = \mathbb{S}^1 \times \mathbb{S}^1$.

Lemma 5.1.29. *Let* M *and* N *be* n*-manifolds with boundary, and suppose* Σ *and* $\Sigma' N$ *are homeomorphic open and closed sets in* ∂M *and* ∂N*, respectively, and* $\phi\colon \Sigma \to \Sigma'$ *a homeomorphism. Then* $M \coprod_\phi N$ *is an* n*-manifold with boundary. Moreover, if* $\Sigma = \partial M$ *and* $\Sigma' = \partial N$*, then* $M \coprod_\phi N$ *is an* n*-manifold.*

Proof. Exercise. □

Example 5.1.30. *Let* $\phi\colon \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$, $(z,w) \mapsto (w,z)$*. Then* ϕ *is a self homeomorphism on the boundary of a solid* 3*-torus* $\bar{B}^2 \times \mathbb{S}^1$*. By Lemma 5.1.29,* $(\bar{B}^2 \times \mathbb{S}^1) \coprod_\phi (\bar{B}^2 \times \mathbb{S}^1)$ *is a* 3*-manifold. Infact, it is homeomorphic to* $\mathbb{S}^3$ *(!).*

Example 5.1.31. *Let M be an n-manifold with boundary. Then $M \coprod_{\mathrm{id}} M$, where* $\mathrm{id}\colon \partial M \to \partial M$*, is called the* double of M*. By Lemma 5.1.29, it is an n-manifold.*

As a final example of this type we introduce connected sums.

Theorem 5.1.32. *Let M_1 and M_2 be n-manifolds and $\psi_i\colon B^n(0,2) \to M_i$ be embeddings. Let $\Sigma_i = \psi_i(\mathbb{S}^{n-1})$, $B_i = \psi_i(B^n)$, and $\phi\colon \Sigma_1 \to \Sigma_2$ the homeomorphism $\phi = \psi_2 \circ \psi_1^{-1}|\Sigma_1$. Then $(M_1 \setminus B_1) \coprod_\phi (M_2 \setminus B_2)$ is an n-manifold.*

Proof. Exercise. □

The n-manifold $(M_1 \setminus B_1) \coprod_\phi (M_2 \setminus B_2)$ is called a *connected sum of M_1 and M_2* and usually denoted simply as $M_1 \# M_2$, with data ψ_1 and ψ_2 being understood from the context.

5.1.2 Bundles

Definition 5.1.33. Let F be a topological space. A *fiber bundle with a fiber F over manifold B* is a triple $\xi = (E, B, \pi)$, where E and B are manifolds and $\pi\colon E \to B$ is a continuous map, so that

(B1) $\pi^{-1}(x) \approx F$ is homeomorphic with F for every $x \in B$ and,

(B2) for every $x \in B$ there exists a neighborhood U of x and a homomeomorphism $\psi\colon \pi^{-1}U \to U \times F$ satisfying

$$\begin{tikzcd} \pi^{-1}U \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

where $\mathrm{pr}_1\colon U \to F \to U$ is the projection $(y, v) \mapsto y$.

The manifolds E and B are called a *total and bases spaces of the bundle* ξ. The preimages $\pi^{-1}(x)$ is called *fibers*. Since the map $\pi\colon E \to B$ contains all the essential information, it is also typical to say for short that $\pi\colon E \to B$ is a bundle. When the map π is understood from the context, it is typical to refer to E as a bundle over the base B.

Example 5.1.34. *Both $\mathbb{S}^1 \times (-1, 1)$ and the Möbius strip are fiber bundles with fiber $\mathbb{R}$ over $\mathbb{S}^1$. In fact, all products $M \times N$ are fiber bundles over M (with fiber N) and over N (with fiber M).*

Definition 5.1.35. Let (E, M, π) be a fiber bundle. A map $s\colon M \to E$ is a *section* if $\pi \circ s = \mathrm{id}_M$;

$$\begin{tikzcd} E \arrow[d, "\pi"] \\ M \arrow[u, bend left, "s"] \end{tikzcd}$$

Definition 5.1.36. Let (E_1, M_1, π_1) and (E_2, M_2, π_2) be bundles. A map $F\colon E_1 \to E_2$ is a *bundle map* if there exists a map $f\colon M_1 \to M_2$ so that

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "f"] & M_2 \end{tikzcd}$$

(i.e. F maps fibers into fibers).

For the definition of a vector bundle, we make a simple observation. Let V be a vector space and U an open set on a manifold. Then $\mathrm{pr}_1\colon U \times V \to U$ is a product bundle and the projection $\mathrm{pr}_2\colon U \times V \to V$, $(x, v) \mapsto v$, restricts to a homeomorphism $\{x\} \times V \to V$ for each fiber $\mathrm{pr}_1^{-1}(x) = \{x\} \times V$. Thus $\{x\} \times V$ has a strucuture of a vector space with

$$(x, v) + a(x, w) = (x, v + aw)$$

for $v, w \in V$, $a \in \mathbb{R}$, and $x \in U$.

Definition 5.1.37. Let V be a vector space. A fiber bundle $\xi = (E, B, \pi)$ with fiber V is a *vector bundle* if

(VB1) each $\pi^{-1}(x)$ is a vector space isomophic to V,

(VB2) each $x \in B$ has a neighborhood U and a homeomorphism $\psi\colon \pi^{-1}(U) \to U \times V$ as in (B2) so that $\psi|\pi^{-1}(y)\colon \pi^{-1}(x) \to \{y\} \times V$ is a linear isomorphism for each $y \in U$.

Definition 5.1.38. Sections of a vector bundle are called *vector fields*.

Interesting examples of vector bundles are tangent and exterior bundles. For these, we need the notion of smoothness on a manifold.

5.2 Smooth manifolds

Topological manifolds have topology, and hence the continuity of maps between manifolds (and other topological spaces) is understood. We introduce now C^∞-smooth manifolds.

Definition 5.2.1. Let M be an n-manifold and suppose (U,ψ) and (V,ϕ) are charts on M so that $U\cap V\neq\emptyset$. Homeomorphism

$$\begin{tikzcd} & U\cap V \arrow[dl, "\psi|"'] \arrow[dr, "\phi|"] & \\ \psi(U\cap V) \arrow[rr, "\phi\circ\psi^{-1}|"] & & \phi(U\cap V) \end{tikzcd}$$

is called a *transition map from* (U,ψ) *to* (V,ϕ).

Definition 5.2.2. Let M be an n-manifold. A collection $\mathcal{A}=\{(U_\alpha,\psi_\alpha)\}_{\alpha\in I}$ of charts of M is an *atlas* if $\bigcup_\alpha U_\alpha = M$.

Definition 5.2.3. An atlas $\mathcal{A}$ of an n-manifold M is a *smooth atlas* if all transition maps between charts in $\mathcal{A}$ are C^∞-smooth. A chart (U,ψ) on M is *compatible with a smooth atlas* $\mathcal{A}$ if all transition maps between (U,ψ) and charts in $\mathcal{A}$ are C^∞-smooth.

Example 5.2.4 (Canonical first example)**.** *Let $(S_{i,\pm},\pi_i|S_{i,\pm})$ be as in Example 5.1.12. Then $\{(S_{i,\pm},\pi_i|S_{i,\pm})\colon i=1,\ldots,n+1\}$ is a smooth atlas on $\mathbb{S}^n$. (Exercise to check the smoothness.)*

Example 5.2.5 (Cautionary second example)**.** *Let $X=\partial[-1,1]^2\subset\mathbb{R}^2$ (i.e. the topological boundary of the square $[-1,1]^2$) and let $h\colon\mathbb{S}^1\to X$ be the homeomorphism $x\mapsto x/|x|_\infty$. Clearly, X is a 1-manifold. Moreover, $\mathcal{A}=\{(hS_{i,\pm},\pi_i|S_{i,\pm}\circ h^{-1})\colon i=1,\ldots,n+1\}$ is an atlas on X. It is, in fact, a smooth atlas.*

Definition 5.2.6. A smooth atlas $\mathcal{A}$ on an n-manifold is a *smooth structure* if it contains all compatible smooth charts.

Since all charts on a fixed manifold form a set, we have the following existence result.

Lemma 5.2.7. *If an n-manifold M admits a smooth atlas, it admits a smooth structure.*

Definition 5.2.8. A smooth manifold is a pair $(M,\mathcal{A})$, where $\mathcal{A}$ is a smooth structure on manifold M.

Definition 5.2.9. A map $f\colon M\to N$ between smooth manifolds $(M,\mathcal{A}_M)$ and $(N,\mathcal{A}_N)$ is *smooth* if for every $x\in M$ there exists charts $(U,\psi)\in\mathcal{A}_M$ and $(V,\phi)\in\mathcal{A}_N$ so that $fU\subset V$ and $\phi\circ f\circ\psi^{-1}\colon\psi(U)\to\phi(V)$ is C^∞-smooth map;

$$\begin{tikzcd} U \arrow[r, "f|U"] \arrow[d, "\psi"] & V \arrow[d, "\phi"] \\ \phi(U) \arrow[r, "\phi\circ f\circ\psi^{-1}"] & \psi(V) \end{tikzcd}$$

A simple application of the chain rule shows that the smooth does not depend on a chosen charts.

Lemma 5.2.10. *Let $f\colon M \to N$ be a smooth map between smooth manifolds. Suppose (U,ψ) and (V,ϕ) are smooth charts on M and on N, respectively so that $fU \cap V \neq \emptyset$. Then $\phi \circ f \circ \psi^{-1}\colon \psi(U \cap f^{-1}(V)) \to \phi V$ is C^∞-smooth map.*

Proof. Let $x \in U \cap f^{-1}(V)$. We need to show that $\psi(x)$ has a neighborhood where $\phi \circ f \circ \phi^{-1}$ is C^∞-smooth. Since f is smooth, there exists charts (U_0,ψ_0) and (V_0,ϕ_0) so that $x \in U_0$, $fU_0 \subset V_0$, and $\phi_0 \circ f \circ \psi_0^{-1}$ is C^∞-smooth. Let $W_0 = U_0 \cap U$. By compatibility of the charts, $\psi_0 \circ \psi^{-1}|\psi(W_0)$ is C^∞-smooth. Thus

$$\phi_0 \circ f \circ \psi^{-1}|\psi(W_0) = (\phi_0 \circ f \circ \psi_0^{-1}) \circ (\psi_0 \circ \psi^{-1}|\psi(W_0))$$

is C^∞-smooth. Since $fW_0 \subset V_0 \cap V$, we observe that a similar argument for $\phi \circ \phi_0|\phi_0(fW_0)$ concludes now the proof. □

Definition 5.2.11. A homeomorphism $f\colon M \to N$ between smooth manifolds $(M,\mathcal{A}_M)$ and $(N,\mathcal{A}_N)$ is a *diffeomorphism* if f and f^{-1} are smooth maps.

Convention 5.2.12. *It is typical, at this point, to stop paying attention to the chosen smooth structure. From now on we will assume that a smooth manifold has a fixed smooth structure (which we won't give a name) and that charts have C^∞-smooth transition mappings. These charts are called* smooth charts *from now on.*

Remark 5.2.13 (Warning!)**.** *Dispite our relaxed attitude towards smooth structures, it is a surprising (and extremely difficult!) result of Donaldson that there are manifolds which do not admit smooth atlases. There is no uniqueness either, even $\mathbb{R}^4$ has infinitely many different (non-diffeomorphic) smooth atlases.*

5.3 Tangent bundle

5.3.1 Tangent space

Let M be a smooth manifold and $x \in M$. Let $P(x)$ be the set of all smooth paths $\gamma\colon (-h,h) \to M$ satisfying $\gamma(0) = p_0$, where $h > 0$; note that $(-h,h)$ is a smooth 1-manifold and thus γ is smooth if and only if for each $t \in (-h,h)$ there exist a smooth chart (U,ϕ) for whic $\phi \circ \gamma|(t-\delta,t+\delta)$ is C^∞-smooth, where $\delta > 0$ satisfies $\gamma(t-\delta,t+\delta) \subset U$.

Let $\sim_x$ be the equivalence relation on $P(x)$ so that $\gamma \sim_x \sigma$ if and only if $(\phi \circ \gamma)'(0) = (\phi \circ \sigma)'(0)$ for all chart (U,ϕ) of M at x; note that it suffices to check with one chart.

Definition 5.3.1. Let M be a smooth manifold and $x \in M$. The tangent space of M at x is the space $T_xM = P(x)/\sim_x$.

The tangent space has a natural linear structure.

Lemma 5.3.2. *Let M be a smooth manifold and $x \in M$. The operators $+\colon T_xM \times T_xM \to T_xM$ and $\cdot\colon \mathbb{R} \times T_xM \to T_xM$ given by formulas*

$$[\gamma_1] + [\gamma_2] = [\phi^{-1} \circ (\phi \circ \gamma_1 + \phi \circ \gamma_2)]$$

and

$$a[\gamma] = [\phi^{-1} \circ (a(\phi \circ \gamma))],$$

where $a \in \mathbb{R}$, $\gamma, \gamma_1, \gamma_2 \in P(x)$, and (U, ϕ) is a chart at x satisfying $\phi(x) = 0$, are well-defined.

Proof. We show first that the right hand sides does not depend on the choice of the chart (U, ϕ) at x.

Let $\gamma_1, \gamma_2 \in P(x)$ and let (U, ϕ) and (V, ψ) be charts of M at $x \in M$. By restricting the domain intervals of γ_1 and γ_2 if necessary, we may assume that the images of γ_1 and γ_2 are contained in $U \cap V$. By restricting further, we may assume that $\phi\circ\gamma_1 + \phi\circ\gamma_2 \subset \phi(U \cap V)$ and $\psi\circ\gamma_1 + \psi\circ\gamma_2 \subset \psi(U \cap V)$.

Since $\psi(x) = \phi(x) = 0$, we have, by the chain rule,

$$\begin{aligned} \left(\phi \circ \left(\psi^{-1} \circ (\psi \circ \gamma_1 + \psi \circ \gamma_2)\right)\right)'(0) &= \left((\phi \circ \psi^{-1}) \circ (\psi \circ \gamma_1 + \psi \circ \gamma_2)\right)'(0) \\ &= D(\phi \circ \psi^{-1})_0 (\psi \circ \gamma_1 + \psi \circ \gamma_2)'(0) \end{aligned}$$

On the other hand,

$$\begin{aligned} (\psi \circ \gamma_i)'(0) &= ((\psi \circ \phi^{-1}) \circ (\phi \circ \gamma_i))'(0) \\ &= D(\psi \circ \phi^{-1})_0 (\phi \circ \gamma_i)'(0). \end{aligned}$$

and

$$D(\phi \circ \psi^{-1})_0 D(\psi \circ \phi^{-1})_0 = D(\phi \circ \psi^{-1} \circ \psi \circ \phi^{-1})_0 = \mathrm{id}.$$

Thus, by linearity,

$$\left(\phi \circ \left(\psi^{-1} \circ (\psi \circ \gamma_1 + \psi \circ \gamma_2)\right)\right)'(0) = (\phi \circ \gamma_1 + \phi \circ \gamma_2)'(0)$$

Thus

$$[\psi^{-1} \circ (\psi \circ \gamma_1 + \psi \circ \gamma_2)] = [\phi^{-1} \circ (\phi \circ \gamma_1 + \phi \circ \gamma_2)].$$

and the addition is well-defined. It is similar to show that the scalar multiplication is well-defined.

To show that the right hand sides do not depend on the choice of representatives, let $\tilde{\gamma}_1 \sim \gamma_1$ and $\tilde{\gamma}_2 \sim \gamma_2$ and let (U, ϕ) be a chart. Since $(\phi \circ \tilde{\gamma}_i)'(0) = (\phi \circ \gamma_i)'(0)$ by definition, the claim clearly follows. $\square$

Corollary 5.3.3. *Let M be a smooth n-manifold, $x \in M$, and (U, ϕ) a chart of M at x. Then $\tilde{D}_x\phi \colon T_xM \to \mathbb{R}^n$, $[\gamma] \mapsto (\phi \circ \gamma)'(0)$ is a linear isomorphism. In particular, $\dim T_xM = n$.*

Proof. Since $\tilde{D}_x\phi$ is linear, it suffices to show that it has an inverse. Define $\Phi \colon \mathbb{R}^n \to T_xM$ by $v \mapsto [\phi^{-1} \circ \alpha_v]$, where $\alpha_v \colon t \mapsto \phi(x) + tv$. We observe first that

$$\tilde{D}_x\phi \circ \Phi(v) = D_x\phi[\phi^{-1} \circ \alpha_v] = \alpha_v'(0) = v$$

for all $v \in \mathbb{R}^n$.

Let $\gamma \in P(x)$ and set $v = (\phi \circ \gamma)'(0)$. Then

$$\Phi \circ \tilde{D}_x\phi[\gamma] = \Phi((\phi \circ \gamma)'(0)) = \Phi(v) = [\phi^{-1} \circ \alpha_v] = [\gamma].$$

Thus Φ is an inverse of $D_x\phi$. □

Remark 5.3.4. *Note that there is no "canonical" identification of T_xM with $\mathbb{R}^n$ but there are isomorphisms $\tilde{D}_x\phi$ induced by charts $\phi \colon U \to \phi U$. We stress that the linear structure, however, is independent on the choice of the chart.*

Remark 5.3.5. *There are several different ways to construct the tangent space of a smooth manifold. This is just one of them.*

Definition 5.3.6. Let M and N be smooth manifold and $f \colon M \to N$ a smooth map. Given $x \in M$, we denote by $D_xf \colon T_xM \to T_{f(x)}N$ the map $[\gamma] \to [f \circ \gamma]$. We call D_xf the *tangent map (or differential) of f at x.*

Remark 5.3.7. *Dispite Remark 5.3.4, we may canonically identify $T_x\mathbb{R}^n$ with $\mathbb{R}^n$ by $\tilde{D}_x\mathrm{id} \colon T_x\mathbb{R}^n \to \mathbb{R}^n$, $[\gamma] \mapsto (\mathrm{id} \circ \gamma)'(0)$, where $\mathrm{id} \colon \mathbb{R}^n \to \mathbb{R}^n$.*

Remark 5.3.7 yields the following observation.

Remark 5.3.8. *(a) Let (U, ϕ) be a chart on a smooth manifold. Then $\tilde{D}_{\phi(x)}\mathrm{id} \circ D_x\phi = \tilde{D}_x\phi$ for $x \in U$.*

(b) Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets and $f \colon U \to V$ a map. If we consider f as a smooth maps between smooth manifolds, the map $(\tilde{D}_{f(x)}\mathrm{id}) \circ D_xf \circ (\tilde{D}_x\mathrm{id})^{-1} \colon \mathbb{R}^n \to \mathbb{R}^m$ is the (usual) differential of f given by partial derivatives of coordinate functions.

Convention 5.3.9. *Given a map $f \colon M \to \mathbb{R}^m$, where M is a smooth n-manifold, we denote $D_xf \colon T_xM \to \mathbb{R}^m$, where we have identified $T_x\mathbb{R}^m$ with $\mathbb{R}^m$ as in Remark 5.3.7. Note that, in particular, given a chart (U, ϕ) on a smooth n-manifold M, mappings $D_x\phi$ and $\tilde{D}_x\phi$ agree.*

5.3.2 Tangent bundle

Let M be a smooth n-manifold. In this section, we show that the set

$$TM = \bigcup_{x\in M} T_xM$$

has natural structure as a smooth manifold and that (TM, M, π_M), where $\pi_M\colon TM \to M$, $[\gamma] \mapsto \gamma(0)$, is a vector bundle.

Given a chart (U, ϕ) of M, let

$$TU = \pi_M^{-1}(U) = \bigcup_{x\in U} T_xM.$$

Corollary 5.3.3 readily yields the following observation.

Lemma 5.3.10. *Let (U, ϕ) be chart on a smooth n-manifold M. Then the map $\tilde{D}\phi\colon TU \to \phi(U) \times \mathbb{R}^n$ defined by*

$$[\gamma] \mapsto (\phi(\gamma(0)), D_{\gamma(0)}\phi[\gamma])$$

is a bijection;

$$\begin{tikzcd} TU \arrow[rr, "\tilde{D}\phi"] \arrow[dr, "\phi\circ\pi_M|U"'] & & \phi(U)\times\mathbb{R}^n \arrow[dl, "\mathrm{pr}_1"] \\ & \phi(U) & \end{tikzcd}$$

Furthermore, if smooth charts (U, ϕ) and (V, ψ) intersect, the map

$$\tilde{D}\phi \circ (\tilde{D}\psi)^{-1}\colon \tilde{D}\psi(TU \cap TV) \to \tilde{D}\phi(TU \cap TV)$$

is diffeomorpism.

Proof. Corollary 5.3.3 readily yields the first claim. For the second claim, we make first the following observation. Let $(x, v) \in (U \cap V) \times \mathbb{R}^n$ and α_v be the path $t \mapsto \psi(x) + tv$. Then

$$D_x\phi \circ (D_x\psi)^{-1}(v) = D_x\phi[\psi^{-1} \circ \alpha_v] = (\phi \circ \psi^{-1} \circ \alpha_v)'(0) = D_{\psi(x)}(\phi \circ \psi^{-1})v,$$

where the last step is the chain rule in the Euclidean space. Thus

$$\tilde{D}\phi \circ (\tilde{D}\psi)^{-1}(x, v) = \big(\phi \circ \psi^{-1}(x), D(\phi \circ \psi^{-1})_x v\big)$$

for all $(x, v) \in \psi(U \cap V) \times \mathbb{R}^n$. Since $\phi \circ \psi^{-1}|\psi(U \cap V)$ is C^∞-smooth, $D\phi \circ (D\psi)^{-1}$ is C^∞-smooth. $\square$

Topology and smooth structure on TM

Recall that a topology of a space X is *induced* a map $f\colon X \to Y$ if open sets of X are of the form $f^{-1}V$, where V is an open set in Y. A topology of X is *co-induced* by the family $\{f_\alpha\colon Y_\alpha \to X\}_{\alpha\in I}$ of maps if a set $V \subset X$ is open only if $f_\alpha^{-1}(V)$ is open in Y_α for each $\alpha \in I$.

We give now TM a topology as follows. For every chart (U,ϕ), we consider TU as a topological space with the topology τ_ϕ induced by the map $\bar{\Phi}^\phi\colon TU \to U \times \mathbb{R}^n$. Using Lemma 5.3.10, it is now easy to see that all charts $\phi\colon U \to \phi(U)$, with the same domain U, induce the same topology τ_{TU} on TU.

We give TM the topology co-induced by all inclusion maps $TU \to TM$, where (U,ϕ) is a chart on M.

Remark 5.3.11. *A set $V \subset TM$ is open in TM if and only if $D\phi(V \cap TU)$ is open in $U \times \mathbb{R}^n$ for all smooth charts (U,ϕ).*

Lemma 5.3.12. *Let (U,ϕ) be a smooth chart. The relative topology of TU in TM is the topology τ_{TU}. In particular, $D\phi$ is a homeomorphism for every (U,ϕ) and TM is an $2n$-manifold.*

Proof. Let $W \subset TU$ be an open set in the relative topology of TU. Then, by definition of the relative topology, there exists an open set W' in TM so that $W' \cap TU = W$. Since W' is open in TM, $W' \cap TU$ is open in topology τ_{TU} by definition. Thus V is open in topology τ_{TU}. On the other hand, if $W \subset TU$ is open in topology τ_{TU}. Then, by Lemma 5.3.10, $W \cap TU'$ is open in TV for all smooth charts (V,ψ). Thus V is open in TM and we shown that the topologies coincide.

Since the relative topology of TU is τ_{TU}, which is induced by $\tilde{D}\phi$, we observe that $\tilde{D}\phi$ is a homeomorphism.

To show that TM is an $2n$-manifold, we note first that TM is clearly a Hausdorff space. To show that TM has a countable basis, we fix a countable collection $\{(U_\alpha,\phi_\alpha)\}_{\alpha\in I}$ of charts of M so that $\bigcup_\alpha U_\alpha = M$. Since each TU_α is an $2n$-manifold, it has a countable basis. Thus, by countability of the collection, TM has a countable basis. Finally, we observe that $\{TU_\alpha\}_\alpha$ is an open cover of TM. Since each TU_α is a $2n$-manifold, every point in TM has a neighborhood homeomorphic to an open subset of $\mathbb{R}^{2n}$. This concludes the proof. $\square$

Convention 5.3.13. *We identify $T\mathbb{R}^n$ with $\mathbb{R}^n \times \mathbb{R}^n$ via $\tilde{D}\mathrm{id}\colon TM \to \mathbb{R}^n \times \mathbb{R}^n$, where $\mathrm{id}\colon \mathbb{R}^n \to \mathbb{R}^n$.*

Lemma 5.3.10 shows also that TM has a natural smooth structure containing charts $(TU, \tilde{D}\phi)$. We formalize this immediate consequence as follows.

Lemma 5.3.14. *Let M be a smooth manifold and $\mathcal{A}$ a smooth atlas of M. Then TM has a smooth structure containing the smooth atlas $T\mathcal{A} = \{(TU, \tilde{D}\phi) \colon (U,\phi) \in \mathcal{A}\}$.* □

Definition 5.3.15. Let M be a smooth manifold. The triple (TM, M, π_M) is the *tangent bundle of M*.

This smooth structure is natural in the following sense.

Definition 5.3.16. Let $f \colon M \to N$ be a smooth map between smooth manifolds M and N. The map $Df \colon TM \to TN$, $[\gamma] \to [f \circ \gamma]$, is the *tangent map (or differential) of f*.

Lemma 5.3.17. *Let $f \colon M \to N$ be a smooth map between smooth manifolds M and N. Then $Df \colon TM \to TN$ is smooth bundle map, that is,*

$$\begin{tikzcd} TM \arrow[r, "Df"] \arrow[d, "\pi_M"'] & TN \arrow[d, "\pi_N"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Proof. Exercise. □

As a final remark, we note the following.

Remark 5.3.18. *Given a chart (U,ϕ) on M but considering U as an n-manifold, the inclusion $\iota \colon U \to M$ induces an embedding $D\iota \colon TU \to TM$ satisfying*

$$\begin{tikzcd} TU \arrow[r, "D\iota"] \arrow[d, "\pi_M|U"'] & TM \arrow[d, "\pi_M"] \\ U \arrow[r, "\iota"] & M \end{tikzcd}$$

Thus $D\iota$ gives a natural identification of T_xU and T_xM via $[\gamma] \mapsto [\iota \circ \gamma]$.

Vectorfields on TM

Let M be a smooth n-manifold and (U,x) be a chart on M; here $x = (x_1, \ldots, x_n) \colon U \to V$, where $V \subset \mathbb{R}^n$ is an open set. Note that TU is a smooth $2n$-manifold and $(TU, U, \pi_M|TU)$ is a vector bundle.

We define tangent vector fields $\frac{\partial}{\partial x_j} \colon U \to TU$ for $j = 1, \ldots, n$ on U as follows. Let $p \in U$ and $\alpha_i^p \colon t \mapsto x(p) + te_i$ be a path in xU. We set $p \in U$,

$$\frac{\partial}{\partial x_j}(p) = [x^{-1} \circ \alpha_{e_i}^p].$$

Thus, by definition of Dx,

$$\frac{\partial}{\partial x_j}(p) = (Dx)^{-1}(p, e_i).$$

Since Dx is a smooth map, we obtain the following corollary by combining Lemmas 5.3.10 and 5.3.17.

Corollary 5.3.19. *Given a chart (U, x) on a smooth n-manifold M. Then $\frac{\partial}{\partial x_j}$ is a smooth vector field on U for each j, and $\left(\frac{\partial}{\partial x_1}(x), \ldots, \frac{\partial}{\partial x_n}(x)\right)$ is a basis of T_pM at every $p \in U$.*

5.4 Cotangent bundle

Let M be a smooth n-manifold. Similarly as we defined the tangent bundle (TM, M, π) of M, we may define the *cotangent bundle* (T^*M, M, π), where

$$T^*M = \bigcup_{x \in M} (T_x^*M)$$

and $\pi\colon T^*M \to M$ is the map satisfying $\pi\left(T_x^*M\right) = \{x\}$; here $T_x^*M = (T_xM)^*$.

Given a chart (U, ϕ) on M, we define $T^*U = \bigcup_{x \in U} T_x^*U$. Corresponding to Lemma 5.3.10, we begin with the following observation

Lemma 5.4.1. *Let M be a smooth n-manifold and (U, ϕ) a chart on M. The mapping $T^*(\phi)\colon T^*U \to \phi(U) \times (\mathbb{R}^n)^*$,*

$$T^*(\phi)(f) = (\phi \circ \pi(f), f \circ D_{\pi(f)}\phi^{-1})$$

is a bijection. Furthermore, if (U, ϕ) and (V, ψ) are overlapping charts then

$$T^*(\phi) \circ T^*(D\psi)^{-1}|\psi(T^*U \cap T^*V)\colon \psi(U \cap V) \times (\mathbb{R}^n)^* \to \phi(U \cap V) \times (\mathbb{R}^n)^*$$

is given by formula

$$T^*(\phi) \circ T^*(D\psi)^{-1}(x, L) = (\phi \circ \psi^{-1}(x), L \circ D_x(\phi \circ \psi^{-1}))$$

for $x \in \psi(U \cap V)$ and $L \in (\mathbb{R}^n)^*$. $\square$

Remark 5.4.2. *We make the following observation. Let $\theta\colon \mathbb{R}^n \to (\mathbb{R}^n)^*$ be the isomorphism $v \mapsto \langle \cdot, v \rangle$. We give $(\mathbb{R}^n)^*$ the topology induced by θ. Then $(\mathbb{R}^n)^*$ is an n-manifold and, in fact, a smooth n-manifold.*

The transition map $T^*(\phi) \circ T^*(\psi)^{-1}|\psi(T^*U \cap T^*V)$ in Lemma 5.4.1 is then a smooth map between manifolds. (Exercise)

5.4.1 Topology and smooth structure on T^*M

Similarly, as for the tangent bundle, we now use maps $T^*(\phi)\colon T^*U \to \phi(U) \times (\mathbb{R}^n)^*$ to induce T^*M a topology so that the inclusion $T^*U \to T^*M$ is an embedding for every chart (U, ϕ). Then T^*M is an $2n$-manifold and, by Remark 5.4.2, $\{(T^*U, T^*(\phi))\colon (U, \phi) \in \mathcal{A}\}$ is a smooth atlas on T^*M, where $\mathcal{A}$ is a smooth atlas on M.

Definition 5.4.3. The bundle (T^*M, M, π) is called the *cotangent bundle of M*. The smooth structure is the one containing the smooth atlas $\{(T^*U, T^*(\phi)) \colon U, \phi) \in cA\}$.

5.4.2 1-forms

Definition 5.4.4. A smooth section $M \to T^*M$ is called a *smooth (differential) 1-form.*

Remark 5.4.5. *Since we identify $T^*\Omega = \Omega \times (\mathbb{R}^n)^*$ for open sets $\Omega \subset \mathbb{R}^n$, we have that the 1-form $\omega \colon \Omega \to T^*\Omega$ corresponds to a unique (Euclidean) 1-form $\mathrm{pr}_2 \circ \omega \colon \Omega \to (\mathbb{R}^n)^*$ as defined in Chapter 2. From now on we identify 1-forms $\Omega \to T^*\Omega$ with (Euclidean) 1-forms $\Omega \to (\mathbb{R}^n)^*$.*

Definition 5.4.6. Let $f \colon M \to N$ to be a smooth map between smooth manifolds. Let $\omega \colon N \to T^*N$ be a smooth 1-form on N. We define the *pull-back* $f^*\omega \colon M \to T^*M$ by

$$(f^*\omega)_p(v) = \omega_{f(p)}(D_p f(v)) = (D_p f)^* \omega_{f(p)}$$

for all $p \in M$ and $v \in T_pM$.

Note that this pull-back agrees with the pull-back introduced in Chapter 2 for smooth maps between open sets in Euclidean spaces.

We have now the following characterization.

Lemma 5.4.7. *A section $\omega \colon M \to T^*M$ is smooth if and only if for each chart (U, ϕ), $(\phi^{-1})^*\omega \colon \phi U \to (\mathbb{R}^n)^*$ is C^∞-smooth.*

Proof. We observe first that the map $T^*(\phi) \circ \omega \circ \phi^{-1}$ satisfies

$$T^*(\phi)(\omega_{\phi^{-1}(x)}) = (x, \omega_{\phi^{-1}(x)} \circ (D_x \phi^{-1})) = (x, \left((\phi^{-1})^*\omega\right)_x)$$

for $x \in \phi U$. Thus ω is smooth if and only if $T(\phi) \circ \omega \circ \phi^{-1}$ if and only if $(\phi^{-1})^*\omega$ is C^∞-smooth. $\square$

Lemma 5.4.8. . *Let $f \colon M \to N$ be a smooth map between manifolds and $\omega \colon N \to T^*N$ a smooth 1-form on N. Then $f^*\omega$ is a smooth 1-form on M.*

Proof. Let $x \in M$. Since f is smooth, there exists charts (U, ϕ) and (V, ψ) so that $fU \subset V$ and $h = \psi \circ f \circ \phi^{-1} \colon \phi U \to \psi V$ is C^∞-smooth.

Since ω is C^∞-smooth, we have that $(\psi^{-1})^*\omega$ is C^∞-smooth. Since

$$(\phi^{-1})^* f^* \omega = (\psi \circ f \circ \phi^{-1})^* \left((\psi^{-1})^*\omega\right)$$

the claim follows from the Euclidean theory and Lemma 5.4.7. $\square$

5.4.3 Exterior derivative of functions

Definition 5.4.9. Let $f\colon M \to \mathbb{R}$ be a smooth function. We define $df\colon M \to T^*M$ by formula

$$df([\gamma]) = (f \circ \gamma)'(0).$$

for $[\gamma] \in TM$.

We record two observations.

Lemma 5.4.10. *Let $f\colon M \to \mathbb{R}$ be a smooth function. Then df is a smooth 1-form.* $\square$

Lemma 5.4.11. *Let (U, x) be a chart on a smooth n-manifold M. Then $((dx_1)_p, \ldots, (dx_n)_p)$ is a basis of T_p^*M for each $p \in M$.* $\square$

5.5 Exterior bundles

We consider T^*M as the first exterior bundle $\mathrm{Alt}^1(TM)$ of M. The other exterior bundles $\mathrm{Alt}^k(TM)$ for $k > 1$ are defined similarly as follows.

Given $k > 1$, let

$$\mathrm{Alt}^k(TM) = \bigcup_{x \in M} \mathrm{Alt}^k(T_xM).$$

Denote by

$$\pi\colon \mathrm{Alt}^k(TM) \to M$$

the map satisfying $\pi(\omega) = x$ for $\omega \in \mathrm{Alt}^k(T_xM)$.

5.5.1 Topology, smoothness, and charts

To show that $\mathrm{Alt}^k(TM)$ has a natural structure as a smooth manifold, we make the following observation. We do not discuss all the details, just merely indicate the ideas (as in the case of T^*M).

Let (U, ϕ) be a chart on M. Denote

$$\mathrm{Alt}^k(TU) = \bigcup_{x \in U} \mathrm{Alt}^k(T_xM);$$

recall that we may naturally identify $T_xU = T_xM$.

Let $\mathrm{Alt}^k(\phi)\colon \mathrm{Alt}^k(TU) \to \phi(U) \times \mathrm{Alt}^k(\mathbb{R}^n)$ be the map

$$\mathrm{Alt}^k(\phi)(\omega) = \left(\phi(\pi(\omega)), (D_{\phi(\pi(\omega))}\phi^{-1})^*\omega\right).$$

Note that here $D_{\phi(\pi(\omega))}\phi^{-1}\colon \mathbb{R}^n \to T_{\pi(\omega)}M$ is a linear map and the map $(D_{\phi(\pi(\omega))}\phi^{-1})^*$ in the definition is a linear map $\mathrm{Alt}^k(T_{\pi(\omega)}M) \to \mathrm{Alt}^k(\mathbb{R}^n)$.

Lemma 5.5.1. *The map* $\mathrm{Alt}^k(\phi)\colon \mathrm{Alt}^k(TU) \to \phi U \times \mathrm{Alt}^k(\mathbb{R}^n)$ *is bijection. Furthermore, if* (U,ϕ) *and* (V,ψ) *are overlapping charts, then*

$$\mathrm{Alt}^k(\phi)\circ \mathrm{Alt}^k(\psi)^{-1}|\psi(U\cap V)\times \mathrm{Alt}^k(\mathbb{R}^n)\colon \psi(U\cap V)\times \mathrm{Alt}^k(\mathbb{R}^n) \to \phi(U\cap V)\times \mathrm{Alt}^k(\mathbb{R}^n)$$

is C^∞*-smooth.*

Since we may fix a linear isomorphism $\mathrm{Alt}^k(\mathbb{R}^n) \to \mathbb{R}^{\binom{n}{k}}$, we may give $\mathrm{Alt}^k(\mathbb{R}^n)$ the (Euclidean) topology and the (Euclidean) smooth structure, which makes all linear isomorphisms $\mathrm{Alt}^k(\mathbb{R}^n) \to \mathbb{R}^{\binom{n}{k}}$ diffeomorphisms. Thus we may consider $\mathrm{Alt}^k(\mathbb{R}^n)$ as a Euclidean space and give $\mathrm{Alt}^k(TM)$ the topology and smooth structure determined by bijections $\mathrm{Alt}^k(\phi)\colon \mathrm{Alt}^k(TU) \to \phi U \times \mathrm{Alt}^k(\mathbb{R}^n)$, where (U,ϕ) is a chart on M, similarly as we did for T^*M.

Then $\mathrm{Alt}^k(TM)$ is a smooth $(n+\dim \mathrm{Alt}^k(\mathbb{R}^n))$-manifold. Furthermore, $(\mathrm{Alt}^k(TM), M, \pi)$ is a smooth vector bundle, where $\pi\colon \mathrm{Alt}^k(TM) \to M$ is the map $\omega \mapsto p$ for $\omega \in \mathrm{Alt}^k(T_pM)$; naturally the linear structure on $\mathrm{Alt}^k(T_pM)$ is induced by maps $\mathrm{Alt}^k(D\phi)$ where (U,ϕ) is a chart of M at x.

Definition 5.5.2. The bundle $(\mathrm{Alt}^k(TM), M, \pi)$ is the *kth exterior bundle of* M. The smooth structure of $\mathrm{Alt}^k(TM)$ is the one containing smooth atlas $\{(\mathrm{Alt}^k(TU), \mathrm{Alt}^k(\phi))\colon (U,\phi) \in \mathcal{A}_M\}$, where $\mathcal{A}_M$ is a smooth structure of M.

5.5.2 Differential k-forms

Definition 5.5.3. A differential k-form on M is a smooth section $M \to \mathrm{Alt}^k(TM)$.

Remark 5.5.4. *Note that* $\mathrm{Alt}^1(TM) = T^*M$. *We also identify* $\mathrm{Alt}^0(TM) = C^\infty(M)$*, i.e. with smooth functions on* M.

Remark 5.5.5. *Let* $U \subset \mathbb{R}^n$ *be an open set. Then the map* $\mathrm{Alt}^k(\mathrm{id})$ *gives a diffeomorphism* $\mathrm{Alt}^k(TU) \to U \times \mathrm{Alt}^k(\mathbb{R}^n)$. *Let* $\omega\colon U \to \mathrm{Alt}^k(TU)$. *Then* $\mathrm{pr}_2 \circ \omega\colon U \to \mathrm{Alt}^k(\mathbb{R}^n)$ *is a* k*-form in the sense of Chapter 2. We identify* k*-forms* $U \to \mathrm{Alt}^k(TU)$ *with (Euclidean)* 1*-forms* $U \to \mathrm{Alt}^k(\mathbb{R}^n)$ *this way.*

Definition 5.5.6. We denote by $\Omega^k(M)$ the set of all smooth k-forms on a smooth manifold M for $k \in \mathbb{Z}$; as usual $\Omega^k(M) = \{0\}$ for $k < 0$.

Definition 5.5.7. Let $f\colon M \to N$ be a smooth map between smooth manifolds. We define the pull-back $f^*\colon \Omega^k(N) \to \Omega^k(M)$ by

$$(f^*\omega)_p(v_1,\ldots,v_k) = \omega_{f(p)}(D_pf(v_1),\ldots,D_p(v_k)) = \left((D_pf)^*\omega_{f(p)}\right)(v_1,\ldots,v_k)$$

for all $\omega \in \Omega^k(N)$, $p \in M$ and $v_1,\ldots,v_k \in T_pM$.

Clearly, this definition extends (naturally) the Euclidean definition in Chapter 2. In particular, we have the following familiar rules; the smoothness of $f^*\omega$ is left to the interested reader.

Lemma 5.5.8. *Let $f\colon M \to N$ be a smooth map. Then $f^*\colon \Omega^k(N) \to \Omega^k(M)$, $\omega \mapsto f^*\omega$, is a (well-defined) linear map. Furthermore, if $g\colon N \to P$ is a smooth map, then $f^* \circ g* = (g \circ f)^*\colon \Omega^k(P) \to \Omega^k(M)$.* $\square$

5.6 Exterior derivative

To introduce the exterior derivative of a smooth k-form on a smooth manifolds, we make the following observation.

Lemma 5.6.1. *Let $\omega \in \Omega^k(M)$ and let (U,ϕ) and (V,ψ) be overlapping charts on M. Then*

$$\left(\psi^* d(\psi^{-1})^*\omega)\right)|U \cap V = \left(\phi^* d(\phi^{-1})^*\omega\right)|U \cap V.$$

Proof. The proof is a calculation using the composition and properties of the exterior derivative:

$$\begin{aligned}
&\left(\psi^* d(\psi^{-1})^*\omega)\right)|U \cap V \\
&\quad = \Big(\phi^* \circ (\psi \circ \phi^{-1})^* d\left((\psi \circ \phi^{-1})^{-1}\right)^* \circ (\phi^{-1})^*\omega\Big)|U \cap V \\
&\quad = \left(\phi^* d(\phi^{-1})\omega\right)|U \cap V.
\end{aligned}$$

$\square$

By Lemma 5.6.1, we have a well-defined exterior derivative for smooth forms.

Definition 5.6.2. Let $\omega \in \Omega^k(M)$. The *exterior derivative of* ω is the unique $(k+1)$-form $d\omega \in \Omega^{k+1}(M)$ satisfying

$$(d\omega)|U = \phi^* d(\phi^{-1})^*\omega \tag{5.6.1}$$

in each chart (U,x) of M.

The exterior derivative satisfies the usual properties which we now recall. The proofs are immediate consequences of the corresponding Euclidean claims.

Lemma 5.6.3. *Let M be a smooth manifold. Then, for every $k \geq 0$, $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ is a linear operator satisfying*

(a) $d \circ d = 0$,

(b) $d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^k \omega \wedge d\tau$ *for* $\omega \in \Omega^k(M)$ *and* $\tau \in \Omega^\ell(M)$,

(c) if $f\colon M \to N$ *is a smooth map then* $f^* d\omega = df^*\omega$ *for each* $\omega \in \Omega^k(N)$.

Local representations of differential forms

The Euclidean results have the following consequences.

Corollary 5.6.4. *Let (U, x) be a chart on a smooth n-manifold M. Then dx_j is a smooth 1-form on U for each j. Furthermore, $((dx_1)_p, \ldots, (dx_n)_p)$ is a basis of T_p^*M for each M. In particular, given a k-form $\omega \in \Omega^k(U)$, there exists smooth functions ω_I for $I = (i_1, \ldots, i_k)$, $1 \le i_1 < \cdots < i_k \le n$, so that*

$$\omega = \sum_I \omega_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Corollary 5.6.5. *Let (U, x) be a chart on M and*

$$\omega = \sum_{\substack{I=(i_1,\ldots,i_k)\\ 1\le i_1<\ldots,i_k\le n}} \omega_I dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

a smooth form. Then

$$d\omega = \sum_I d\omega_I \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

5.7 de Rham cohomology

Definition 5.7.1. Let M be a smooth manifold. The kth de Rham cohomology (group) $H^k(M)$ is the vector space

$$H^k(M) = \frac{\{\omega \in \Omega^k(M) \colon d\omega = 0\}}{\{d\tau \in \Omega^k(M) \colon \tau \in \Omega^{k-1}(M)\}}.$$

As in the Euclidean case, $(\Omega^k(M), d)_{k\in\mathbb{Z}}$ is a chain complex and a smooth mapping $f \colon M \to N$ between smooth manifolds induces a chain map $f^* \colon \Omega^k(N) \to \Omega^k(M)$.

Similarly, the map $f^* \colon H^k(N) \to H^k(M)$, $[\omega] \to [f^*\omega]$, is a well-defined linear map for each k.

5.7.1 Properties of de Rham cohomology

Meyer–Vietoris

Theorem 5.7.2. *Let M be a smooth manifold and let U_1 and U_2 be open sets in M, $U = U_1 \cup U_2$, and let $i_m \colon U_m \to U_1 \cup U_2$ and $j_m \colon U_1 \cap U_2 \to U_m$ be inclusions. Let*

(5.7.1)

$$0 \longrightarrow \Omega^k(U_1 \cup U_2) \xrightarrow{I_k} \Omega^k(U_1) \oplus \Omega^k(U_2) \xrightarrow{J_k} \Omega^k(U_1 \cap U_2) \longrightarrow 0$$

be a short exact sequence, where $I(\omega) = (i_1^*(\omega), i_2^*(\omega))$ *and* $J(\omega_1, \omega_2) = j_1^*\omega_1 - j_2^*(\omega_2)$.

Then the sequence

$$\cdots \longrightarrow H^k(U) \xrightarrow{I_*} H^k(U_1) \oplus H^k(U_2) \xrightarrow{J_*} H^k(U_1 \cap U_2) \xrightarrow{\partial_k} H^{k+1}(U) \longrightarrow \cdots$$

where ∂_k *is the boundary operator for the short exact sequence* (5.7.1)*, is a long exact sequence.*

Proof. The claim follows from the exactness of (5.7.1) and Theorem 3.3.1. The exactness of (5.7.1) follows from the partition of unity as in the Euclidean case. (Partition of unity is an exercise). □

Smooth homotopy theory

Definition 5.7.3. Smooth maps $f_0, f_1 \colon M \to N$ are *smoothly homotopic* if there exists a smooth map $F \colon M \times \mathbb{R} \to N$ so that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for all $x \in M$.

Similarly as in the Euclidean case we obtain the following homotopy invariance.

Theorem 5.7.4. *Let* f_0 *and* f_1 *be smoothly homotopic smooth maps* $M \to N$*. Then*

$$f_0^* = f_1^* \colon H^k(N) \to H^k(M).$$

Proof. Exercise. (Adaptation of the Euclidean proof.) □

Non-smooth homotopy theory and topological invariance

To obtain the topological invariance of de Rham cohomology, we need the following counterpart of the smoothening lemma. As in the Euclidean case, the partition of unity yields the following result. We leave the details to the interested reader (Exercise!) and refer to the book of Hirsch [6] for a detailed exposition; see also [7] for a proof based on embedding $M \to \mathbb{R}^k$.

Theorem 5.7.5. *Let* $f \colon M \to N$ *be a continuous map between smooth manifolds. Then there exists a smooth map* $g \colon M \to N$ *homotopic to* f*. Furthermore, if* $g_0, g_1 \colon M \to N$ *smooth maps which are homotopic then they are smoothly homotopic.*

Corollary 5.7.6. *Let* $f \colon M \to N$ *be a homotopy equivalence between smooth manifolds. Then* $H^k(N) \cong H^k(M)$ *for all* $k \geq 0$.

Chapter 6

Orientable manifolds

We two related (elementary) definitions.

Definition 6.0.7. A smooth n-manifold M is *orientable* if there exists a smooth n-form $\omega_M \in \Omega^n(M)$ so that $(\omega_M)_p \neq 0$ for every $p \in M$. The form ω_M is then called an *orientation form of* M. A pair (M, ω_M) is an *oriented manifold.*

Convention 6.0.8. *We denote* $\mathrm{vol}_{\mathbb{R}^n} = dx_1 \wedge \cdots \wedge dx_n$ *and consider* $\mathbb{R}^n$ *as the oriented manifold* $(\mathbb{R}^n, \mathrm{vol}_{\mathbb{R}^n})$ *unless otherwise mentioned.*

Definition 6.0.9. Let M be a smooth n-manifold. A smooth atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ on M is *positive* if the transition functions $\phi_\alpha \circ \phi_\beta^{-1}|\psi(U_\alpha \cap U_\beta)$ satisfy

$$J_{\phi_\alpha \circ \phi_\beta^{-1}}(x) := \det(D_x(\phi_\alpha \circ \phi_\beta^{-1})) > 0$$

for each $x \in \phi_\beta^{-1}(U_\alpha \cap U_\beta)$.

We have the following characterization of orientability by charts.

Lemma 6.0.10. *Let M be a smooth n-manifold. Then the following are equivalent:*

(a) M is orientable and

(b) there exists a positive atlas on M.

Proof. Suppose M is orientable and let $\omega \in \Omega^n(M)$ be an orientation form on M. Let (U, φ) be a connected chart on M. Then $\varphi^*(dx_1 \wedge \cdots \wedge dx_n)$ is an orientation form on U. Thus there exists a function $\lambda_\phi \colon U \to \mathbb{R}$ so that $\varphi^*(dx_1 \wedge \cdots \wedge dx_n) = \lambda_\phi \omega | U$. Since $\varphi^*(dx_1 \wedge \cdots \wedge dx_n)_p \neq 0$ for every $p \in U$, we have $\lambda_\phi(p) \neq 0$ for each $p \in U$.

Let $\mathcal{A} = \{(U, \phi) \colon \lambda_\phi > 0\}$. We show that $\mathcal{A}$ is a positive atlas. First, let $p \in M$ and (V, ψ) a chart on M at p. If $\lambda_\psi < 0$ let $\rho \colon \mathbb{R}^n \to \mathbb{R}^n$ be the map

$(x_1, \dots, x_n) \mapsto (-x_1, x_2, \dots, x_n)$. Then $\lambda_{\tilde{\psi}} > 0$ for $\tilde{\psi} = \rho_1 \circ \psi \colon V \to \tilde{\psi} V$. Thus $(V, \tilde{\psi}) \in \mathcal{A}$ and $\mathcal{A}$ is an atlas.

Let $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{A}$ be overlapping charts. Then

$$\begin{aligned} \left(\det D(\phi \circ \psi^{-1})\right) \mathrm{vol}_{\mathbb{R}^n} | \psi(U \cap V) &= (\psi^{-1})^* \phi^* \mathrm{vol}_{\mathbb{R}^n} | \psi(U \cap V) \\ &= (\psi^{-1})^* \left(\lambda_\phi \omega\right) \\ &= \frac{\lambda_\phi}{\lambda_\psi} (\psi^{-1})^* \mathrm{vol}_{\mathbb{R}^n} | \psi(U \cap V). \end{aligned}$$

Thus $\mathcal{A}$ is a positive atlas.

Suppose now that $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ is a positive atlas. Let $\{u_\alpha\}$ be a partition of unity with respect to $\{U_\alpha\}_{\alpha \in I}$. Define $\omega_\alpha \in \Omega^n(M)$ by $\omega_\alpha(p) = u_\alpha(p) \phi_\alpha^* \mathrm{vol}_{\mathbb{R}^n}$ for $p \in U$ and $\omega_\alpha(p) = 0$ otherwise. Let

$$\omega = \sum_\alpha \omega_\alpha \in \Omega^n(M).$$

It suffices to show that $\omega_p \neq 0$ for each $p \in M$. Let $p \in M$ and $\alpha \in I$ so that $u_\alpha(p) > 0$. Suppose now that $\beta \in I$ is such that $u_\beta(p) > 0$. Then

$$\begin{aligned} u_\beta(p) \left(\phi_\beta^* \mathrm{vol}_{\mathbb{R}^n}\right)_p &= u_\beta(p) \left(\phi_\alpha^* (\phi_\beta \circ \phi_\alpha^{-1})^* \mathrm{vol}_{\mathbb{R}^n}\right)_p \\ &= u_\beta(p) \left(\phi_\alpha^* J_{\phi_\beta \circ \phi_\alpha^{-1}} \mathrm{vol}_{\mathbb{R}^n}\right)_p \\ &= u_\beta(p) (J_{\phi_\beta \circ \phi_\alpha^{-1}} \circ \phi_\alpha)(p) \omega_\alpha(p). \end{aligned}$$

Thus $\omega_\beta | U \cap V = u_{\alpha\beta} \omega_\alpha | U \cap V$ where $u_{\alpha\beta} \colon U \cap V \to \mathbb{R}$ is a positive function. $\square$

Definition 6.0.11. Let (M, ω_M) and (N, ω_N) be oriented manifolds of the same dimension. Let $f \colon M \to N$ be a smooth map and let $\lambda_f \colon M \to \mathbb{R}$ be the function $f^* \omega_N = \lambda_f \omega_M$. We say that f is *orientation preserving* if $\lambda_f > 0$ and *orientation reversing* if $\lambda_f < 0$.

Remark 6.0.12. *Let (M, ω_M) be an oriented manifold. Then $(M, -\omega_M)$ is an oriented manifold and* $\mathrm{id} \colon (M, \omega_M) \to (M, -\omega_M)$ *is orientation reversing. Indeed, $-\omega_M$ is an orientation form on M and $\mathrm{id}^*(-\omega_M) = -\omega_M$.*

Lemma 6.0.13. *Let (M, ω_M) be a connected and oriented n-manifold and $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ a positive atlas. Consider each $(U_\alpha, \omega_M | U_\alpha)$ and $(\psi_\alpha U_\alpha, \mathrm{vol}_{\mathbb{R}^n} | \psi_\alpha U_\alpha)$ as an oriented manifold. Then either of the following is true:*

(a) All maps $\phi_\alpha \colon U_\alpha \to \psi U_\alpha$ are orientation preserving or

(b) All maps $\phi_\alpha \colon U_\alpha \to \psi U_\alpha$ are orientation revsersing.

Proof. Exercise. $\square$

Definition 6.0.14. Let (M, ω_M) be an oriented manifold. A positive atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ on M is *compatible with* ω_M if each map ψ_α is orientation preserving $(U_\alpha, \omega_M | U_\alpha) \to (\phi_\alpha U_\alpha, \mathrm{vol}_{\mathbb{R}^n} | \phi_\alpha U_\alpha)$.

6.0.2 Excursion: oriented vector spaces

Definition 6.0.15. Let V be an n-dimensional vector space and $\omega \in \mathrm{Alt}^n(V)$ a non-zero element. The pair (V, ω) is an *oriented vector space*. A basis $(e_1, \ldots, e_n)$ of V is *positively oriented with respect to* ω if $\omega(e_1, \ldots, e_n) > 0$ and *negatively oriented otherwise*.

Definition 6.0.16. Let $L\colon (V, \omega_V) \to (W, \omega_W)$ be a linar map between oriented vectorspaces of the same dimension and $\lambda \in \mathbb{R}$ so that $L^*\omega_W = \lambda\omega_V$. We say that L is *orientation preserving* if $\lambda > 0$, and *orientation reversing* if $\lambda < 0$.

Lemma 6.0.17. *Let (V, ω) be an oriented vector space and $L\colon V \to V$ an isomorphism. Then L is orientation preserving if and only if $\det A_L > 0$ where A_L is a metrix of L with respect to any basis of V.*

6.0.3 Examples

Example 6.0.18. The manifold $\mathbb{S}^{n-1}$ is orientable. *We construct $\omega_{\mathbb{S}^{n-1}}$ directly. Let $\omega \in \Omega^n(\mathbb{R}^n \backslash \{0\})$ be the form $\omega_x(v_1, \ldots, v_{n-1}) = \det(x, v_1, \ldots v_{n-1})$. Let $\iota\colon \mathbb{S}^n \to \mathbb{R}^n \setminus \{0\}$ be the standard embedding. Then $\omega_{\mathbb{S}^{n-1}} = \iota^*\omega$ is an orientation form on $\mathbb{S}^{n-1}$. (Exercise)*

Example 6.0.19. *Let $A\colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ be the map $x \mapsto -x$ and $\Gamma = \{\mathrm{id}, A\}$ be a subgroup of diffeomorphisms of $\mathbb{S}^n$. Let $\mathbb{R}P^{n-1} = \mathbb{S}^{n-1}/\{A, -A\}$. Then $\mathbb{R}P^{n-1}$ is a smooth manifold and the canonical map $\pi\colon \mathbb{S}^n \to \mathbb{R}P^{n-1}$ is a smooth covering map. Furthermore, $\mathbb{R}P^{n-1}$ has a positive atlas for n even (Exercise!). Thus $\mathbb{R}P^{n-1}$ is orientable for n even.*

On the other hand, $\mathbb{R}P^{n-1}$ is not orientable for n odd. This can be seen as follows. Suppose $\mathbb{R}P^{n-1}$ is orientable and $\omega_{\mathbb{R}P^{n-1}}$ is an orientation form on $\mathbb{R}P^{n-1}$. Then $\pi^*\omega$ is an orientation form on $\mathbb{S}^{n-1}$ (Exercise!). Thus there exists a function $\lambda\colon \mathbb{S}^{n-1} \to \mathbb{R}$ so that $\pi^*\omega = \lambda \mathrm{vol}_{\mathbb{S}^{n-1}}$ and $\lambda(p) \neq 0$ for each $p \in \mathbb{S}^{n-1}$.

We observe that $A^*\omega_{\mathbb{S}^{n-1}} = (-1)^n\omega_{\mathbb{S}^{n-1}}$. Indeed, since $D_x(\iota \circ A(v)) = -D_x\iota(v)$ for all $x \in \mathbb{S}^{n-1}$ and $v \in T_x\mathbb{S}^{n-1}$ (Exercise!), we have

$$\begin{aligned}
(A^*\omega_{\mathbb{S}^{n-1}})_x(v_1, \ldots, v_{n-1}) &= (\omega_{\mathbb{S}^{n-1}})_{A(x)}(D_xA(v_1), \ldots, D_xA(v_{n-1})) \\
&= \omega_{A(x)}(D_x(\iota \circ A)v_1), \ldots, D_x(\iota \circ A)v_{n-1})) \\
&= (-1)^{n-1}\det(A(x), (D_x\iota)v_1, \ldots, (D_x\iota)v_{n-1}) \\
&= (-1)^n \det(x, (D_x\iota)v_1, \ldots, (D_x\iota)v_{n-1}) \\
&= (\omega_{\mathbb{S}^{n-1}})_x(v_1, \ldots, v_{n-1}),
\end{aligned}$$

that is,

$$A^*\omega_{\mathbb{S}^{n-1}} = (-1)^n\omega_{\mathbb{S}^{n-1}}.$$

On the other hand,

$$A^*\pi^*\omega_{\mathbb{R}P^{n-1}} = (\pi \circ A)^*\omega_{\mathbb{R}P^{n-1}} = \pi^*\omega_{\mathbb{R}P^{n-1}},$$

Thus, by combining the indentities, we have

$$(-1)^n(\lambda \circ A)\omega_{\mathbb{S}^{n-1}} = A^*(\lambda\omega_{\mathbb{S}^{n-1}}) = A^*\pi^*\omega_{\mathbb{R}P^{n-1}} = \pi^*\omega_{\mathbb{R}P^{n-1}} = \lambda\omega_{\mathbb{R}P^{n-1}}.$$

Since $(\lambda \circ A)/\lambda > 0$ *and* $(-1)^n = -1$ *for* n *odd, we have a contradiction. Thus* $\mathbb{R}P^{n-1}$ *is not orientable.*

Chapter 7

Integration

We use the following notations and terminology

Definition 7.0.20. Let M be a smooth manifold. Then $\Omega_c^k(M)$ is the space of all compactly supported smooth k-forms on M, that is, forms $\omega \in \Omega^k(M)$ for which $\mathrm{spt}(\omega) = \mathrm{cl}\{p \in M : \omega_p \neq 0\}$ is compact.

Definition 7.0.21. Let M and N be manifolds. The map $f : M \to N$ is *proper* if $f^{-1}E$ is compact for each compact set $E \subset N$.

Remark 7.0.22. *Clearly, $d : \Omega_c^k(M) \to \Omega_c^{k+1}(M)$ for each k and each proper map $f : M \to N$ induces a chain map $f^* : \Omega_c^k(N) \to \Omega_c^k(M)$.*

Remark 7.0.23. *Let M be a smooth manifold and $U \subset M$ an open set. Then each $\omega \in \Omega_c^k(U)$ extends trivially to a compectly supported form on M, i.e. we may consider ω as an element in $\Omega_c^k(M)$.*

7.1 Euclidean case

Definition 7.1.1. Let $U \subset \mathbb{R}^n$ be an open set and $u \in C_0^\infty(U)$. We define

$$\int_U u \mathrm{vol}_{\mathbb{R}^n} = \int_U u \, \mathrm{d}\mathcal{L}^n$$

where $\mathcal{L}^n$ is the Lebesgue measure in $\mathbb{R}^n$.

We take the change of variables formula as a fact; see e.g. [8, Theorem 7.26].

Lemma 7.1.2. *Let U and V be open sets in $\mathbb{R}^n$ and $f : U \to V$ a smooth proper map. Then, for each $u \in C_0^\infty(V)$,*

$$\int_U (u \circ f)|J_f| \, \mathrm{d}\mathcal{L}^n = \int_V u \, \mathrm{d}\mathcal{L}^n,$$

where J_f is the Jacobian determinant *$J_f(x) = \det(D_x f)$.*

Lemma 7.1.3. *Let $f: U \to V$ be a diffeomorphism between connected open sets in $\mathbb{R}^n$. Then J_f has constant sign, that is, either $J_f > 0$ in U or $J_f < 0$ in U.*

Proof. Let $p \in U$. Since $(D_p f)^{-1} = D_{f(p)} f^{-1}$, we have that $J_f(p) \neq 0$. Since U is connected, we have either $J_f > 0$ or $J_f < 0$ in U. $\square$

Lemma 7.1.4. *Let U and V be connected open sets in $\mathbb{R}^n$ and $f: U \to V$ a diffeomorphism. Then, for each $\omega \in \Omega^n(V)$,*

$$\int_U f^*\omega = \delta_f \int_V \omega.$$

where $\delta_f = J_f/|J_f|$ is the sign of the Jacobian of f.

Proof. Let $\omega = u\mathrm{vol}_{\mathbb{R}^n}$. Then, by linear algebra, $f^*\omega = (u \circ f) f^*(\mathrm{vol}_{\mathbb{R}^n}) = (u \circ f) f^*(dx_1 \wedge \cdots \wedge dx_n) = (u \circ f) \det(Df) dx_1 \wedge \cdots \wedge dx_n = (u \circ f) J_f \mathrm{vol}_{\mathbb{R}^n}$. Thus

$$\int_U f^*\omega = \int_U (u \circ f) J_f \mathrm{d}\mathcal{L}^n = \delta_f \int_V u \mathrm{d}\mathcal{L}^n = \delta_f \int_V \omega.$$

$\square$

The main result in the Euclidean case is the following theorem.

Theorem 7.1.5. *The sequence*

$$\Omega_c^{n-1}(\mathbb{R}^n) \xrightarrow{d} \Omega_c^n(\mathbb{R}^n) \xrightarrow{\int_{\mathbb{R}^n}} \mathbb{R}$$

is exact.

Remark 7.1.6. *The claim holds also for all open sets in $\mathbb{R}^n$. This will be covered by a manifold version of the theorem in the next section.*

We prove Theorem 7.1.5 in two parts.

Lemma 7.1.7. *Let $\tau \in \Omega_c^{n-1}(\mathbb{R}^n)$. Then $\int_{\mathbb{R}^n} d\tau = 0$.*

Lemma 7.1.8. *Suppose $\omega \in \Omega_c^n(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}^n} \omega = 0$. Then there exists $\tau \in \Omega_c^{n-1}(\mathbb{R}^n)$ so that $\omega = d\tau$.*

Proof of Theorem 7.1.5. The surjectivity of the map $\omega \mapsto \int_{\mathbb{R}^n} \omega$ is immediate. It suffices to fix a function $f \in C_c^\infty(\mathbb{R}^n)$ so that $\int_{\mathbb{R}^n} f d\mathcal{L}^n = 1$. Then $\int_{\mathbb{R}^n} \lambda f \mathrm{vol}_{\mathbb{R}^n} = \lambda$ for every $\lambda \in \mathbb{R}$.

By Lemma 7.1.7, the image of d is contained in the kernel of $\int_{\mathbb{R}^n}$, and by Lemma 7.1.8, the kernel of $\int_{\mathbb{R}^n}$ is contained in the image of d. Thus the sequence is exact. $\square$

Proof of Lemma 7.1.7. The proof is an application of Fubini's theorem. Let

$$\tau = \sum_{i=1}^{n} g_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

Then

$$d\tau = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial g_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n.$$

Let $C > 0$ be such that $\operatorname{spt} \tau \subset [-C, C]^n$. Then

$$\begin{aligned} \int_{\mathbb{R}^n} d\tau &= \sum_{i=1}^{n} (-1)^{i-1} \int_{\mathbb{R}^n} \frac{\partial g_i}{\partial x_i} d\mathcal{L}^n \\ &= \sum_{i=1}^{n} (-1)^{i-1} \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial g_i}{\partial x_i}(y_1, \ldots, y_i, \ldots, y_n) dy_i \right) d\mathcal{L}^{n-1}(y_1, \ldots, \widehat{y_i}, \ldots, y_n). \end{aligned}$$

On the other hand, for each i and $(y_1, \ldots, y_n) \in \mathbb{R}^n$,

$$\begin{aligned} &\int_{\mathbb{R}} \frac{\partial g_i}{\partial x_i}(y_1, \ldots, t, \ldots, y_n) dt \\ &\quad = \int_{-C}^{C} \frac{\partial g_i}{\partial x_i}(y_1, \ldots, t, \ldots, y_n) dt \\ &\quad = g_i(y_1, \ldots, C, \ldots, y_n) - g_i(y_1, \ldots, -C, \ldots, y_n) = 0. \end{aligned}$$

The proof is complete. □

Proof of Lemma 7.1.8. It remains to show that for every $\omega \in \Omega_c^n(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \omega = 0$ there exists $\tau \in \Omega_c^n(\mathbb{R}^n)$ so that $\omega = d\tau$; note that ω is always exact, but we have to show that we can find τ with compact support.

We prove the claim by induction on dimension.

Suppose $n = 1$. Let $\omega = f dx \in \Omega_c^1(\mathbb{R})$ be such that $\int_{\mathbb{R}} \omega = \int_{\mathbb{R}} f dx 0$. Since f has compact support, there exists $M > 0$ so that $\operatorname{spt} f \subset [-M, M]$. Let $F \colon \mathbb{R} \to \mathbb{R}$ be an integral function of f, that is,

$$F(x) = \int_{-M}^{x} f(t) dt.$$

Then $F(x) = 0$ for $x < -M$ and $x > M$. Thus F has compact support. Moreover, $dF = \omega$ by the fundamental theorem of calculus. This complates the case $n = 1$.

Suppose now that the claim holds for $n - 1$ for some $n \geq 2$, that is, if $\zeta \in \Omega_c^{n-1}(\mathbb{R}^{n-1})$ satisfies $\int_{\mathbb{R}^{n-1}} \zeta = 0$ then there exists $\xi \in \Omega_c^{n-2}(\mathbb{R}^{n-1})$ so that $\zeta = d\xi$.

Let $\omega = f dx_1 \wedge \cdots \wedge dx_n \in \Omega^n_c(\mathbb{R}^n)$. We fix $M > 0$ such that $\operatorname{spt}\omega \subset [-M, M]^n$ and define $g\colon \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$g(x_1, \dots, x_{n-1}) = \int_{-M}^{M} f(x_1, \dots, x_{n-1}, t)dt.$$

Now g is C^∞-smooth, since $f \in C^\infty$-smooth. Clearly, $(x_1, \dots, x_{n-1}) \notin [-M, M]^{n-1}$, $g(x_1, \dots, x_{n-1}) = 0$. By Fubini's theorem,

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} g dx_1 \wedge \cdots \wedge dx_{n-1} &= \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{\infty} f(y,t)dt \right) d\mathcal{L}^{n-1}(y) \\ &= \int_{\mathbb{R}^n} f(x) d\mathcal{L}^n(x) = \int_{\mathbb{R}^n} \omega = 0. \end{aligned}$$

Thus, by induction assumption, there exists a form

$$\xi = \sum_{i=1}^{n-1} (-1)^{i-1} g_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n-1} \in \Omega^{n-2}_c(\mathbb{R}^{n-1})$$

so that $g dx_1 \wedge \cdots \wedge dx_{n-1} = d\xi$. Note that

$$\sum_{i=1}^{n-1} \frac{\partial g_i}{\partial x_i} = g.$$

We find now the form $\tau \in \Omega^{n-1}_c(\mathbb{R}^n)$ in two steps. First, let $\lambda \in C^\infty_c(\mathbb{R})$ be a function with integral 1. For $i = 1, \dots, n-1$, we set $f_i\colon \mathbb{R}^n \to \mathbb{R}$ by

$$f_i(x_1, \dots, x_n) = g_i(x_1, \dots, x_{n-1})\lambda(x_n).$$

Then, for each i, f_i is a C^∞-smooth function with a compact support. Recall that $\omega = f dx_1 \wedge \cdots \wedge dx_n$ where f has compact support and define $h\colon \mathbb{R}^n \to \mathbb{R}$ by

$$h = f - \sum_{i=1}^{n-1} \frac{\partial f_i}{\partial x_i}.$$

Then h has compact support. Let $f_n\colon \mathbb{R}^n \to \mathbb{R}$ be a "partial integral function of h", that is, we set f_n by

$$f_n(x_1, \dots, x_n) = \int_{-\infty}^{x_n} h(x_1, \dots, x_{n-1}, t)dt$$

and obtain

$$\frac{\partial f_n}{\partial x_n} = h.$$

Let

$$\tau = \sum_{i=1}^{n} (-1)^{i-1} f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

Then

$$\begin{aligned} d\tau &= \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n \\ &= \left(\sum_{i=1}^{n-1} \frac{\partial f_i}{\partial x_i} + \left(f - \sum_{i=1}^{n-1} \frac{\partial f_i}{\partial x_i} \right) \right) dx_1 \wedge \cdots \wedge dx_n \\ &= f dx_1 \wedge \cdots \wedge dx_n = \omega. \end{aligned}$$

It remains to show that function f_n has a compact support. Since h has compact support, there exists $C > 0$ so that $\operatorname{spt} h \subset [-C, C]^n$. Thus $\operatorname{spt} f \subset [-C, C]^{n-1} \times \mathbb{R}$ and f_n is zero on $\{x\} \times (-\infty, -C]$ and constant on $\{x\} \times [C, \infty)$ for each $x \in \mathbb{R}^{n-1}$.

On the other hand, for every $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, we have

$$\begin{aligned} f_n(x_1, \ldots, x_{n-1}, C) &= \int_{\mathbb{R}} h(x_1, \ldots, x_{n-1}, t) dt \\ &= \int_{\mathbb{R}} f(x_1, \ldots, x_{n-1}, t) dt - \sum_{i=1}^{n-1} \int_{\mathbb{R}} \frac{\partial f_i}{\partial x_i}(x_1, \ldots, x_{n-1}, t) dt \\ &= \int_{\mathbb{R}} f(x_1, \ldots, x_{n-1}, t) dt - \sum_{i=1}^{n-1} \frac{\partial g_i}{\partial x_i}(x_1, \ldots, x_{n-1}) \int_{\mathbb{R}} \lambda(t) dt \\ &= \int_{\mathbb{R}} f(x_1, \ldots, x_{n-1}, t) dt - \sum_{i=1}^{n-1} \frac{\partial g_i}{\partial x_i}(x_1, \ldots, x_{n-1}) \\ &= \int_{\mathbb{R}} f(x_1, \ldots, x_{n-1}, t) dt - g(x_1, \ldots, x_{n-1}) = 0. \end{aligned}$$

Thus f_n has compact support. This concludes the induction step and the proof. □

7.2 Manifold case

Lemma 7.2.1. *Let M be a smooth orientable n-manifold, (U, ϕ) and (V, ψ) overlapping charts in a smooth positively oriented atlas $\mathcal{A}$ of M. Then, for every $\omega \in \Omega^n_c(U \cap V)$, we have*

$$\int_{\phi(U \cap V)} (\phi^{-1})^* \omega = \int_{\psi(U \cap V)} (\psi^{-1})^* \omega.$$

Proof. Let $f = \psi \circ \phi^{-1} | \phi(U \cap V) \colon \phi(U \cap V) \to \psi(U \cap V)$. Then, by Lemma 7.1.4,

$$\int_{\psi(U \cap V)} (\psi^{-1})^* \omega = \int_{\phi(U \cap V)} (\psi \circ \phi^{-1})^* (\psi^{-1})^* \omega = \int_{\phi(U \cap V)} (\phi^{-1})^* \omega.$$

□

Definition 7.2.2. Let (M, ω_M) be a smooth oriented manifold and $\omega \in \Omega^n_c(M)$. We define the *integral of* ω by

$$\int_M \omega = \int_{(M,\omega_M)} \omega = \sum_\alpha \int_{\phi_\alpha U_\alpha} (\phi_\alpha^{-1})^*(\lambda_\alpha \omega),$$

where $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ is a positive atlas on M compatible with ω_M and $\{\lambda_\alpha\}_\alpha$ a partition of unity with respect to $\{U_\alpha\}_\alpha$.

Lemma 7.2.3. *The integral* $\int_M \omega$ *of* ω *depends neither on the positive compatible atlas* $\mathcal{A}$ *nor the partition of unity* $\{\lambda_\alpha\}_\alpha$ *.*

Proof. Let $\mathcal{B} = \{(V_\beta, \psi_\beta)\}_\beta$ be a positive compatible atlas and $\{\mu_\beta\}_\beta$ a partition of unity with respect to $\{V_\beta\}_\beta$.

Let $\omega_\alpha = \lambda_\alpha \omega$, $\omega_\beta = \mu_\beta \omega$, and $\omega_{\alpha\beta} = \lambda_\alpha \mu_\beta \omega$. Then, for $U_\alpha \cap V_\beta \neq \emptyset$, we have

$$\int_{\phi_\alpha(U_\alpha \cap V_\beta)} (\phi_\alpha^{-1})^* \omega_{\alpha\beta} = \int_{\psi_\beta(U_\alpha \cap V_\beta)} (\psi_\alpha^{-1})^* \omega_{\alpha\beta}.$$

Since

$$\omega_\alpha = \sum_\beta \mu_\beta \omega_\alpha = \sum_\beta \omega_{\alpha\beta},$$

we have

$$\begin{aligned} \sum_\alpha \int_{\phi_\alpha U_\alpha} (\phi_\alpha^{-1})^* \omega_\alpha &= \sum_\alpha \sum_\beta \int_{\phi_\alpha(U_\alpha \cap V_\beta)} (\phi_\alpha^{-1})^* \omega_{\alpha\beta} \\ &= \sum_\beta \sum_\alpha \int_{\psi_\beta(U_\alpha \cap V_\beta)} (\psi_\beta^{-1})^* \omega_{\alpha\beta} \\ &= \sum_\beta \int_{\psi_\beta V_\beta} (\psi_\beta^{-1})^* \omega_\beta. \end{aligned}$$

The claim follows. $\square$

Lemma 7.2.4. *Let* (M, ω_M) *and* (N, ω_N) *be oriented* n*-manifolds and* $f \colon M \to N$ *a diffeomorphism. Then*

$$\int_M f^* \omega = \delta_f \int_N \omega$$

for each $\omega \in \Omega^n(N)$*, where*

$$\delta_f = \begin{cases} +1, & f \text{ is orientation preserving,} \\ -1, & f \text{ is orientation reversing.} \end{cases}$$

In particular,

$$\int_{(M,-\omega_M)} \omega = -\int_{(M,\omega_M)} \omega$$

for each $\omega \in \Omega^n(M)$*.*

Proof. Exercise. □

The goal for this section is to show the following theorem.

Theorem 7.2.5. *Let (M, ω_M) be an oriented n-manifold. Then the sequence*

$$\Omega_c^{n-1}(M) \xrightarrow{d} \Omega_c^n(M) \xrightarrow{\int_M} \mathbb{R} \longrightarrow 0$$

is exact.

It has the following corollary in terms of the compactly supported cohomology.

Definition 7.2.6. Let M be a smooth manifold. The vector space

$$H_c^k(M) = \frac{\{\omega \in \Omega_c^k(M) \colon d\omega = 0\}}{\{d\tau \in \Omega_c^k(M) \colon \tau \in \Omega_c^{k-1}(M)\}}$$

is the *kth compactly supported cohomology of M.*

In terms of the compactly supported cohomology Theorem 7.2.5 has the following consequences.

Corollary 7.2.7. *Let M be a connected orientable n-manifold. Then the linear map*

$$\int_M \colon H_c^n(M) \to \mathbb{R}, \quad [\omega] \mapsto \int_M \omega,$$

is a well-defined isomorphism. In particular, $H^n(M) \cong \mathbb{R}$ for a compact, connected and orientable n-manifold M.

We split the proof of Theorem 7.2.5 into parts (as in the Euclidean case).

Lemma 7.2.8. *Let (M, ω_M) be an oriented manifold and $\tau \in \Omega_c^{n-1}(M)$. Then*

$$\int_M d\tau = 0.$$

Proof. Let $\{(U_i, \phi_i)\}_{i \geq 0}$ be a countable positive atlas of M and let $\{\lambda_i\}$ be a partition of unity with respect to $\{U_i\}$. Let $\tau_i = \lambda_i \tau \in \Omega_c^{n-1}(U_i)$ for each i. Since $(\phi_i^{-1})^* \tau_i \in \Omega_c^n(\phi_i U_i)$, we have, by Lemma 7.1.7, that

$$\int_M d\tau = \sum_{i=1}^{\infty} \int_{U_i} d\tau_i = \sum_{i=1}^{\infty} \delta \int_{\phi_i U_i} (\phi_i^{-1})^* d\tau_i = \sum_{i=1}^{\infty} \delta \int_{\phi_i U_i} d(\phi_i^{-1})^* \tau_i = 0.$$

where $\delta_{\phi_i} = \pm 1$ depending whether ϕ_i's are orientable preserving or reversing. □

The converse to Lemma 7.2.8 is based on two auxiliary lemmas.

Lemma 7.2.9 (Chain Lemma)**.** *Let M be a connected manifold, $(U_\alpha)_{\alpha\in I}$ an open cover of M, and $p, q \in M$. Then there exists indiced $\alpha_1, \ldots, \alpha_k$ so that*

(i) $p \in U_{\alpha_1}$, $q \in U_{\alpha_k}$, and

(ii) $U_{\alpha_i} \cap U_{\alpha_{i+1}} \neq \emptyset$ for $i = 1, \ldots, k-1$.

Proof. Let $C \subset M \times M$ be the set of those pairs (p, q) for which such chains exist. Then C is an open set. Indeed, let $(p, q) \in C$ and $U_{\alpha_1}, \ldots, U_{\alpha_k}$ a connecting chain. Then the same chain can be used for all $(p', q') \in U_{\alpha_1} \times U_{\alpha_k}$. On the other hand C is closed. Indeed, let $(p, q) \in \bar{C}$. Let $\alpha_0, \alpha_{k+1} \in I$ so that $p \in U_{\alpha_0}$ and $q \in U_{\alpha_k}$. Then there exists $p \in U_{\alpha_0} \cap C$ and $q \in U_{\alpha_{k+1}} \cap C$. Thus there exists a chain $U_{\alpha_1}, \ldots, U_{\alpha_k}$ from p' to q'. Hence $U_{\alpha_0}, \ldots, U_{\alpha_{k+1}}$ is a chain from p and q and (after reindexing the chain) $(p, q) \in C$. Since $M \times M$ is connected, $C = M \times M$. □

Lemma 7.2.10 (Local concentration)**.** *Let M be a smooth n-manifold, $U \subset M$ an open set diffeomorphic to $\mathbb{R}^n$, and $W \subset U$ an open subset. Given $\omega \in \Omega^n_c(U)$ there exists $\tau \in \Omega^{n-1}_c(U)$ so that* $\operatorname{spt}(\omega - d\tau) \subset W$.

Proof. Let $\phi \colon U \to \mathbb{R}^n$ be a diffeomorphism and define $\zeta = (\phi^{-1})^*\omega \in \Omega^n_c(\mathbb{R}^n)$. Let $f \in C^\infty(\phi(W))$ be a function so that

$$\int_{\mathbb{R}^n} \zeta = \int_{\mathbb{R}^n} f d\mathcal{L}^n$$

Then there exists $\xi \in \Omega^{n-1}_c(\mathbb{R}^n)$ so that $\zeta - f\mathrm{vol}_{\mathbb{R}^n} = d\xi$. Then $(\zeta - d\xi)_x = f(x)\mathrm{vol}_{\mathbb{R}^n} = 0$ for $x \notin \phi(W)$. Thus $\tau = \phi^*\xi$ satisfies the required conditions. □

Corollary 7.2.11 (Chain concentration)**.** *Let M be a connected smooth n-manifold and let $U_1, \ldots, U_k$ open subsets of M diffeomorphic to $\mathbb{R}^n$ so that $U_i \cap U_{i+1} \neq \emptyset$ for each $i = 1, \ldots, k-1$. Then, for each $\omega \in \Omega^n_c(U_1)$ there exists $\tau \in \Omega^{n-1}_c(U_1 \cup \cdots \cup U_k)$ so that* $\operatorname{spt}(\omega - d\tau) \subset U_k$.

Proof. Exercise. □

Lemma 7.2.12. *Let (M, ω_M) be an oriented n-manifiold and $\omega \in \Omega^n_c(M)$ so that*

$$\int_M \omega = 0.$$

Then there exists $\tau \in \Omega^{n-1}_c(M)$ so that $\omega = d\tau$.

Proof. Let (again) $\{(U_i, \phi_i)\}_{i\geq 0}$ be a countable positive atlas of M so that $\phi_i U_i = \mathbb{R}^n$.

Since ω has compact support, there exists a finite collection $\{U_{i_j}\}_{j=1,\ldots,\ell}$ which is an open cover of $\operatorname{spt}\omega$. We may assume that $i_j = j$ in our indexing.

Let $\{\lambda_i\}$ be a partition of unity with respect to $\{U_i\}_{i=1,\ldots,\ell}$ and let $\omega_i = \lambda_i \omega$ for $i = 1, \ldots, \ell$.

For each i, we may fix a chain $U_{i_1}, \ldots, U_{i_k}$, as in the chain lemma, so that $i_1 = i$ and $i_k = 0$. Then, by Corollary 7.2.11, there exists for each i an $(n-1)$-form $\xi_i \in \Omega_c^{n-1}(M)$ so that $\mathrm{spt}(\omega_i - d\xi_i) \subset U_0$. Let $\xi = \sum_{i=1}^{\ell} \xi_i$. Then, by Lemma 7.2.8,

$$\begin{aligned} \int_{\phi_0 U_0} (\phi_0^{-1})^*(\omega - d\xi) &= \int_{U_0} (\omega - d\xi) = \int_M (\omega - d\xi) \\ &= \left(\int_M \omega\right) - \left(\int_M d\tau\right) = -\int_M d\tau = 0. \end{aligned}$$

Then, by Lemma 7.1.8, there exists $\kappa \in \Omega_c^n(\phi_0 U_0)$ so that $(\phi_0^{-1})^*(\omega - d\tau) = d\kappa$. Thus

$$\omega = d\xi + d(\phi_0)^*\kappa.$$

The claim is proven. $\square$

Proof of Theorem 7.2.5. Since $\int_M$ is clearly surjective, Theorem 7.2.5 follows from Lemmas 7.2.8 and 7.2.12. $\square$

7.3 Integration on domains with smooth boundary

Definition 7.3.1. Let M be a smooth n-manifold. A subset $N \subset M$ is a *domain with smooth boundary* if for every $p \in N$ there exists a chart (U, ϕ) of M at p so that

$$\phi(U \cap N) = \phi(U) \cap \mathbb{R}^n_-. \tag{7.3.1}$$

where $\mathbb{R}^n_- = (-\infty, 0] \times \mathbb{R}^{n-1}$.

Remark 7.3.2. *In particular, a domain with smooth boundary in a smooth n-manifold is an n-manifold with boundary and ∂N (the topological boundary of N in M) is a smooth $(n-1)$-manifold with smooth atlas $\{(U \cap N, \phi|(U \cap N)) \colon (U, \phi) \in \mathcal{A}_M,\ U \cap N \neq \emptyset\}$, where $\mathcal{A}_M$ is a smooth structure of M.*

Definition 7.3.3. Let M be a smooth n-manifold and $N \subset M$ a domain with smooth (non-empty) boundary. A tangent vector $w \in T_pM$ at $p \in \partial N$ is *outward directed* if there exists a chart (U, ϕ) of M at p satisfying (7.3.1) so that $\mathrm{pr}_1(D\phi(w)) > 0$, where $\mathrm{pr}_1 \colon \mathbb{R}^n \to \mathbb{R}$, $(x_1, \ldots, x_n) \mapsto x_1$.

Lemma 7.3.4. *Let M be an orientable n-manifold for $n \geq 2$, $N \subset M$ a domain with smooth boundary, and $w \in T_pM$ an outward directed tangent vector at $p \in \partial N$. Then, for each chart (U, ϕ) satisfying* (7.3.1), $\mathrm{pr}_1(D_p\phi(w)) > 0$.

Proof. Exercise. □

Definition 7.3.5. Let M be a smooth n-manifold and $N \subset M$ a domain with smooth boundary. A smooth map $X\colon \partial N \to TM$ is a *vector field (with values in TM)* if $X(p) \in T_pM$ for each $p \in \partial N$.

Lemma 7.3.6. *Let M be an orientable n-manifold for $n \geq 2$ and $N \subset M$ a domain with smooth boundary. Then there exists a vector field$\nu_N\colon N \to TM$ with values in TM so that $\nu_N(p) \in T_pM$ is an outward directed vector at each $p \in N$.*

Proof. Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in I}$ be a (countable) collection of charts of M covering N so that $U_\alpha \cap N \neq \emptyset$ and satisfying (7.3.1) for each α. Let $\{u_\alpha\}_{\alpha\in I}$ be a partition of unity with respect to $\{U_\alpha\}_{\alpha\in I}$.

For each α, let $X_\alpha\colon U_\alpha \to TM$ be the smooth map, $p \mapsto (D\phi_\alpha)^{-1}(e_1)$. Then $X_\alpha(p)$ is outward directed at each $p \in U_\alpha$. Let $\nu_\alpha\colon N \to TM$ be the map, $\nu_\alpha(p) = u_\alpha(p)X_\alpha(p)$ for every $p \in U_\alpha$ and $\nu_\alpha(p) = 0$, otherwise. We define $\nu_N\colon N \to TM$ by $\nu_N = \sum_\alpha \nu_\alpha$. By Lemma 7.3.4, $\nu_N(p)$ is outward directed for each $p \in N$. The smoothness is left as an exercise for the reader. □

Definition 7.3.7. Let M be a smooth n-manifold and $N \subset M$ a domain with smooth boundary. The *contraction of $\omega \in \Omega^n(M)$ by a vector field $X\colon N \to TM$* is the $(n-1)$-form $(X \llcorner \omega) \in \Omega^{n-1}(N)$ defined by

$$(X \llcorner \omega)_p(v_1, \ldots, v_{n-1}) = \omega_p(X, v_1, \ldots, v_{n-1}).$$

Theorem 7.3.8. *Let (M, ω_M) be an oriented n-manifold for $n \geq 2$ and $N \subset M$ a domain with smooth boundary. Let $\nu_N\colon N \to TM$ be outward directed vector field. Then $\omega_N = \nu_N \llcorner \omega_M \in \Omega^{n-1}(N)$ is an orientation form of N. In particular, ∂N is orientable.*

Proof. Let $p \in N$. Since $\nu_N(p) \neq 0$, we find vectors $v_2, \ldots, v_n \in T_pM$ so that $(\nu_N(p), v_2, \ldots, v_n)$ is a basis of T_pM. Since $(\omega_M)_p \neq 0$, we have

$$(\omega_N)_p(v_2, \ldots, v_n) = (\omega_M)_p(\nu_N(p), v_2, \ldots, v_n) \neq 0.$$

We leave it to the reader to check that ω_N is smooth. □

Definition 7.3.9. Let (M, ω_M) is be an oriented n-manifold and $N \subset M$ a domain with smooth boundar, and $\nu_N\colon N \to TM$ an outward directed vector field. The form $\omega_N = \nu_N \llcorner \omega_M \in \Omega^{n-1}(N)$ is an *induced orientation on ∂N.*

Lemma 7.3.10. *Let (M, ω_M) be an oriented n-manifold, (N, ω_N) a domain with smooth boudnary and induced orientation. Let (U, ϕ) be a chart on M satisfying* (7.3.1) *which is orientation preserving map from $(U, \omega_M|U)$ to*

$(\mathbb{R}^n, dx_1 \wedge \cdots \wedge dx_n)$. *Then* $\phi|(U \cap \partial N)\colon (U \cap \partial N) \to \phi(U) \cap \{0\} \times \mathbb{R}^{n-1})$ *is a orientation preserving map from* $(U \cap N, \omega_N|(U \cap N))$ *to* $(\phi(U) \cap (\{0\} \times \mathbb{R}^{n-1}, dx_2 \wedge \cdots \wedge dx_n)$.

Proof. Since $\phi\colon U \to \phi U$ is orientation preserving, there exists positive $\lambda \in C^\infty(U)$ so that $\phi^*(dx_1 \wedge \cdots \wedge dx_n) = \lambda \omega_M|U$. Let ν_N be the outward directed vector field on N so that $\omega_N = \nu_N \llcorner \omega_M$.

Let $X\colon \phi(U) \cap (\{0\} \times \mathbb{R}^{n-1}) \to \mathbb{R}^n$ the vector field, $X = D\phi \circ \nu_N \circ \phi^{-1}$. Then $X = \sum_i u_i e_i$, where $u_1 \in C^\infty(\phi(U) \cap (\{0\} \times \mathbb{R}^{n-1}))$ is a positive function, since ν_N is outward directed. Thus

$$\begin{aligned}\omega_N &= \nu_N \llcorner \omega_M = (\phi|(\partial N \cap U))^* \left(X \llcorner dx_1 \wedge \cdots \wedge dx_n\right) \\ &= \lambda (u_1 \circ \phi) \left(\phi|(\partial N \cap U)\right)^* (dx_2 \wedge \cdots \wedge dx_n).\end{aligned}$$

Hence $\phi|(\partial N \cap U)$ is orientation preserving. $\square$

Theorem 7.3.11 (Stokes' theorem)**.** *Let* (M, ω_M) *be an oriented* n*-manifold,* $n \geq 2$*, and* (N, ω_N) *a domain with smooth boundary in* M *with an induced orientation. Let* $\iota\colon partial N \to M$ *be the inclusion. Then, for each* $\omega \in \Omega^{n-1}(M)$ *having* $N \cap \mathrm{spt}(\omega)$ *compact,*

$$\int_{\partial N} \iota^* \omega = \int_N d\omega.$$

The proof is based on a Euclidean lemma; proof is almost identical to Lemma 7.1.7.

Lemma 7.3.12. *Let* $\tau \in \Omega_c^{n-1}(\mathbb{R}^n)$*. Then*

$$\int_{\mathbb{R}^{n-1}} \iota_1^* \tau = \int_{\mathbb{R}^n_-} d\tau,$$

where $\iota_1\colon \mathbb{R}^{n-1} \to \mathbb{R}^n$ *is the inclusion.*

Proof. Let $f_i \in C_c^\infty(\mathbb{R}^n)$ be the functions satisfying

$$\tau = \sum_{i=1}^n f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

We observe first that

$$\iota_1^* \tau = f_1 dx_2 \wedge \cdots \wedge dx_n,$$

where $dx_j \in \Omega^1(\mathbb{R}^{n-1})$.

Then, by the fundamental theorem of calculus

$$\int_{-\infty}^0 \frac{\partial f_1}{\partial x_1}(t, x_2, \ldots, x_n) dt = f_1(0)$$

and

$$\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x_i}(x_1, \ldots, t, \ldots, x_n) dt = 0$$

for every $i = 2, \ldots, n$ and all $x_1, x_2, \ldots, x_n \in \mathbb{R}$.

Hence, by Fubini's theorem,

$$\begin{aligned}
\int_{\mathbb{R}^n_-} d\tau &= \sum_{i=1}^{n} \int_{\mathbb{R}^n_-} \frac{\partial f_i}{\partial x_i} \mathrm{d}\mathcal{L}^n \\
&= \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{0} \frac{\partial f_1}{\partial x_1}(t, x_2, \ldots, x_n) dt \right) \mathrm{d}\mathcal{L}^{n-1} \\
&= \int_{\mathbb{R}^{n-1}} f_1(x_2, \ldots, x_n) \mathcal{L}^{n-1} \\
&= \int_{\{0\} \times \mathbb{R}^{n-1}} \iota^* \tau.
\end{aligned}$$

□

Proof. Let $\mathcal{A}$ be a positive atlas on M so that $(U, \phi) \in \mathcal{A}$ satisfies (7.3.1), let $\mathcal{A}_N = \{(U, \phi) \in \mathcal{A} \colon U \cap N \neq \emptyset\} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ and $\mathcal{A}_{\partial N} = \{(U, \phi) \in \mathcal{A} \colon U \cap \partial N \neq \emptyset\} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in J}$, where $J \subset I$. Let $\{u_\alpha\}_{\alpha \in I}$ be a partition of unity with respect to $\{U_\alpha\}_{\alpha \in I}$.

For each $\alpha \in I$, let $\omega_\alpha = u_\alpha \omega$. Then

$$d\omega = \sum_{\alpha \in I} d\omega_\alpha$$

and

$$\iota^* \omega = \sum_{\alpha \in J} \iota^* \omega_\alpha.$$

Thus it suffices to show that

$$\int_{\partial N} \iota^* \omega_\alpha = \int_N d\omega_\alpha \tag{7.3.2}$$

for each $\alpha \in I$.

Suppose $\alpha \in I \setminus J$. Then $d\omega_\alpha \in \Omega^n_c(U_\alpha)$. Then

$$\int_N d\omega_\alpha = \int_{U_\alpha} d\omega_\alpha = 0$$

by Lemma 7.2.8. Thus (7.3.2) holds in this case

Suppose $\alpha \in J$. Let $\iota_1 \colon \mathbb{R}^{n-1} \to \mathbb{R}^n$ be the inclusion. Then $\iota_1 \circ \phi_\alpha | (\partial N \cap U_\alpha) = \phi_\alpha \circ \iota | (\partial N \cap U_\alpha)$.

Thus, by Lemma 7.3.12 and Lemma 7.3.10, we have

$$\begin{aligned}
\int_N d\omega_\alpha &= \int_{N\cap U_\alpha} d\omega_\alpha = \int_{\mathbb{R}^n_-\cap\phi_\alpha U_\alpha} (\phi_\alpha^{-1})^*\omega_\alpha \\
&= \int_{\mathbb{R}^{n-1}\cap\phi_\alpha U_\alpha} \iota_1^*(\phi_\alpha^{-1})^*\omega_\alpha \\
&= \int_{\partial N\cap U_\alpha} (\phi_\alpha|(\partial N\cap U_\alpha))^*\,\iota_1^*(\phi_\alpha^{-1})^*\omega_\alpha \\
&= \int_{\partial N\cap U_\alpha} (\iota|(\partial N\cap U_\alpha))^*\,\omega_\alpha \\
&= \int_{\partial N} \iota^*\omega_\alpha
\end{aligned}$$

The proof is complete. □

Chapter 8

Poincaré duality

Lemma 8.0.13. *Let M be an oriented n-manifold and $0 \leq k \leq n$. Suppose $\xi \in \Omega^k(M)$ and $\zeta \in \Omega_c^{n-k}(M)$ are both closed. Then for all $\alpha \in \Omega^{k-1}(M)$ and $\beta \in \Omega_c^{n-k-1}(M)$,*

$$\int_M (\xi + d\alpha) \wedge (\zeta + d\beta) = \int_M \xi \wedge \zeta.$$

Proof. Since $\xi \wedge \beta$ and $\alpha \wedge (\zeta + d\beta)$ have compact support, we have, by Lemma 7.2.8,

$$\begin{aligned}
\int_M (\xi + d\alpha) \wedge (\zeta + d\beta) &= \int_M \xi \wedge \zeta + \int_M \xi \wedge d\beta + \int_M d\alpha \wedge (\zeta + d\beta) \\
&= \int_M \xi \wedge \zeta + (-1)^k \int_M d(\xi \wedge \beta) + \int_M d(\alpha \wedge (\zeta + d\beta)) \\
&= \int_M \xi \wedge \zeta.
\end{aligned}$$

□

Thus the values of $I\colon \Omega^k(M) \times \Omega_c^{n-k}(M) \to \mathbb{R}$, defined by

$$(\xi, \zeta) \mapsto \int_M \xi \wedge \zeta,$$

depend only on the cohomology classes of ξ and ζ. Thus the (bilinear map) $\bar{I}\colon H^k(M) \times H_c^{n-k}(M) \to \mathbb{R}$,

$$([\xi], [\zeta]) \mapsto \int_M \xi \wedge \zeta$$

is well-defined. It is an interesting fact that $\bar{I}$ is a non-degenerate pairing.

Theorem 8.0.14. *Let M be a connected orientable n-manifold. Let $[\xi] \in H^k(M)$ be non-zero. Then there exists $[\zeta] \in H_c^{n-k}(M)$ so that $\bar{I}([xi], [\zeta]) \neq 0$. Similarly, given $[\zeta] \in H_c^{n-k}(M)$ non-zero, there exist $[\xi] \in H^k(M)$ so that $\bar{I}([\xi], [\zeta]) \neq 0$.*

This fact is better formulated as follows. Let $H_c^{n-k}(M)^*$ be the dual (vector) space of $H_c^{n-k}(M)$ and let $P_M\colon H^k(M) \to H_c^{n-k}(M)^*$ be the map

$$D_M([\xi])[\zeta] = \bar{I}([\xi],[\zeta]) = \int_M \xi \wedge \zeta.$$

Theorem 8.0.15 (Poincaré duality)**.** *Let M be a connected orientable n-manifold. Then*

$$D_M\colon H^k(M) \to H_c^{n-k}(M)^*$$

is an isomorphism.

Corollary 8.0.16. *Let M be a compact connected orientable n-manifold. Then*

$$H^k(M) \cong H^{n-k}(M)$$

for each $0 \le k \le n$.

Proof. Since $H_c^{n-k}(M) = H^{n-k}(M)$, it suffices to observe that $\dim H^k(M) < \infty$ for each k; for a proof using tubular neighborhoods, see [7, Proposition 9.25]. Then

$$H^{n-k}(M) \cong H_c^{n-k}(M) \cong H_c^{n-k}(M)^* \cong H^k(M).$$

□

Corollary 8.0.17. *Let M be an open connected orientable n-manifold. Then $H^n(M) = 0$.*

Proof. Since M is open, $H_c^0(M) = 0$ (Exercise). Thus $H^n(M) \cong H_c^0(M) = 0$. □

We dedicate the rest of this section for the proof of Theorem 8.0.15. The proof consists of three parts which are discussed in separate sections and the proof is then completed in a separate section.

8.1 Special case

Lemma 8.1.1. *Let M be a smooth connected and orientable n-manifold. Then $D_M\colon H^0(M) \to (H_c^n(M))^*$ is an isomorphism.*

Proof. Since $\dim H_c^n(M)^* = 1$ and $\int_M\colon H_c^n(M) \to \mathbb{R}$ is non-trivial, $\int_M$ spans $H_c^n(M)$. On the other hand, $[\chi_M]$ spans $H^0(M)$ and $D_{\mathbb{R}^n}[\chi_M] = \int_M$. The claim follows. □

Lemma 8.1.2. *For each $n \ge 1$,*

$$H_c^k(\mathbb{R}^n) \cong \begin{cases} \mathbb{R}, & k = n \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Exercise. □

Corollary 8.1.3. *Let M be a smooth orientable n-manifold and $U \subset M$ diffeomorphic to $\mathbb{R}^n$. Then $D_M \colon H^k(U) \to \left(H_c^{n-k}(U)\right)^*$ is an isomorphism for every $k = 0, \ldots, n$.*

8.2 Exact sequence for compactly supported cohomology

Definition 8.2.1. Let M be a smooth manifold, $U \subset M$ an open set, and $\iota \colon U \to M$ the inclusion. The map $\iota_* \colon \Omega_c^k(U) \to \Omega_c^k(M)$ defined by

$$(\iota_*\omega)_p = \begin{cases} \omega_p, & p \in U \\ 0, & p \notin U \end{cases}$$

is called a *push-forward*. The map $\iota_* \colon H_c^k(U) \to H_c^k(M)$, $[\omega] \mapsto [\iota_*\omega]$ is called the *direct image homomorphism*.

Theorem 8.2.2. *Let M be a smooth n-manifold and let U_1 and U_2 be open sets in M. Let $i_\nu \colon U_\nu \to U_1 \cup U_2$ and $j_\nu \colon U_1 \cap U_2 \to U_\nu$ be inclusions, and define $J_k \colon \Omega_c^k(U_1 \cap U_2) \to \Omega_c^k(U_1) \oplus \Omega_c^k(U_2)$ and $I_K \colon \Omega_c^k(U_1) \oplus \Omega_c^k(U_2) \to \Omega_c^k(U_1 \cup U_2)$ by $J_k(\omega) = ((j_1)_*\omega, -(j_2)_*\omega)$ and $I_k(\omega_1, \omega_2) = (i_1)_*\omega_1 + (i_2)_*\omega_2$.*

Then the sequence

(8.2.1)

$$0 \longrightarrow \Omega_c^k(U_1 \cap U_2) \xrightarrow{J_k} \Omega_c^k(U_1) \oplus \Omega_c^k(U_2) \xrightarrow{I_k} \Omega_c^k(U_1 \cup U_2) \longrightarrow 0$$

is exact.

Proof. Exercise. □

We define now the connecting homomorphism

$$\partial_* \colon H_c^k(U_1 \cup U_2) \to H_c^{k+1}(U_1 \cap U_2)$$

for the short exact sequence (8.2.1) in the usual manner as follows. Let $\omega \in \Omega_c^k(U_1 \cup U_2)$. Let $(\tau_1, \tau_2) \in \Omega_c^k(U_1) \oplus \Omega_c^k(U_2)$ be such that $I_k(\tau_1, \tau_2) = \omega$. Let also $\kappa \in \Omega_c^k(U_1 \cap U_2)$ be such that $J_{k+1}(\kappa) = (d\tau_1, d\tau_2)$. Then κ is closed and the cohomology class of κ in $H_c^{k+1}(U_1 \cap U_2)$ depends only the cohomology class of ω in $H_c^k(U_1 \cup U_2)$. (The general proof for chain complexes applies.) We define $\partial_*[\omega] = [\kappa]$.

By the theorem on Long Exact Sequence (Theorem 3.3.1), we have the following corollary.

Corollary 8.2.3. *Using the notation of Theorem 8.2.2, there is an exact sequence*

$$\longrightarrow H_c^k(U_1 \cap U_2) \xrightarrow{J_*} H_c^k(U_1) \oplus H_c^k(U_2) \xrightarrow{I_*} H_c^k(U_1 \cup U_2) \xrightarrow{\partial_*} H_c^{k+1}(U_1 \cap U_2) \longrightarrow$$

8.2.1 Duality of connecting homomorphisms

We need an observation which we formalize as a lemma.

Lemma 8.2.4. *Let M be a smooth manifold, $U \subset M$ an open set, and $\iota\colon U \to M$ the inclusion. Then*

$$(\iota^*\omega \wedge \tau)_p = (\omega \wedge \iota_*\tau)_p$$

for all $\omega \in \Omega^k(M)$ and $\tau \in \Omega_c^{n-k}(U)$ and $p \in U$. In particular,

$$\int_U \iota^*\omega \wedge \tau = \int_M \omega \wedge \iota_*\tau$$

Proof. The first claim follows from observation that $D_p\iota = \mathrm{id}$ and the second claim from the observation that $\omega \wedge \iota_*\tau = 0$ for $p \notin U$. $\square$

Lemma 8.2.5. *Let M be an oriented and $U_1 \subset M$ and $U_2 \subset M$ be open sets. Let $\partial^*\colon H^k(U_1 \cap U_2) \to H^{k+1}(U_1 \cup U_2)$ and $\partial_*\colon H_c^{n-k}(U_1 \cup U_2) \to H_c^{n-(k+1)}(U_1 \cap U_2)$ be connecting homomorphisms for the corresponding Meyer–Vietoris sequences. Then*

$$\int_{U_1\cup U_2} \partial^*[\omega] \wedge [\tau] = (-1)^{k+1} \int_{U_1\cap U_2} [\omega] \wedge \partial_*[\tau]$$

for $[\omega] \in H^k(U_1 \cap U_2)$ and $[\tau] \in H_c^{n-(k+1)}(U_1 \cup U_2)$.

Proof. Let $\omega \in \Omega^k(U_1 \cap U_2)$ and $\tau \in \Omega_c^{n-(k+1)}(U_1 \cup U_2)$ be closed forms.

By definition of ∂^*, $\partial^*[\omega] = [\kappa]$ where $\kappa \in \Omega^{k+1}(U_1 \cup U_2)$ is defined by $i_\nu^*(\kappa) = d\omega_\nu$, where forms $\omega_\nu \in \Omega^k(U_\nu)$ satisfy $\omega = j_2^*(\omega_2) - j_1^*(\omega_1)$. Here $i_\nu\colon U_\nu \to U_1 \cup U_2$ and $j_\nu\colon U_1 \cap U_2 \to U_\nu$ are the inclusions in the Mayer–Vietoris theorem.

Similarly, by definition of ∂_*, $\partial_*[\tau] = [\sigma]$, where $\sigma \in \Omega_c^{n-(k-1)}(U_1 \cup U_2)$ satisfies $-(j_1)_*\sigma = (j_2)_*\sigma = d\tau_\nu$, where $\tau_\nu \in \Omega_c^{n-k}(U_\nu)$ satisfy $(i_2)_*\tau_2 + (i_1)_*\tau_1 = \tau$.

We need to show that

$$\int_{U_1\cup U_2} \kappa \wedge \tau = (-1)^{k+1} \int_{U_1\cap U_2} \omega \wedge \sigma.$$

We observe first that

$$\int_{U_1\cup U_2} \kappa \wedge \tau = \int_{U_1} \kappa \wedge \tau_1 + \int_{U_2} \kappa \wedge \tau_2 = \int_{U_1} d\omega_1 \wedge \tau_1 + \int_{U_2} d\omega_2 \wedge \tau_2.$$

Since

$$d(\omega_\nu \wedge \tau_\nu) = d\omega_\nu \wedge \tau_\nu + (-1)^k \omega_\nu \wedge d\tau_\nu$$

for $\nu = 1, 2$ and $(j_2)_*\sigma = -d\tau_2$, we have, by Lemma 8.2.4,

$$\begin{aligned}
\int_{U_1\cup U_2} \kappa\wedge\tau &= \int_{U_1} d\omega_1\wedge\tau_1 + \int_{U_2} d\omega_2\wedge\tau_2 \\
&= -(-1)^k\int_{U_1}\omega_1\wedge d\tau_1 - (-1)^k\int_{U_2}\omega_2\wedge d\tau_2 \\
&= (-1)^{k+1}\int_{U_1}\omega_1\wedge (j_1)_*\sigma - (-1)^{k+1}\int_{U_2}\omega_2\wedge (j_2)_*\sigma \\
&= (-1)^{k+1}\int_{U_1\cap U_2}(j_1)^*\omega_1\wedge\sigma - (-1)^{k+1}\int_{U_1\cap U_2}(j_2)^*\omega_2\wedge\sigma \\
&= (-1)^{k+1}\int_{U_1\cap U_2}((j_1)^*\omega_1 - (j_2)^*\omega_2)\wedge\sigma \\
&= (-1)^{k+1}\int_{U_1\cap U_2}\omega\wedge\sigma.
\end{aligned}$$

This completes the proof. □

8.3 Exact sequence for dual spaces

We use the following notation. Let $\iota\colon U\to M$ be an inclusion of an open set $U\subset M$ to an n-manifold M. We denote by $\iota^!\colon \Omega_c^k(M)^*\to\Omega_c^k(U)^*$ the dual map of $\iota_*\colon\Omega_c^k(U)\to\Omega_c^k(M)$, that is, $\iota^! = (\iota_*)^*$ and more precisely, for $L\in\Omega_c^k(M)^*$ and $\omega\in\Omega_c^k(U)$, we have

$$(\iota^!(L))(\omega) = L(\iota_*\omega).$$

Theorem 8.3.1. *Let M be a smooth n-manifold, and let $U_1\subset M$ and $U_2\subset M$ be open sets. Let $J\colon H_c^k(U_1\cap U_2)\to H_c^k(U_1)\oplus H_c^k(U_2)$, $I\colon H_c^k(U_1)\oplus H_c^k(U_2)\to H_c^k(U_1\cup U_2)$, and $\partial_*\colon H_c^k(U_1\cup U_2)\to H_c^{k+1}(U_1\cap U_2)$ be as in Theorem 8.2.2. Then the sequence*

$$\cdots\longrightarrow H_c^{k+1}(U_1\cap U_2)^*\xrightarrow{\partial^!} H_c^k(U_1\cup U_2)^*\xrightarrow{I^!} H_c^k(U_1)^*\oplus H_c^k(U_2)^*\xrightarrow{J^!} H_c^k(U_1\cap U_2)^*\longrightarrow\cdots$$

is exact, where $I^!(\alpha) = (i_1^!(\alpha), i_2^!(\alpha))$, $J^!(\alpha_1,\alpha_2) = j_1^!(\alpha_1) - j_2^!(\alpha_2)$, and $\partial^! = (\partial_*)^*$.

This theorem is an immediate corollary of the following algebraic lemma.

Lemma 8.3.2. *Let $A\xrightarrow{\varphi} B\xrightarrow{\psi} C$ be an exact sequence of vector spaces. Then $C^*\xrightarrow{\psi^*} B^*\xrightarrow{\varphi^*} A^*$ is an exact sequence of dual spaces.*

Recall that if $\varphi\colon A\to B$ is a linear map between vector spaces, then $\varphi^*\colon B^*\to A^*$ is the map $f\mapsto f\circ\varphi$.

Proof of Lemma 8.3.2. Since $\varphi^* \circ \psi^* = (\psi \circ \varphi)^* = 0$, the image of ψ^* is contained in the kernel of φ^*. Suppose now that $f \in \ker \varphi^*$, that is, $f : B \to \mathbb{R}$ a linear map so that $f \circ \varphi = 0$. Thus imege of φ is contained in $\ker f$. Let $\bar{\psi}: B/\ker\psi \to \mathrm{Im}\psi$ be isomorphism

$$\begin{tikzcd} B \arrow[rr, "\psi"] \arrow[dr] & & \mathrm{Im}\psi \\ & B/\ker\psi \arrow[ur, "\bar{\psi}"'] & \end{tikzcd}$$

Since $\ker\psi = \mathrm{Im}\varphi \subset \ker f$, there exists a linear map $\bar{f}: B/\ker\psi \to \mathbb{R}$ so that

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr] & & \mathbb{R} \\ & B/\ker\psi \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

and define $\tilde{f}: \mathrm{Im}\psi \to \mathbb{R}$ by $\tilde{f} = \bar{f} \circ \bar{\psi}^{-1}$. Then $\tilde{f} \circ \psi = f$.

Extend $\tilde{f}$ to a linear map $\hat{f}: C \to \mathbb{R}$ so that $\hat{f}|\mathrm{Im}\psi = \tilde{f}$. (Extend the basis of $\mathrm{Im}\psi$ to a basis of C and the map f.) Now $\psi^*(\hat{f}) = \hat{f} \circ \psi = \tilde{f} \circ \psi = f$. Thus $f \in \mathrm{Im}\psi^*$ and the sequence is exact. $\square$

8.4 Main steps

8.4.1 From intersections to unions

In this section, we prove that if open sets U_1, U_2 and $U_1 \cap U_2$ satisfy the Poincaré duality, the so does $U_1 \cup U_2$. We formalize this as follows.

Proposition 8.4.1. *Let M be an orientable n-manifold. Let U_1 and U_2 be open sets in M so that $D_{U_\nu}: H^k(U_\nu) \to H_c^{n-k}(U_\nu)^*$ and $D_{U_1\cap U_2}: H^k(U_1 \cap U_2) \to H_c^{n-k}(U_1 \cap U_2)^*$ are isomorphisms for each k. Then $D_{U_1\cup U_2}: H^k(U_1 \cup U_2) \to H_c^{n-k}(U_1 \cup U_2)^*$ is an isomorphism for each k.*

We show that the diagram

$$\begin{tikzcd} H^k(U_1 \cup U_2) \arrow[r, "I_*"] \arrow[d, "D_{U_1\cup U_2}"] & H^k(U_1) \oplus H^k(U_2) \arrow[r, "J_*"] \arrow[d, "D_{U_1}\oplus D_{U_2}"] & H^k(U_1 \cap U_2) \arrow[r, "\partial_*"] \arrow[d, "D_{U_1\cap U_2}"] & H^{k+1}(U_1 \cup U_2) \arrow[d, "D_{U_1\cup U_2}"] \\ H_c^{n-k}(U_1 \cup U_2)^* \arrow[r, "I^!"] & H_c^{n-k}(U_1)^* \oplus H_c^{n-k}(U_2)^* \arrow[r, "J^!"] & H_c^{n-k}(U_1 \cap U_2)^* \arrow[r, "(-1)^{k+1}\partial^!"] & H_c^{n-(k+1)}(U_1 \cup U_2)^* \end{tikzcd}$$

commutes. The claim follows then from the 5-lemma. We begin with an auxiliary result.

Lemma 8.4.2. *Let M be an oriented n-manifold, $V \subset U \subset M$ open sets and $i\colon V \to U$ the inclusion. Then the diagram*

$$\begin{tikzcd} H^k(U) \arrow[r, "i^*"] \arrow[d, "D_U"] & H^k(V) \arrow[d, "D_V"] \\ H_c^{n-k}(U)^* \arrow[r, "i^!"] & H_c^{n-k}(V)^* \end{tikzcd}$$

commutes.

Proof. Let $\omega \in \Omega^k(U)$ and $\tau \in \Omega_c^{n-k}(V)$. Then

$$((D_V \circ i^*)[\omega])\,([\tau]) = D_V\,([i^*\omega])\,[\tau] = \int_V i^*\omega \wedge \tau = \int_U \omega \wedge i_*\tau.$$

by Lemma 8.2.4. On the other hand,

$$\left(i^! \circ D_U[\omega]\right)[\tau] = D_U[\omega](i_*[\tau]) = \int_U \omega \wedge i_*\tau.$$

The claim follows. □

Lemma 8.4.3. *Let M be an oriented n-manifold. Then, for open sets U_1 and U_2 in M, the diagram*

$$\begin{tikzcd} H^k(U_1 \cap U_2) \arrow[r, "\partial^*"] \arrow[d, "D_{U_1\cap U_2}"] & H^{k+1}(U_1 \cup U_2) \arrow[d, "D_{U_1\cup U_2}"] \\ H_c^{n-k}(U_1 \cap U_2)^* \arrow[r, "(-1)^{k+1}\partial^!"] & H_c^{n-(k+1)}(U_1 \cup U_2)^* \end{tikzcd}$$

is commutes.

Proof. We need to show that

$$D_{U_1\cup U_2}\partial^*[\omega]([\tau]) = (-1)^{k+1}\left(\partial^! D_{U_1\cap U_2}[\omega]\right)[\tau]$$

for $[\omega] \in H^k(U_1 \cap U_2)$ and $[\tau] \in H_c^{n-(k+1)}(U_1 \cup U_2)$.

Since

(8.4.1) $$(D_{U_1\cup U_2}\partial^*[\omega])\,[\tau] = \int_{U_1\cup U_2} \partial^*[\omega] \wedge \tau$$

and

(8.4.2) $$\partial^!\,(D_{U_1\cap U_2}[\omega])\,[\tau] = D_{U_1\cap U_2}([\omega])(\partial_*[\tau]) = \int_{U_1\cap U_2} [\omega] \wedge \partial_*[\tau],$$

the claim follows from Lemma 8.2.5. □

Proof of Proposition 8.4.1. To show the commutativity of (8.4.1), it suffises to apply Lemma 8.4.2 to diagrams

$$\begin{tikzcd}
H^k(U_1 \cup U_2) \arrow[r, "i_\nu^*"] \arrow[d, "D_{U_1\cup U_2}"] & H^k(U_\nu) \arrow[d, "D_{U_\nu}"] \\
H_c^{n-k}(U_1 \cup U_2)^* \arrow[r, "i^!"] & H_c^{n-k}(U_\nu)^*
\end{tikzcd}$$

and

$$\begin{tikzcd}
H^k(U_\nu) \arrow[r, "j_\nu^*"] \arrow[d, "D_{U_\nu}"] & H^k(U_1 \cup U_2) \arrow[d, "D_{U_1\cup U_2}"] \\
H_c^{n-k}(U_\nu)^* \arrow[r, "j_\nu^!"] & H_c^{n-k}(U_1 \cup U_2)^*
\end{tikzcd}$$

for $\nu = 1, 2$ and combine results.

Since $D_{U_1} \oplus D_{U_2}$ and $D_{U_1 \cap U_2}$ are isomorphisms, so is $D_{U_1 \cap U_2}$ by the 5-lemma. $\square$

8.4.2 Disjoint unions

Proposition 8.4.4. *Let M be an orientable n-manifold. Suppose $\{U_\alpha\}_\alpha$ is a collection of pair-wise disjoint sets so that $D_{U_\alpha}\colon H^k(U_\alpha) \to H_c^{n-k}(U_\alpha)^*$ is an isomorphism for each k. Then $D_U\colon H^k(U) \to H_c^{n-k}(U)^*$ is an isomorphism for each k, where $U = \bigcup_\alpha U_\alpha$.*

Again we need a linear algebraic fact. Let I be a set and V_α be a vector space for each $\alpha \in I$. For $v = (v_\alpha)_{\alpha\in I} \in \prod_{\alpha \in I} V_\alpha$, denote by $\mathrm{spt}(v) = \{\alpha \in I\colon v_\alpha \neq 0\}$ the *support of v*. Recall that

$$\bigoplus_{\alpha\in I} V_\alpha = \{(v_\alpha)_\alpha \in \prod_\alpha V_\alpha\colon \#\,\mathrm{spt}(v_\alpha)_\alpha < \infty\}.$$

For each $\beta \in I$, we have the natural inclusion $i_\beta\colon V_\beta \to \bigoplus_{\alpha\in I} V_\alpha$ by $v \mapsto (v_\alpha)_\alpha$, where $v_\alpha = v$ if $\alpha = \beta$ and 0 otherwise. We also have the natural projection $p_\beta\colon \bigoplus_{\alpha\in I} V_\alpha \to V_\beta$, $(v_\alpha)_\alpha \mapsto v_\beta$. Clearly, $p_\beta \circ i_\beta = \mathrm{id}$ and $p_\gamma \circ i_\beta = 0$ for each $\beta, \gamma \in I$ and $\beta \neq \gamma$. Moreover,

$$\sum_\alpha i_\alpha \circ p_\alpha = \mathrm{id}\colon \bigoplus_\alpha V_\alpha \to \bigoplus_\alpha V_\alpha.$$

Note that given a $L_\alpha \in V_\alpha^*$ for each $\alpha \in I$, $(L_\alpha)_\alpha \in \prod_{\alpha\in I} (V_\alpha)^*$.

Lemma 8.4.5. *Let I be a set and V_α be a vector space for each $\alpha \in I$. Then*

$$\delta\colon \left(\bigoplus_{\alpha\in I} V_\alpha\right)^* \to \prod_{\alpha\in I} V_\alpha^*, \quad L \mapsto (L \circ i_\alpha)_\alpha$$

is an isomorphism with the inverse

$$\delta' \colon \prod_{\alpha\in I} V_\alpha^* \to \left(\bigoplus_{\alpha\in I} V_\alpha\right)^*, \quad (L_\alpha)_\alpha \mapsto \sum_{\alpha\in I} L_\alpha \circ p_\alpha$$

Proof. We observe first that the mapping δ' is well-defined. Since each $v = (v_\alpha)_\alpha \in \bigoplus_{\alpha\in I} V_\alpha$ has finite support, $L_\alpha(p_\alpha(v)) \neq 0$ only for finitely many $\alpha \in I$. Thus $\delta'(L)(v)$ is well-defined for each $L \in \prod_\alpha V_\alpha^*$ and $v \in \bigoplus_\alpha V_\alpha$.

Secondly, we observe that

$$\delta \circ \delta' \left((L_\alpha)_\alpha\right) = \delta\left(\sum_\beta L_\beta \circ p_\beta\right) = \left(\sum_\beta L_\beta \circ p_\beta \circ i_\alpha\right)_\alpha = (L_\alpha)_\alpha$$

for $(L_\alpha)_\alpha \in \prod_\alpha V_\alpha^*$, and

$$\delta' \circ \delta\,(L) = \delta'\left((L \circ i_\alpha)\right) = \sum_\alpha L \circ i_\alpha \circ p_\alpha = L\left(\sum_\alpha i_\alpha \circ p_\alpha\right) = L$$

for $L \in (\bigoplus_\alpha V_\alpha)^*$.

Since δ and δ' are clearly linear, the claim follows. $\square$

Proof of Proposition 8.4.4. Let $\iota_\alpha \colon U_\alpha \to U$. By Lemma 8.4.2, we have for each $\alpha \in I$,

(8.4.3)
$$\begin{tikzcd} H^k(U) \arrow[r, "\iota_\alpha^*"] \arrow[d, "D_U"] & H^k(U_\alpha) \arrow[d, "D_{U_\alpha}"] \\ H_c^{n-k}(U)^* \arrow[r, "\iota_\alpha^!"] & H_c^{n-k}(U_\alpha)^* \end{tikzcd}$$

We show that there exists a commutative diagram

$$\begin{tikzcd} H^k(U) \arrow[rr, "\theta"] \arrow[d, "D_U"] & & \prod_\alpha H^k(U_\alpha) \arrow[d, "\prod D_{U_\alpha}"] \\ H_c^{n-k}(U)^* \arrow[r, "\rho^*"] & \left(\bigoplus_\alpha H_c^{n-k}(U_\alpha)\right)^* \arrow[r, "\delta"] & \prod_\alpha H_c^{n-k}(U_\alpha)^* \end{tikzcd}$$

where θ and ρ^* are isomorphisms and δ is the isomorphism in Lemma 8.4.5. Since $\prod_\alpha D_{U_\alpha}$ is an isomorphism, so is then D_U.

Since sets U_α are components of U, we have that the linear map

$$\theta \colon H^k(U) \to \prod_\alpha H^k(U_\alpha), \quad [\omega] \to \left(\iota_\alpha^*[\omega]\right),$$

is an isomorphism. Indeed, suppose $\theta[\omega] = 0$. Then $\iota_\alpha^*\omega$ is exact, that is, $\omega|U_\alpha$ is exact for each α. But then ω is exact and $[\omega] = 0$. Suppose

$([\omega_\alpha])_\alpha \in \prod_\alpha H^k(U_\alpha)$. Define $\omega \in \Omega^k(U)$ so that $\omega|U_\alpha = \omega_\alpha$. Since sets U_α are pair-wise disjoint, ω is well-defined, closed, and satisfies $\theta[\omega] = ([\omega_\alpha])_\alpha$.

Similarly, let

$$\rho\colon \bigoplus_\alpha H_c^k(U_\alpha) \to H_c^k(U), \quad ([\omega_\alpha])_\alpha \mapsto \sum_\alpha (\iota_\alpha)_*[\omega_\alpha];$$

note that the sum is finite by definition of the direct sum. Also ρ is an isomorphism. Indeed, suppose $\rho([\omega_\alpha])_\alpha = 0$. Then $\sum_\alpha (\iota_\alpha)_*\omega_\alpha$ is exact. But then also each ω_α is exact and $[\omega_\alpha] = 0$ for each α. Let $[\omega] \in H_c^{n-k}(U)$. Since ω is compactly supported, the support meets only finitely many sets U_α. Thus $([\omega_\alpha])_\alpha$, where $\omega_\alpha = \omega|U_\alpha$, is in $\oplus_\alpha H_c^k(U_\alpha)$.

Furthermore, $\rho^*(L) = \sum_\alpha \iota_\alpha^!(L)\circ p_\alpha$, where $p_\alpha\colon \bigoplus_\beta H_c^{n-k}(U_\beta) \to H_c^{n-k}(U_\alpha)$ is the canonical projection. Indeed, let $([\omega_\alpha])_\alpha \in \oplus_\alpha H_c^k(U_\alpha)$. Then

$$\begin{aligned}
\rho^*(L)\,([\omega_\alpha])_\alpha &= L\left(\sum_\alpha (\iota_\alpha)_*[\omega_\alpha]\right) = \sum_\alpha L((\iota_\alpha)_*[\omega_\alpha]) \\
&= \sum_\alpha (\iota_\alpha^! L)[\omega_\alpha] = \left(\sum_\alpha (\iota_\alpha^! L)\circ p_\alpha\right)([\omega_\alpha])_\alpha.
\end{aligned}$$

Then, for $[\omega] \in H^k(U)$,

$$\begin{aligned}
\left(\prod_\alpha D_{U_\alpha}\right)(\theta[\omega]) &= \left(\prod_\alpha D_{U_\alpha}\right)(\iota_\alpha^*[\omega])_\alpha = (D_{U_\alpha}\iota_\alpha^*[\omega])_\alpha \\
&= \left(\iota_\alpha^! D_U[\omega]\right)_\alpha = \delta\left(\sum_\alpha (\iota_\alpha^! D_U[\omega])\circ p_\alpha\right) \\
&= \delta\circ\rho^*(D_U[\omega]).
\end{aligned}$$

The claim follows. □

8.4.3 Proof of the Poincaré duality

We gather first the fact we have gathered so far. Let M be a smooth oriented n-manifold and $U \subset M$ an open set. We say that U *satisfies the Poincaré duality if $D_U\colon H^k(U) \to H_c^{n-k}(U)^*$ is an isomorphism for each k.*

Proposition 8.4.6. *Let M be a smooth oriented n-manifold and $U \subset M$ an open set. Then U satisfies the Poincaré duality if*

(i) U is diffeomorphic to $\mathbb{R}^n$,

(ii) $U = V \cup V'$, where V, V', and $V \cap V'$ satisfy the Poincaré duality, or

(iii) U is a disjoint union $\bigcup_i U_i$, where each U_i satisfies the Poincaré duality.

Proof. This is a combination of Corollary 8.1.3 and Propositions 8.4.1 and 8.4.4. $\square$

Let

$$\mathcal{U} = \{U \subset M \colon U \text{ is an open set satisfying the Poincaré duality}\}.$$

It suffices now to show that $M \in \mathcal{U}$. This is a consequence of a general result in topology; we apply this theorem to $\mathcal{U}$ above and to $\mathcal{V} = \{M\}$ to obtain the Poincaré duality.

Theorem 8.4.7. *Let M be a smooth n-dimensional manifold and $\mathcal{V} = \{V_i\}_{i\in I}$ an open cover of M. Suppose $\mathcal{U}$ is a collection of open sets of M satisfying the following conditions:*

(a) $\emptyset \in \mathcal{U}$,

(b) for every $i \in I$, any open set $U \subset V_i$ diffeomorphic to $\mathbb{R}^n$ belongs to $\mathcal{U}$,

(c) if $U, U', U \cap U' \in \mathcal{U}$ then $U \cup U' \in \mathcal{U}$,

(d) if $U_k \in \mathcal{U}$, $k \geq 0$, are disjoint sets then $\bigcup_k U_k \in \mathcal{U}$.

Then $M \in \mathcal{U}$.

We need a lemma.

Lemma 8.4.8. *Let $\mathcal{U}$ be as in Theorem 8.4.7 and $U_1, U_2, \ldots$ a sequence of open relatively compact subsets of M with*

(i) $\bigcap_{j\in J} U_j \in \mathcal{U}$ for every finite $J \subset \mathbb{N}$, and

(ii) $(U_i)_{i\geq 1}$ is locally finite.

Then $\bigcup_{i\geq 1} U_i \in \mathcal{U}$.

Proof. (See [7, Lemma 13.10].)

Let $I_1 = \{1\}$ and $W_1 = U_1$. For $m > 1$ define $W_m = \bigcup_{i\in I_m} U_i$, where

$$I_m = \{m\} \cup \{i > m \colon U_i \cap W_{m-1} \neq \emptyset\} \setminus \bigcup_{j=1}^{m-1} I_j \subset \mathbb{N}.$$

We make first some observations on sets I_m and W_m:

(1) each I_m is finite,

(2) $\mathbb{Z}_+ = \{1, 2, \ldots\} = \bigcup_{m=1}^{\infty} I_m$,

(3) sets I_m are disjoint, and

(4) $W_m \cap W_k = \emptyset$ for $|k-m| \geq 2$.

Here (1) is an induction: Clearly, I_1 is finite. Suppose I_{m-1} is finite. Then W_{m-1} is relatively compact. Thus by local finiteness of $(U_i)_{i\geq 1}$, W_{m-1} intersects only finitely many set U_i. Thus I_m is finite.

For (2), we note that each $m \in \mathbb{Z}_+$ belongs either to I_m or to some I_k for $k < m$ by construction.

Condition (3) is clearly true by construction, and, finally, for (4), we may assume that $k \geq m+2$. If $W_m \cap W_k \neq \emptyset$, there exists $i \in I_k$ so that so that $U_i \cap W_m \neq \emptyset$. But then either $i \in I_{m+1}$ or alredy in I_j for some $j \leq m$ by construction. This is a contradiction since sets I_m are disjoint.

Let $V = \bigcup_{m\geq 1} W_{2m}$ and $V' = \bigcup_{m\geq 1} W_{2m-1}$. Note that by (4) these are disjoint unions.

We show first that sets W_m belong to $\mathcal{U}$ for each $m \geq 1$. Then V and V' belong to $\mathcal{U}$ by condition (d) in Theorem 8.4.7. We show then that $V \cap V'$ belongs to $\mathcal{U}$. Then V, V', and $V \cap V'$ belong to $\mathcal{U}$. Hence also $V \cup V' = \bigcup_{i\geq 1} U_i$ belongs to $\mathcal{U}$, which concludes the proof of this lemma.

We prove these claims with following two auxiliary statements.

Claim 1: For every finite set $J \subset \mathbb{Z}_+$, the union $\bigcup_{j\in J} U_j$ belongs to $\mathcal{U}$.

Proof: The proof is an induction on the size of J. By (i) and (c) in Theorem 8.4.7, $\bigcup_{j\in J} U_j \in \mathcal{U}$ for $\#J \leq 2$. Suppose now that the claim holds for all sets J with cardianality at most $m \geq 2$. Let $J = \{j_1, \ldots, j_{m+1}\} \subset \mathbb{Z}_+$ be a set of cardinality $m+1$. Define $U' = U_{j_1} \cup \cdots U_{j_m}$. Then, by induction assumption $U' \in \mathcal{U}$. Since $U' \cap U_{m+1} \in \mathcal{U}$ by (i), we have that $U' \cup U \in \mathcal{U}$ by (c) in Theorem 8.4.7. Thus $\bigcup_{j\in J} U_j \in \mathcal{U}$. $\square$

Thus $W_m \in \mathcal{U}$.

Claim 2: Let $J = \{j_1, \ldots, j_m\}$ and $J' = \{j'_1, \ldots, j'_m\}$ be finite sets of at most m elements in $\mathbb{Z}_+$, then $\left(\bigcup_{k=1}^m U_{j_k} \cap U_{j'_k}\right) \in \mathcal{U}$.

Proof: Again the proof is by induction on m. By (i), the claim holds for $m = 1$. Suppose now that it holds for $m \geq 1$. Let $J = \{j_1, \ldots, j_{m+1}\}$ and $J' = \{j'_1, \ldots, j'_{m+1}\}$ be sets of $m+1$ elements and define $A = \bigcup_{k=1}^m U_{j_k} \cap U_{j'_k}$. Then $A \in \mathcal{U}$ by induction assumption. Since $B = U_{j_{m+1}} \cap U_{j'_{m+1}} \in \mathcal{U}$ and $A \cap B \in \mathcal{U}$ by (i), we have that $A \cup B \in \mathcal{U}$ by (c) in Theorem 8.4.7. $\square$

To show that $W_m \cap W_{m+1} \in \mathcal{U}$, let $\mu = \max\{\#I_m, \#I_{m+1}\}$. We may assume that $\#I_{m+1} > \#I_m$. Thus, by repeating the indices in I_m, Claim 2 yields $W_m \cap W_{m+1} \in \mathcal{U}$. This concludes the proof of the lemma. $\square$

Proposition 8.4.9. *Theorem 8.4.7 holds for manifolds M which are open subsets of Euclidean spaces.*

Proof of Theorem 8.4.7. Let $M = W \subset \mathbb{R}^n$ be an open set. We cover W with open subsets diffeomorphic to $\mathbb{R}^n$ so that the finite intersections of these sets are also diffeomorprhic to $\mathbb{R}^n$. The claim the follows from Lemma 8.4.8.

(By a modification of an old HW problem) there exists a locally finite sequence U_j, $j \geq 0$, of open balls, in the sup-norm of $\mathbb{R}^n$, so that

(a) $W = \bigcup_{j\geq 0} U_j = \bigcup_{j \geq 0} \overline{U_j}$,

(b) Each U_j is contained in at least one V_β for $\beta \in I$.

Since $U_j = \prod_{i=1}^n (a_{j,i}, b_{j,i})$ for some $a_{j,i} < b_{j,i}$, we have that $U_{j_1} \cap \cdots \cap U_{j_k}$ is either empty or of the form $\prod_i (a_i, b_i)$. Thus, for every finite $J \subset \mathbb{N}$, $\bigcap_{j\in J} U_j$ is diffeomorphic to $\mathbb{R}^n$ (or empty). Hence, by Lemma 8.4.8, $W = \bigcup_{j\geq 0} U_j \in \mathcal{U}$. □

Proof of Theorem 8.4.7. The idea is to show that all coordinate neighborhoods belong to $\mathcal{U}$. Note that intersections of coordianate neigborhoods are also coordinate neigbhorhoods.

Let (V, h) be a chart in M and denote $W = hV$. Let $\mathcal{W} = \{W' \subset W : h^{-1}W' \in \mathcal{U}\}$ and $\mathcal{V}' = \{h^{-1}V_i\}_i$. Then $\mathcal{V}'$ is a cover of W and $W \in \mathcal{W}$ by the argument above. Thus $V \in \mathcal{U}$, that is, each coordinate patch belongs to $\mathcal{U}$.

Suppose now that M is compact. Then M is a union of finitely many coordinate patches $V_1, \ldots, V_k$. But then by Lemma 8.4.8, $M \in \mathcal{U}$. If M is non-compact,

Fix a countable locally finite collection $U_1, U_2, \ldots$ of coordinate patches covering M (old exercise); if M is compact, we may take any finite covering of M with coordinate patches. Let I be the index set of coordinate patches.

Let $J \subset I$ be a finite subset. Then $\bigcap_{j\in J} U_j$ is either empty or a coordinate patch. Thus $\bigcap_{j \in J} U_j \in \mathcal{U}$.

By Lemma 8.4.8, $M = \bigcup_{j\in I} U_j \in \mathcal{U}$. This concludes the proof. (And this course.) □