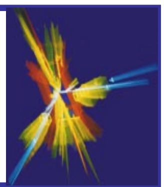

Particle Physics Phenomenology

Hiukkasfysiikan fenomenologia

PAP327: A 5 ECTS course spring 2026

https://www.mv.helsinki.fi/home/osterber/phenomenology/

Lectures Mon 12-14

weeks 3-7, 9, 11-14, 16-18 in Physicum D114

given by

Prof. Kenneth Österberg,

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exercise sessions Thu 16-17

starting week 5 in Physicum D116

given by

MSc Anna Milieva,

email: anna.milieva@helsinki.fi

Course Content

Relativistic kinematics part

- Special relativity and frequently used reference frames (laboratory & center-of-mass frame)
- Phase space integrals & cross section
- Two particle final states: two-particle decay and scattering, Mandelstam variables
- Three- and multi-particle reactions

Phenomenology part

- Standard Model (SM): theoretical framework, principle of gauge invariance, quantum electrodynamics (QED) and chromodynamics (QCD), electroweak unification and Higgs mechanism & Higgs boson
- Beyond SM (BSM): SM flaws, dark matter, basics of BSM, Grand Unified Theories, supersymmetric and extra dimensional models
- Hadron colliders: Deep inelastic scattering & hadron collider physics
- LHC phenomenology: soft physics, QCD, electroweak, top, Higgs and beyond SM.

Recommended prerequisites

- **Introduction to Particle Physics I & II**
- Quantum Field Theory I & II / QM IIa & IIb (beneficial)

Literature

Text books

M. Thomson: Modern Particle Physics, Cambridge University Press 2013 (for the Standard model part)

Other resources

- W. von Schlippe: Lectures on relativistic kinematics, St. Petersburg State University (see course homepage)
- A. Pich: The Standard Model of Electroweak Interactions, arXiv:1201.0537
- J. Iliopoulus: Introduction to the Standard Model of Electroweak Interactions, arXiv:1305.6779
- Particle Data Group Reviews, pdg.lbl.gov

Course Grading

- Exercises: max 24 points (weight 40 %)
- Presentation of experiment and/or physics subject: max 9 points (weight 15 %) **– a must if credit wanted**
- Oral course exam: max 27 points (weight 45 %).

Course Exercises

- Exercises given latest Wed, returned next Wed 16.00 (first exercise paper will be given next week)

Use of generative AI:

- ✓ Follow University of Helsinki general rules on AI usage: https://studies.helsinki.fi/instructions/article/using-ai-support-learning?
- ✓ Usage of Large Language Models (LLMs) encouraged.
- ✓ **However: the usage of LLMs should be clearly stated including how the LLMs were used.**
- ✓ Recommendation: use Copilot provided by the university (https://copilot.cloud.microsoft/).
- ✓ **LLMs to be used as a tool** (& NOT as a black box).
- ✓ Can help you with ideas & methods (= library), coding & repetetive tasks (e.g. partial derivates).
- ✓ Always check yourself that the LLM suggestion makes sense. Explore the capabilities of LLMs.

define the Mandelstam variables

Copilot

The **Mandelstam variables** are a set of three Lorentz-invariant quantities commonly used in particle physics to describe the kinematics of scattering processes involving two incoming and two outgoing particles. They are defined in terms of the four-momenta of the particles:

- Let the incoming particles have four-momenta p_1 and p_2, and the outgoing particles have p_3 and p_4.
- The Mandelstam variables are:

$$s = (p_1 + p_2)^2,$$
$$t = (p_1 - p_3)^2,$$
$$u = (p_1 - p_4)^2.$$

Interpretation

- s: The square of the total energy in the center-of-mass frame (often called the "center-of-mass energy squared").
- t: The square of the momentum transfer between particle 1 and particle 3.
- u: The square of the momentum transfer between particle 1 and particle 4.

• In High Energy Physics (HEP) the particles (e.g. protons, pions, electrons) we are concerned with usually move at speeds close to the speed of light.

• The classical relationship for kinetic energy of particle in terms of its mass and velocity is not valid:

kinetic energy $T \neq mv^2/2$

• Thus we must use *special relativity* to describe the energies and momenta of the particles.

• The total energy (E = rest + kinetic) of a particle with rest mass, m_0, is:

$$E = m_0c^2 + T \iff E = mc^2 = \frac{m_0 c^2}{\sqrt{1-(v/c)^2}} = \gamma m_0 c^2$$

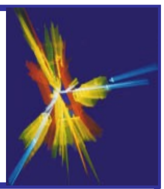
• Here v is its speed, c = speed of light, and m is sometimes called the relativistic mass. The total momentum, p, of a particle with rest mass, m_0, is:

$$\bar{p} = m\bar{v} = \frac{m_0 \bar{v}}{\sqrt{1-(v/c)^2}} = \gamma m_0 \bar{v}$$

• We can also relate the total energy, E, to a particle's total momentum, p:

$$E^2 = (\bar{p}c)^2 + (m_0c^2)^2$$

Natural units & 4 vectors



$E^2 = (\bar{p}c)^2 + (mc^2)^2$ NB! from now m = rest mass

3 fundamental units: length L, time T and energy E

2 constants: $c = 3.0 \cdot 10^8 \frac{\text{m}}{\text{s}}, \hbar = 6.6 \cdot 10^{-25}$ GeVs

Set c = 1 = [L] / [T] $\Rightarrow$ [T] = [L] (e.g. in 4-vectors)

Also $\hbar$ = 1 = [E] · [T] $\Rightarrow$ [L] = [T] = 1/ [E] (= GeV^{-1})

One degree-of-freedom left so choose [E] = GeV

Now $E^2 = \bar{p}^2 + m^2 \Rightarrow$ [E] = [p] = [m] = GeV

Define $\beta = \frac{v}{c}$ $(0 \le \beta < 1)$ $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ $(1 \le \gamma < \infty)$

e.g. $\beta = p/E$, $\gamma = E/m$ & for unstable particles:
<lifetime> = $\gamma\tau_0$ & <decay length> = $\beta\gamma c\tau_0$

τ_0 = proper lifetime

Convenient to describe a particle by a <u>4-vector</u>. The components of the momentum and energy 4-vector, p, are given by:

$$p = (E, p_x, p_y, p_z) \text{ or } p = (E, \bar{p}) \text{ with } c = 1$$

- The length of the 4-vector is given by:

$$m^2 = E^2 - P^2 = E^2 - p_x{}^2 - p_y{}^2 - p_z{}^2$$

True in ALL reference frames (lab, center of mass,…) due of Lorentz invariance

Particle "on mass shell" if m = true rest mass, "virtual" in case not.

4-vector with length L^2 is:

- Time like if $L^2 > 0$
- Space like if $L^2 < 0$

special case: Light like if L^2 = 0 (m = 0 if $P = E$ i.e. photon!)

Speed of light (c) same in all inertial reference frames $\Rightarrow$ Lorentz transformation i.e. time & distance between events may differ among frames but a scalar distance

$$s^2 = (\tau_2 - \tau_1)^2 + (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2; \tau \equiv ict$$

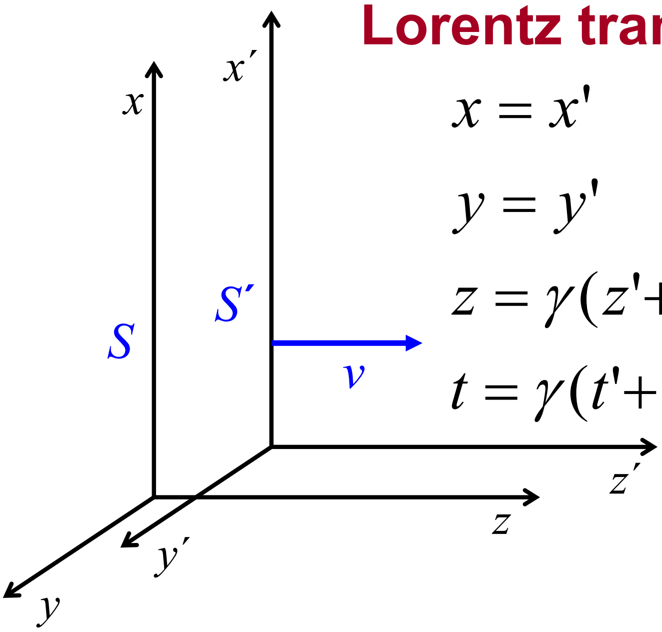
between 2 events is the same in all inertial frames. If translations excluded, only transformations leaving s invariant are rotations connected with τ e.g. z–τ plane

$$\left.\begin{aligned} z &= z'\cos\alpha - \tau'\sin\alpha \\ \tau &= z'\sin\alpha + \tau'\cos\alpha \end{aligned}\right\} \quad \text{trivially} \quad x = x' \text{ and } y = y'$$

determine α by being in frame S & observing z'=0 of S':

$$\left.\begin{aligned} z &= -\tau'\sin\alpha \\ \tau &= \tau'\cos\alpha \end{aligned}\right\} \quad -\frac{z}{\tau} = -\frac{v}{ic} = \tan\alpha = i\frac{v}{c} = i\beta \quad \Rightarrow$$

$$\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{1}{\sqrt{1-\beta^2}} \equiv \gamma \quad \sin\alpha = \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} = i\gamma\beta$$



Lorentz transformations:

$$x = x' \qquad x' = x$$

$$y = y' \qquad y' = y$$

$$z = \gamma(z' + \beta ct') \qquad z' = \gamma(z - \beta ct)$$

$$t = \gamma(t' + \beta z'/c) \qquad t' = \gamma(t - \beta z/c)$$

S' to S Lorentz transformations obtained by change of β to $-\beta$

Four-vectors

Equations on previous page form a special class of Lorentz transformations but that's all what is needed. The most general Lorentz transformation equations have the simplest form in four-vector space $x = (x^0, x^1, x^2, x^3) = (x^0, \bar{x}) = (ct, x, y, z)$. For any four-vector the general Lorentz transformation is given as:

$$a' = \mathbf{L}a \qquad a'^{\mu} = \sum_{0}^{3} L^{\mu}_{\nu} a^{\nu}$$, where $\mathbf{L}$ is a real matrix

metric tensor: $$\mathbf{g} = (g_{\mu\nu}) = (g^{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \mathbf{g}^{-1}$$

scalar product: $$a \cdot b = \sum_{\mu} a_{\mu} b^{\mu} = \sum_{\mu,\nu} g_{\mu\nu} a^{\mu} b^{\nu} = a^0 b^0 - \bar{a} \cdot \bar{b}$$

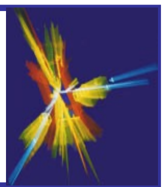
A Lorentz transformation is a linear transformation that leaves scalar product $a \cdot b$ invariant ($\Rightarrow$ $\mathbf{L}$ must satisfy $\mathbf{g}\mathbf{L}^{-1}\mathbf{g} = \mathbf{L}^{T}$). Can be expressed as a boost (see previous page) followed by a 3-dimensional rotation. In addition, Lorentz transformations satisfy following conditions:

$$\det \mathbf{L} = +1, \quad \text{i.e. spatial reflections excluded}$$

$$L^{0}_{0} \geq 1, \ \text{sign of 0-component of timelike vector invariant}$$

The specific Lorentz transformation solution of previous page would give:

$$\begin{aligned} a'^{0} &= \gamma(a^{0} - \beta a^{3}) \\ a'^{1} &= a^{1} \\ a'^{2} &= a^{2} \\ a'^{3} &= \gamma(a^{3} - \beta a^{0}) \end{aligned}$$



Example: Consider a proton with a momentum P of 10 GeV hitting a proton at rest in the lab frame

- What is the energy of incoming proton in lab frame?

For a proton m = 0.938 GeV. Since the rest energy of a particle is a Lorentz invariant we get:

$$E = \sqrt{P^2 + m^2} = \sqrt{10^2 + 0.938^2} = 10.044 \text{ GeV}$$

Thus at high energies ($E >> m$) $E \approx |\boldsymbol{p}|$.

- How fast is the proton moving in lab frame?

We need to remember energy/momentum relationship between rest frame of proton and the lab frame:

$$P_{\text{lab}}/E_{\text{lab}} = \gamma\beta mc/\gamma mc = \beta = 10/10.044 = 0.996$$

Thus v = 0.996c (very fast!)

Colliding beam vs fixed target collisions

- As discussed later cross sections & energy available for new particle production depend on total energy in center of mass (CM) frame (E_{CM}). CM frame = frame where total momentum vector is zero. Define $s = \Sigma p_{i,\text{CM}}^2$

CM frame: $s = (p_1+p_2)^2 = (E_1+E_2, \boldsymbol{p}_1+\boldsymbol{p}_2)^2 = (E_1+E_2, 0)^2$

If masses & E's equal then $s = (E_1+ E_2, 0)^2 = 4E^2$

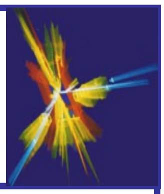
Yet for case above in lab frame: $s = (E_1+m_\text{p}, \boldsymbol{p}_1+0)^2$

s = 20.6 GeV^2 $\Rightarrow$ E_{CM} = 4.54 GeV

$E_{\text{CM}} \approx \sqrt{2m_{\text{target}}E_{\text{beam}}}$ for *fixed target*

$E_{\text{CM}} = 2E_{\text{beam}}$ for *colliding beam*

colliding beams more efficient for producing new & heavy particles



$$s = (E_{beam} + M_{target}, \vec{p}_{beam} + 0)$$

$$s^2 = (E_{beam} + M_{target})^2 - p_{beam}^2$$

$$= \cancel{p_{beam}^2} + m_{beam}^2 + 2E_{beam}M_{target} + M_{target}^2 - \cancel{p_{beam}^2}$$

$$= 2E_{beam}M_{target} + M_{beam}^2 + M_{target}^2$$

$$E_{beam} \gg M_{beam}, M_{target} \Rightarrow$$

$$s^2 \approx 2E_{beam}M_{target}$$

$$s^2 = E_{CM}^2$$

$$\Rightarrow E_{CM} \approx \sqrt{2E_{beam}M_{target}}$$

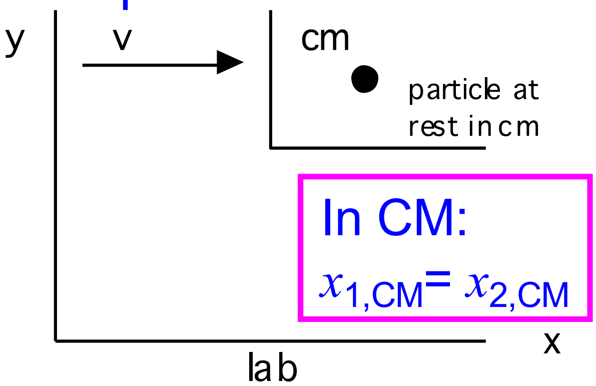
Time dilation

Most HEP particles are not stable, i.e. they decay into other particles after a certain amount of time. E.g.

Lepton	**Mean Lifetime (s)**
electron	stable
muon (μ)	$\approx 2 \times 10^{-6}$

Above table gives lepton average lifetime in <u>their</u> rest frame. However, often must know how long a particle will live (on average) in a frame where it is moving close to the speed of light (c). Use special relativity!

- Consider a particle moving with speed v in lab frame along the x-axis. Then:
  - $t_{\text{lab}} = \gamma(t_{\text{CM}} + \beta x_{\text{CM}})$
  - $x_{\text{lab}} = \gamma(x_{\text{CM}} + \beta t_{\text{CM}})$



In lab frame time between creation & decay of particle:

$$\tau_{\text{lab}} = t_{2,\text{lab}} - t_{1,\text{lab}} = \gamma(t_{2,\text{CM}} + \beta x_{2,\text{CM}}) - \gamma(t_{1,\text{CM}} + \beta x_{1,\text{CM}}) = \gamma(t_{2,\text{CM}} - t_{1,\text{CM}}) = \gamma\tau$$

NB! τ = proper lifetime & $\tau \leq \tau_{\text{lab}}$

A muon (m = 0.106 GeV) with E = 1 GeV in lab frame.

- On average how long does muon live in lab?

In muon's rest frame only lives (on average) $\tau = 2\ \mu$s. But in lab frame lives (on average):

- $\tau_{\text{lab}} = \gamma\tau \approx 20\ \mu$s since $\gamma = E/m = 1/0.106 \approx 10$
- How far does muon travel in lab before decaying?
- $\Delta x_{\text{lab}} = \gamma\beta c\tau \approx (10)(3 \cdot 10^8\ \text{m/s})(2 \cdot 10^{-6}\ \text{s}) = 6 \cdot 10^3\ \text{m}$

Large increase due to special relativity

An alternative description, Minkowski metric, based on introducing an imaginary time $x_4 = it$. Now the 4-vector components of space-time are:

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = it.$$

and the Lorentz invariant square becomes

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = \text{invariant}.$$

The transformation, which leaves this expression invariant, is a rotation. Thus, for instance, a rotation in the (x_1, x_4) plane, leaving x_2 and x_3 unchanged, is of the form

$$x_1' = x_1 \cos\alpha - x_4 \sin\alpha, \qquad x_4' = x_1 \sin\alpha + x_4 \cos\alpha$$

Going back to imaginary time, rotations becomes:

$$x' = x\cos\alpha - it\sin\alpha, \qquad it' = x\sin\alpha + it\cos\alpha$$

and using the identities

$$i\sin\alpha = \sinh i\alpha, \qquad \cos\alpha = \cosh i\alpha$$

since $\cosh x = \cos(ix)$, $\sinh x = -i\sin(ix)$

and setting $y = i\alpha$ we recover a completely real form of the Lorentz transformation, viz

$$x' = x\cosh y + t\sinh y, \qquad t' = x\sinh y + t\cosh y$$

standard Lorenz transformation

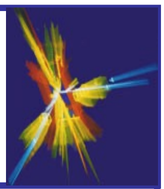
For this to be equivalent with Eq. (4) we must demand that y be real, and hence α is imaginary. Thus the price to pay for a familiar Euclidian form of the rotation in the (x_1, x_4) plane is an imaginary angle of rotation. The real quantity y defined above is called the *rapidity* of the transformation; it is related to the relative velocity v between the two frames and to the relativistic γ factor by $\cosh y = \gamma \qquad \sinh y = \beta\gamma$

hence $\beta = \tanh y$ i.e. $y = 1/2\,\ln(1+\beta/1-\beta)$

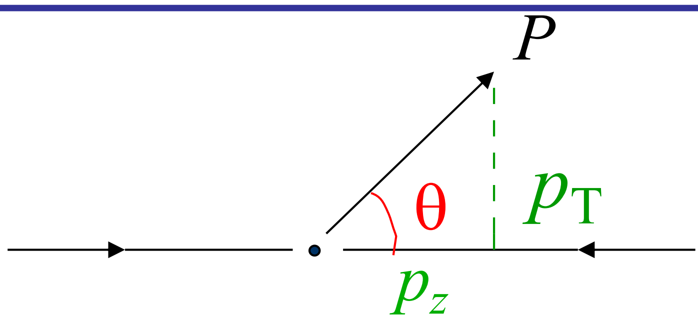
We can similarly write the Lorentz transformation of the 4-momentum as

$$p' = p\cosh y + E\sinh y, \qquad E' = p\sinh y + E\cosh y$$

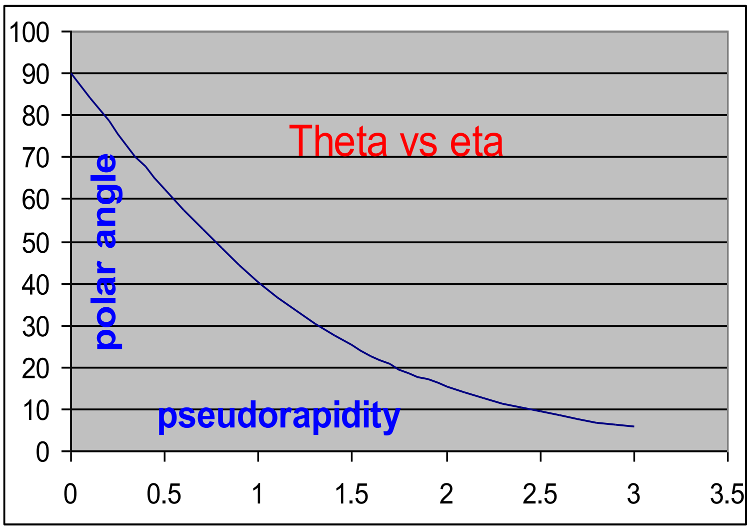
NB! Rapidity is additive i.e. two rapidity transformations can be replaced by a single one $y_3 = y_1 + y_2$



Pseudorapidity



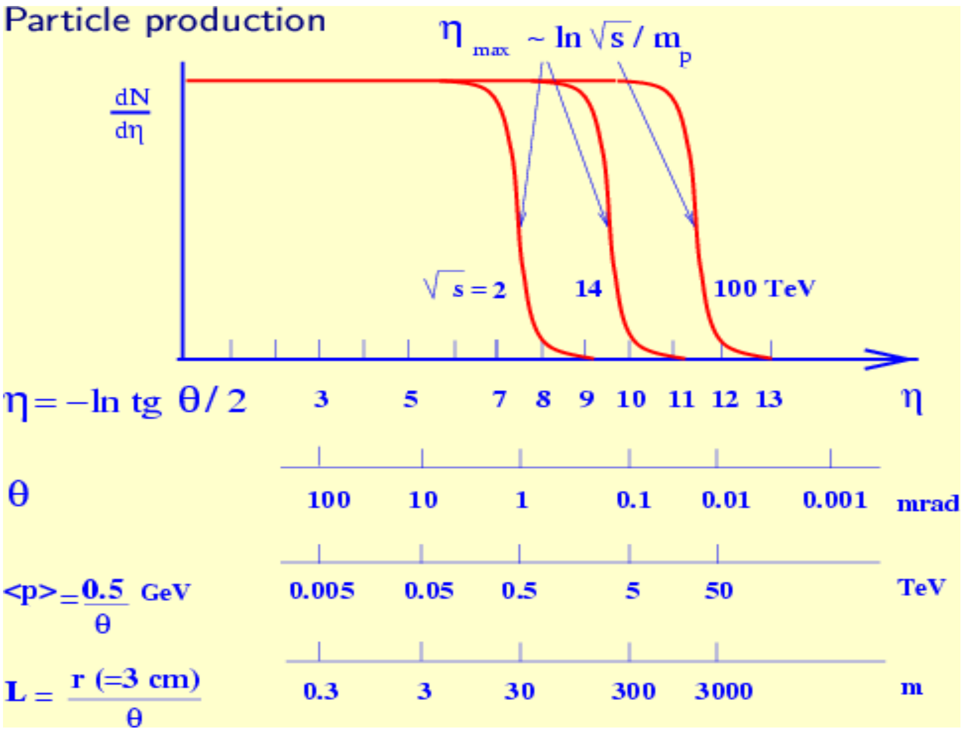
rapidity $y = \ln[(E+p_z)/(E-p_z)]/2$ becomes pseudorapidity $\eta = -\ln(\tan(\theta/2))$ if particle masses neglected (E, p_z » m).



$\theta = 90^o \rightarrow \eta = 0$

$\theta = 10^o \rightarrow \eta \approx 2.4$

$\theta = 170^o \rightarrow \eta \approx -2.4$



Useful frames for scattering processes

Let's introduce some frames, defined by the initial state of a scattering process. In a two-particle process, particles a and b with four-momenta $p_a = (E_a, \bar{p}_a)$ & $p_b = (E_b, \bar{p}_b)$ interact.

1. Laboratory frame (LF) is defined as the frame in which the experiment is carried out and all energies and momenta measured. This is the primary frame from which quantities (usually denoted by an index L or an index lab) are transformed to other systems.

2. Centre-of-mass frame (CMF) is defined as the frame in which $\vec{p}_a + \vec{p}_b = 0$. The CMF quantities are usually denoted by an asterisk or an index CM. Many experiments like at the LHC are colliding beam experiments. If the mass of particles a & b are the same then Laboratory frame (LF) = Centre-of-mass frame (CMF)

3. Target frame (TF) defined as the frame in which $\vec{p}_b^T = 0$. The TF quantities are usually denoted by an index T. Some experiments are fixed target ones, i.e. Laboratory frame (LF) = Target frame (TF).

CMF:

$\bar{p}_a^* = -\bar{p}_b^*$

TF:

$\bar{p}_a^T \qquad \bar{p}_b^T = 0$

Lorentz transformations between frames

Consider first Lorentz transformations between the CMF and TF frames. Initial state can be expressed as follows:

$$p_a^* = (E_a^*, 0, 0, P_a^*) \qquad p_a^T = (E_a^T, 0, 0, P_a^T)$$
$$p_b^* = (E_b^*, 0, 0, -P_a^*) \qquad p_b^T = (m_b, 0, 0, 0)$$

where the direction of motion has been chosen as the z axis. The Lorentz transformation equations are now:

$$P_a^* = \gamma^{CM}(P_a^T - \beta^{CM} E_a^T) \qquad \beta^{CM} \text{ is velocity}$$
$$E_a^* = \gamma^{CM}(E_a^T - \beta^{CM} P_a^T) \qquad \text{of CMF in TF}$$

need to determine β^{CM}. If total energy and momentum of a group of particles in some reference frame: $E_{tot}, \bar{p}_{tot}$ then

$$\bar{\beta}_{tot} = \bar{p}_{tot} / E_{tot} \qquad \gamma_{tot} = E_{tot} / m_{tot} \qquad \gamma_{tot}\bar{\beta}_{tot} = \bar{p}_{tot} / m_{tot}$$

Where $m_{tot} = \sqrt{E_{tot}^2 - \bar{p}_{tot}^2} = \sqrt{s}$ is the invariant mass of the group of particle . For 2-particles in TF this becomes:

$$s \equiv s_{ab} = (E_a + E_b)^2 - (\bar{p}_a + \bar{p}_b)^2 = (E_a^T + m_b)^2 - (P_a^T)^2 = m_a^2 + m_b^2 + 2m_b E_a^T$$

Now the CMF-TF relation can be expressed as:

$$\beta^{CM} = P_a^T / (E_a^T + m_b) \qquad \gamma^{CM} = (E_a^T + m_b)/\sqrt{s} \qquad \gamma^{CM}\beta^{CM} = P_a^T / \sqrt{s}$$

inserting these into the Lorentz transformation equations

$$P_a^* = m_b P_a^T / \sqrt{s} \qquad E_a^* = (m_a^2 + m_b E_a^T)/\sqrt{s}$$
$$P_b^* = m_b P_a^T / \sqrt{s} = P_a^* \qquad E_b^* = m_b(m_b + E_a^T)/\sqrt{s}$$

The Lorentz transformations can be done explicitly as above but in more complicated cases this becomes too tedious (and error prone) so instead noninvariants will be expressed in terms of invariants to make algebra easier.

Relativistic kinematics

$$p_a^* = \gamma^{CM}\left(p_a^T - \beta^{CM} E_a^T\right)$$

$$= \frac{(E_a^T + m_b)}{\sqrt{s}} \cdot p_a^T - \frac{p_a^T}{\sqrt{s}} \cdot E_a^T$$

$$= \frac{E_a^T p_a^T + m_b p_a^T - p_a^T E_a^T}{\sqrt{s}}$$

$$= \frac{m_b p_a^T}{\sqrt{s}} \qquad \square$$

$$E_a^* = \gamma^{CM}\left(E_a^T - \beta^{CM} p_a^T\right)$$

$$= \frac{(E_a^T + m_b)}{\sqrt{s}} \cdot E_a^T - \frac{p_a^T}{\sqrt{s}} \cdot p_a^T$$

$$= \frac{{E_a^T}^2 + m_b E_a^T - {p_a^T}^2}{\sqrt{s}}$$

$$= \frac{{p_a^T}^2 + m_a^2 + m_b E_a^T - {p_a^T}^2}{\sqrt{s}}$$

$$= \frac{m_a^2 + m_b E_a^T}{\sqrt{s}} \qquad \square$$

E, P in terms of invariants: TF

For the target frame (TF) we have $\bar{p}_b^T = P_b^T = 0$ and $E_b^T = m_b$

$$E_a^T = \frac{(s - m_a^2 - m_b^2)}{2m_b} \qquad (P_a^T)^2 = (E_a^T)^2 - m_a^2 = \frac{\{(s - m_a^2 - m_b^2)^2 - 4m_a^2 m_b^2\}}{4m_b^2}$$

To simplify we introduce a kinematical function λ:

$$\lambda(x,y,z) = (x-y-z)^2 - 4yz = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$$
$$= \left\{x - (\sqrt{y} + \sqrt{z})^2\right\}\left\{x - (\sqrt{y} - \sqrt{z})^2\right\}$$
$$= \left\{\sqrt{x} - \sqrt{y} - \sqrt{z}\right\}\left\{\sqrt{x} + \sqrt{y} + \sqrt{z}\right\}\left\{\sqrt{x} - \sqrt{y} + \sqrt{z}\right\}\left\{\sqrt{x} + \sqrt{y} - \sqrt{z}\right\}$$

λ is invariant under all permutations of its arguments (see above). λ is sometimes called the triangle function since $\sqrt{-\lambda(x,y,z)}/4$ is the area of a triangle with sides $\sqrt{x}, \sqrt{y}$ and $\sqrt{z}$

for TF momentum we get: $P_a^T = \sqrt{\lambda(s, m_a^2, m_b^2)}/2m_b$

now: $\lambda(s, m_a^2, m_b^2) = \left\{s - (m_a + m_b)^2\right\}\left\{s - (m_a - m_b)^2\right\}$

thus P_a^T is real if: $\sqrt{s} \geq m_a + m_b$

Threshold value $m_a + m_b$ the smallest value $\sqrt{s}$ can attain.

For the centre-of-mass frame (CMF):

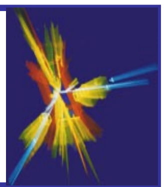
$$P_a^* = P_b^* = P^* \qquad \sqrt{s} = E_a^* + E_b^* \qquad \Rightarrow$$
$$\sqrt{s} = \sqrt{(P^*)^2 + m_a^2} + \sqrt{(P^*)^2 + m_b^2}$$

$\sqrt{s}$ is equal to the total energy in CMF. One obtains the following expression for CMF energy and momentum:

$$E_a^* = (s + m_a^2 - m_b^2)\big/2\sqrt{s}$$
$$P^* = \sqrt{\lambda(s, m_a^2, m_b^2)}\big/2\sqrt{s}$$

$$E_b^* = \sqrt{s} - E_a^* \quad \Rightarrow \quad E_b^* = (s - m_a^2 + m_b^2)\big/2\sqrt{s}$$

Phase space



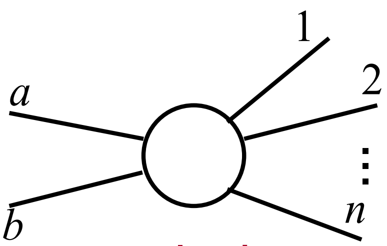
In the reaction $a+b \rightarrow 1+...+n$, the final state is constrained by the initial state via **four-momentum conservation** i.e.

$$p_a^\mu + p_b^\mu = \sum_{i=1}^{n} p_i^\mu \quad \Leftrightarrow \quad E_a + E_b = \sum_{i=1}^{n} E_i \quad , \quad \bar{p}_a + \bar{p}_b = \sum_{i=1}^{n} \bar{p}_i$$

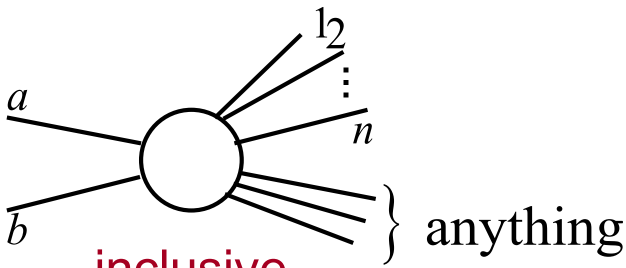
NB! For ”asymptotic” states, intermediate one can violate energy or momentum conversation (Heisenbergs uncertainty relation).

Define the 3n dimensional space of the unconstrained final state momentum vectors $\bar{p}_i$, the **momentum space.** The conditions above define in this space a 3n–4 dimensional surface, which will be called **phase space.**

Need to distinguish 2 types of reactions or measurements:



exclusive



inclusive

The reaction channel is fixed in an exclusive reaction, whereas an inclusive is a sum over several different exclusive channels.

Two types of exclusive processes encountered in practice:

a particle decay, $0 \rightarrow 1+...+m$
a collision of particles, $a+b \rightarrow 1+...+n$
One can call the 1st a 1 $\rightarrow$ m & the latter a 2 $\rightarrow$ n process.

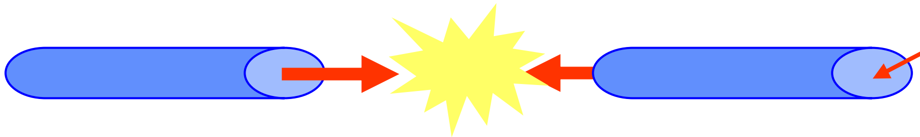
If m = n +1, $p_a = p_0$, $p_b = -p_m$, particle decay and particle collision are related by **crossing**, i.e. collision can be obtained from decay by moving one final state particles to initial state and vice-versa.

The concept of cross sections

Cross sections σ or differential cross sections $d\sigma/d\Omega$ are used to express the probability of interactions between elementary particles.

Example two colliding particle beams

beam spot area A



N_1 f = collision frequency N_2

What is the interaction rate $R_{int.}$?

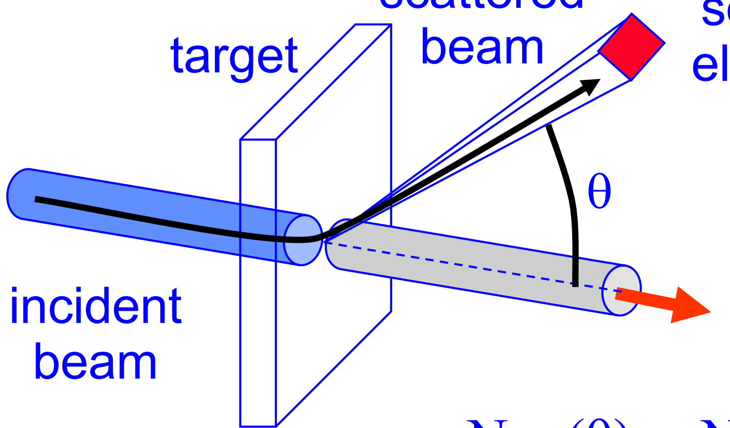
$$R_{int} \propto \underbrace{f\, N_1 N_2 / A}_{} = \sigma \cdot \mathcal{L}$$

Luminosity $\mathcal{L}$ [cm^{-2} s^{-1}]

σ has dimension area !
Practical unit:
1 barn (b) = 10^{-24} cm^2

Example: Scattering from target

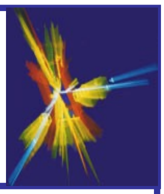
$N_{int} = R_{int}\, t$



assumption: target area >> beam spot area

n_A = area density of scattering centers in target

$$N_{scat}(\theta) \propto N_{inc} \cdot n_A \cdot d\Omega$$
$$= d\sigma/d\Omega\,(\theta) \cdot N_{inc} \cdot n_A \cdot d\Omega$$



Define **luminosity** precisely:

imagine a particle colliding with a bunch of cross section area – A. Probability of collision is:

$$\sigma \cdot N_{part/bunch} / A$$

(E.Wilson)

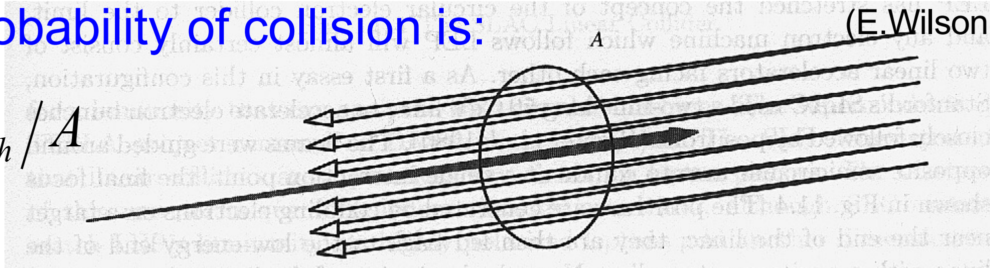


Fig. 11.3 A probe particle encounters a target—a beam of particles with cross sectional area A travelling in the opposite sense.

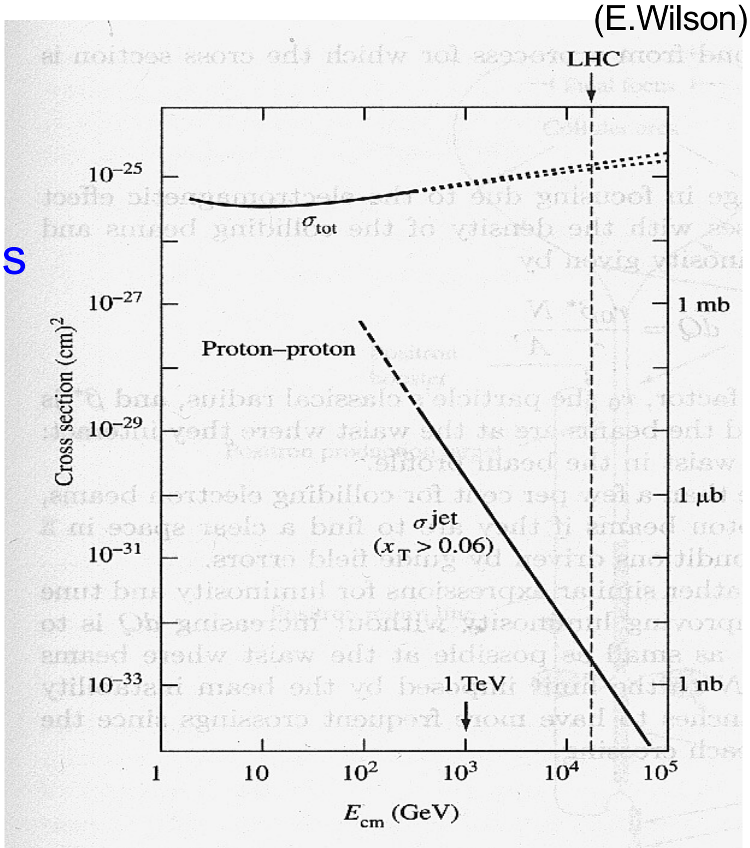
for $N_{part/bunch}$ particles in both beams $\sigma \cdot N^2_{part/bunch} / A$

and finally take into account the bunch crossing frequency f_b = # of bunches multiplied by the revolution frequency.

$$\text{Event rate} = L \cdot \sigma, \text{ where } L = \frac{f_b N^2_{part/bunch}}{A}$$ **(= luminosity)**

Ultimate challenge to high energy colliders: the production rate of "interesting" interactions fall as $1/s$ ($\propto 1/E_{CM}^2$), hence need to improve luminosity a factor 100 for each factor 10 energy increase to benefit from energy increase (distances at which structures probed $\propto 1/\sqrt{s}$).

(E.Wilson)



Cross section, partial decay width & lifetime

Transition probability from initial to final state defined as:

$\langle \bar{p}_1, \ldots, \bar{p}_n | \mathrm{M} | \bar{p}_a, \bar{p}_b \rangle \equiv \mathrm{M}(\bar{p}_i)$ the "matrix element".

Matrix M contains all "physics" of the reaction and will not be discussed more in detail here. Simply noted to be an unknown function of $\bar{p}_i$'s. To obtain measurable quantities, must integrate $|\mathrm{M}(\bar{p}_i)|^2$ over an allowed set of $\bar{p}_i$ values.

Partial cross section obtained by integrating over 3n–4 dimensional phase space for an allowed set of $\bar{p}_i$ values. Corresponding quantity for decay is **partial decay width**.

Cross section: $\sigma_n \equiv \sigma_n(s, m_i) = I_n / F$, where

$F = 4(2\pi)^{3n-4}\sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}$ is the flux factor &

$I_n(s) = \int \prod_{i=1}^{n} \frac{d^3 p_i}{2E_i} \delta^4(p_a + p_b - \sum_{i=1}^{n} p_i) \left|\mathrm{M}(\bar{p}_i)\right|^2$ contains the integration over phase space. NB! definition a convention.

differential cross section: integration restricted to subset of the allowed phase space. Done by inserting δ functions.

$$\frac{d\sigma_n}{dx} = \frac{1}{F} \int \prod_{i=1}^{n} \frac{d^3 p_i}{2E_i} \delta^4(p_a + p_b - \sum_{i=1}^{n} p_i) \delta(x - x(\bar{p}_i)) \left|\mathrm{M}(\bar{p}_i)\right|^2$$

$x = x(\bar{p}_i)$ required . Satisfies $\int dx\,(d\sigma_n / dx) = \sigma_n$. Higher-order $d^2\sigma_n / dx dy$ etc... obtained by inserting just more δ functions.

Partial decay width: $\Gamma_m = I_m(m^2) / 2\sqrt{s}\,(2\pi)^{3m-4}$, where

$$I_m(m^2) = \int \prod_{i=1}^{m} \frac{d^3 p_i}{2E_i} \delta^4(p - \sum_{i=1}^{m} p_i) \left|\langle \bar{p}_1, \ldots, \bar{p}_m | \mathrm{M} | \bar{p} \rangle\right|^2.$$

The **lifetime** τ of an unstable particle is the inverse of Γ_{tot}, the sum of the partial decay widths of all possible decays,

$$1/\tau = \Gamma_{\mathrm{tot}} = \sum_j \Gamma_j, \quad \text{similarly} \quad \sigma_{\mathrm{tot}} = \sum_j \sigma_j$$

Let's return to the integral $I_n(s)$, which includes factors $d^3p_i/2E_i$. They are Lorentz invariant as can be seen by differentiating the 4–momentum transformation formulas.

$$dp_{x(y)} = dp'_{x(y)} \quad dp_z = \gamma(dp'_z + vdE') = \gamma\, dp'_z(1 + vp'_z/E') = dp'_z E/E'$$

since $dE'/dp'_z = p'_z/E'$ and $E = \gamma(E' + vp'_z)$. The volume element $d^3p = dp_x dp_y dp_z$ thus satisfies $d^3p/E = d^3p'/E'$ so d^3p/E is invariant (for exact proof see next page). Rewritten into integral form for a timelike p:

$$d^3p/2E = \int d^4p\,\delta(p^2 - m^2)\Theta(p^0)$$

where $\Theta(p^0)$ is a step function that is zero for $p^0 < 0$ and 1 for $p^0 > 0$. The δ function integration has following property

$$\delta(f(x)) = \delta(x - x_0)/|f'(x_0)|, \quad \text{where } f(x_0) = 0 \qquad \text{so}$$

$$\int d^4p\,\delta(p^2 - m^2)\Theta(p^0) = \frac{d^3p}{2E}\int dp^0\,\delta(p^0 - E)\Theta(p^0)$$

Now the factor 2 that is conventially added gets an explaination. Note that the Θ function is usually omitted.

So now the integral over the phase space is:

$$I_n(s) = \int \prod_{i=1}^{n} d^4p_i\,\delta(p_i^2 - m^2)\Theta(p^0)\delta^4(p_a + p_b - \sum_{i=1}^{n} p_i)\left|\mathrm{M}(\bar{p}_i)\right|^2$$

δ-function, singular function, eliminated in numerical calculations. After, 3n–4 variables only constrained by limits of integration, defined as variables Φ.

$$I_n(s) = \int d\Phi\,\rho_n(\Phi)\left|\mathrm{M}(\Phi)\right|^2$$ $\rho_n(\Phi)$ **phase space density**

Proof of Lorentz invariance

$$d\phi_i(s, p_1, p_2) = \delta^4(p_a + p_b - p_1 - p_2)\frac{d^3\bar{p}_1}{2E_1}\frac{d^3\bar{p}_2}{2E_2}$$

show that $d\phi_i$ is Lorentz invariant

$\delta^4(p_a + p_b - p_1 - p_2)$ Lorentz invariant trivially since four-momentum is conserved

$\frac{d^3\vec{p}}{E}$? $\qquad d^3\vec{p} = dp_x\, dp_y\, dp_z$

Lorentz transformation:
$$\begin{cases} dp_x = dp_x' \\ dp_y = dp_y' \\ dp_z = \gamma(dp_z' + v\,dE') \\ \quad\;\; = \gamma\, dp_z'(1 + v p_z'/E') \end{cases}$$

$$E' = \sqrt{p_x'^2 + p_y'^2 + p_z'^2 + m^2} \Rightarrow$$

$$\frac{dE'}{dp_z'} = \frac{2 p_z'}{2\sqrt{p_x'^2 + p_y'^2 + p_z'^2 + m^2}} = \frac{p_z'}{E'}$$

$$\text{so } dp_z = \frac{dp_z'}{E'}\underbrace{\gamma(E' + v p_z')}_{=E} = \frac{dp_z'}{E'}E$$

$$\text{hence } \frac{d^3p}{E} = \frac{dp_x\, dp_y\, dp_z}{E} = \frac{dp_x'\, dp_y'\, dp_z'}{E'} = \frac{d^3p'}{E'}$$

$\therefore$ $d\phi_i$ Lorentz invariant

so we can choose to do the integration in any frame, let's choose CMF!!

Two-particle phase space integral

When M is set to 1, we define the **phase space integral**

$$R_n(s) = \int \prod_{i=1}^{n} \frac{d^3 p_i}{2E_i} \delta^4(p_a + p_b - \sum_{i=1}^{n} p_i)$$

This integral has no physical meaning but is technically useful since e.g. $\rho_n(\Phi)$ & physical region of Φ independent of M. Thus most kinematics can be done without knowing M. Rest of chapter largely deal with transformations of R_n.

Let's consider the simplest possible process: one particle going into two, thus study two-particle final state without specifying initial state $p=(E,\bar{p})$ except that 4-momentum conserved. Then the two-particle phase space integral is:

$$R_2(p,m_1^2,m_2^2) = \int d^4p_1 d^4p_2 \delta(p_1^2 - m_1^2)\delta(p_2^2 - m_2^2)\delta^4(p - p_1 - p_2)$$

Note that R_2 can only be a function of s (= p^2), m_1 and m_2. First integrate over p_2 in the four-dimensional δ function imposing $p_2 = p - p_1$ and then go to the CMF $p = (\sqrt{s},\bar{0})$

$$R_2 = \int d^4p_1 \delta(p_1^2 - m_1^2)\delta\{(p-p_1)^2 - m_2^2\} = \int \frac{d^3p_1^*}{2E_1^*}\delta\{s - 2\sqrt{s}E_1^* + m_1^2 - m_2^2\}$$

$$= \int \frac{P_1^* d\Omega_1^* dE_1^*}{2}\delta\{s - 2\sqrt{s}E_1^* + m_1^2 - m_2^2\} = \frac{P_1^*}{4\sqrt{s}}\int d\Omega_1^* = \frac{\pi P_1^*}{\sqrt{s}}$$

The special δ function property is used here. In addition, that $d^3p_1/2E_1 = d^3p_1^*/2E_1^*$ (invariant) & $E^2 = P^2 + m^2 \Rightarrow EdE = PdP$ is used. Note further that the last δ integration defines

$$s - 2\sqrt{s}E_1^* + m_1^2 - m_2^2 = 0 \quad \Rightarrow \quad E_1^* = (s + m_1^2 - m_2^2)/2\sqrt{s}$$

From decay kinematics: $P_1^* = \sqrt{\lambda(s,m_1^2,m_2^2)}/2\sqrt{s} = P_2^*$

So finally: $R_2(s) = \pi P_1^*/\sqrt{s} = \pi\sqrt{\lambda(s,m_1^2,m_2^2)}/2s$

Two-particle phase space integral

$$R_2 = \int d^4p_1\, d^4p_2\, \delta(p_1^2 - m_1^2)\, \delta(p_2^2 - m_2^2)\, \delta(p - p_1 - p_2)$$

Integrate over p_2 imposing $p_2 = p - p_1$ $\Rightarrow$

$$R_2 = \int d^4p_1\, \delta(p_1^2 - m_1^2)\, \delta((p-p_1)^2 - m_2^2)$$

go to CMF frame ($p = (\sqrt{s}, \vec{0})$, $p_1 = (E_1^*, \vec{p}_1^{\,*})$) and use invariance of $d^3p_1/2E_1 = d^3p_1^*/2E_1^*$

now $(p-p_1)^2 = p^2 - 2p\cdot p_1 + p_1^2$
$= s - 2\sqrt{s}E_1^* + m_1^2$ $\Rightarrow$

$$R_2 = \int \frac{d^3p_1^*}{2E_1^*}\, \delta(s - 2\sqrt{s}E_1^* + m_1^2 - m_2^2)$$

now $d^3p_1^* = p_1^{*2}\, d\Omega_1^*\, dp_1^*$

and $E^2 = p^2 + m^2 \Rightarrow E_1^* dE_1^* = p_1^* dp_1^*$

$$R_2 = \int \frac{p_1^* \cancel{E_1^*}\, d\Omega_1^*\, dE_1^*}{2\cancel{E_1^*}}\, \delta(s - 2\sqrt{s}E_1^* + m_1^2 - m_2^2)$$

$s - 2\sqrt{s}E_1^* + m_1^2 - m_2^2 = 0 \Rightarrow E_1^* = \dfrac{(s + m_1^2 - m_2^2)}{2\sqrt{s}}$

$$\left|\frac{\partial}{\partial E_1^*}(s - 2\sqrt{s}E_1^* + m_1^2 - m_2^2)\right| = |2\sqrt{s}|$$

← derivative of δ-function

$$R_2 = \frac{p_1^*}{4\sqrt{s}} \underbrace{\int d\Omega_1^*}_{=4\pi} = \frac{\pi p_1^*}{\sqrt{s}} \quad \square.$$

Two-particle phase space

The condition for the process $p \to p_1 + p_2$ to be physical, derived from expressions for the phase space integral, is

$$\lambda(p^2, p_1^2, p_2^2) = \{p^2 - (\sqrt{p_1^2} + \sqrt{p_2^2})^2\}\{p^2 - (\sqrt{p_1^2} - \sqrt{p_2^2})^2\} \geq 0$$

If all four-vectors are timelike, the condition requires:

$$\sqrt{p^2} \geq m_1 + m_2 = \text{threshold}$$

that is a natural condition for a decay

Define symmetric Gram determinants $\Delta_n(p_1,\dots, p_n)$:

$$\Delta_n(p_1,\dots,p_n) \equiv \begin{vmatrix} p_1^2 & p_1 \cdot p_2 \cdots & p_1 \cdot p_n \\ \vdots & \vdots & \vdots \\ p_n \cdot p_1 & p_n \cdot p_2 \cdots & p_n^2 \end{vmatrix} \Rightarrow$$

especially: $\Delta_2(p_1, p_2) = -\lambda\{(p_1 + p_2)^2, p_1^2, p_2^2\}/4$

The process $p \to p_1 + p_2$ physical if (and only if):

$$\Delta_2(p_1, p_2) \leq 0$$

The boundary of the physical region in terms of invariants obtained from the condition $\Delta_2(p_1,p_2) = 0$. Now also the λ function reveals its true significance, as an expanded form of Δ_2. One can call λ **the basic three particle kinematic function**. This follows from the fact that Δ_2 is relevant for reactions, where the total number of four-momenta is three (e.g. a $1 \to 2$ decay).

Mandelstam variables

Let's now introduce a whole set of invariant variables for $2 \to 2$ scattering, the "Mandelstam variables", though we have already used one of them, s. For reasons related to "crossing" one usually defines 2 more invariants t and u. The definitions of the invariants for $p_a + p_b \to p_1 + p_2$ are:

$$s = (p_a + p_b)^2 = (p_1 + p_2)^2$$
$$= \left(E_1^* + E_2^*\right)^2 = (E_a^* + E_b^*)^2 = m_a^2 + m_b^2 + 2m_b E_a^T$$

$$t = (p_a - p_1)^2 = (p_b - p_2)^2$$

$\cos\theta_{a1}^* =$ scattering angle between a and 1 in CMF

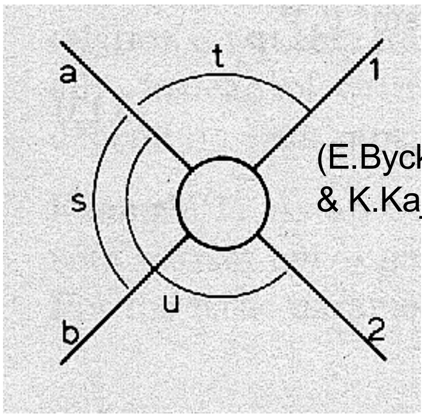
$$= m_a^2 + m_1^2 - 2E_a^* E_1^* + 2P_a^* P_1^* \cos\theta_{a1}^* = m_b^2 + m_2^2 - 2m_b E_2^T$$

$$u = (p_a - p_2)^2 = (p_b - p_1)^2$$
$$= m_a^2 + m_2^2 - 2E_a^* E_2^* + 2P_a^* P_2^* \cos\theta_{a2}^* = m_b^2 + m_1^2 - 2m_b E_1^T$$

There are only two independent variables so s, t and u must be related. Infact, the relation is

$$s + t + u = m_a^2 + m_b^2 + m_1^2 + m_2^2$$

NB! $s \geq 0$ but $t, u \ \leq 0$



(E.Byckling & K.Kajantie)

Crossing: We have sofar treated the reaction $p_a + p_b \to p_1 + p_2$ assuming all energies are positive: $p = (E, \bar{p})$ with $E = +\sqrt{P^2 + m^2} \geq m \geq 0$. But the equation for four-momentum conservation is also analytically valid for any timelike p with a negative 0-component: $p = (E, \bar{p})$ with $E = -\sqrt{P^2 + m^2} \leq 0$. These negative energy states will in QM be seen as the positive energy states of the anti-particle.

Mandelstam variables

$s+t+u = m_a^2 + m_b^2 + m_1^2 + m_2^2$

doesn't include any frame dependent variables, only invariants $\Rightarrow$ can choose any frame and should hold!

Let's choose target frame (TF):

$p_a = (E_a^T, \vec{p}_a^{\,T}),\; p_b = (m_b, \vec{0}),\; p_{1,2} = (E_{1,2}^T, \vec{p}_{1,2}^{\,T})$

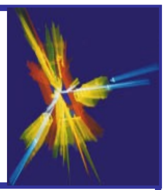
now $s+t+u = m_a^2 + m_b^2 + 2m_b E_a^T$
$+ m_b^2 + m_2^2 - 2m_b E_2^T + m_b^2 + m_1^2 - 2m_b E_1^T =$
$m_a^2 + 3m_b^2 + m_1^2 + m_2^2 +$
$2m_b \underbrace{(E_a^T - E_2^T - E_1^T)}_{-m_b} = m_a^2 + m_b^2 + m_1^2 + m_2^2$ □

Energy conservation:

$$E_a^T + m_b = E_2^T + E_1^T \Rightarrow$$

$$E_a^T - E_2^T - E_1^T = -m_b$$

Mandelstam variables



4-momentum conservation & antiparticle definition give:

$$p_a + p_b = p_1 + p_2 \qquad \text{s-channel}: p_a + p_b = p_1 + p_2$$
$$p_a + (-p_1) = (-p_b) + p_2 \quad \Rightarrow \quad \text{t-channel}: p_a + p_{\bar{1}} = p_{\bar{b}} + p_2$$
$$p_a + (-p_2) = p_1 + (-p_b) \qquad \text{u-channel}: p_a + p_{\bar{2}} = p_1 + p_{\bar{b}}$$

where the "bar" denotes an antiparticle & all 4-momenta now have positive E's. For the kinematics, irrelevant whether a particle is an antiparticle or not but when examing dynamical properties the particle–antiparticle distinction has to be taken into account when a particle is moved from initial state to final state and vice versa.

(E.Byckling & K.Kajantie)

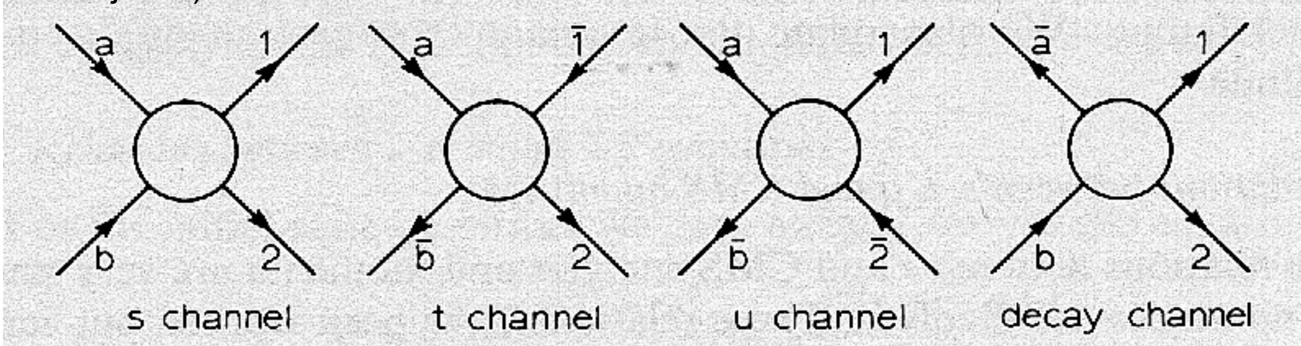


Figure IV.4.2 Various channels for $p_a + p_b \to p_1 + p_2$. The decay channel shown is physical if $m_b > m_1 + m_2 + m_a$

In addition to scattering channels, there may also exist decay channels.

E.g. if $m_b \geq m_a + m_1 + m_2$ the following decay is possible

$$p_b \to p_{\bar{a}} + p_1 + p_2$$

Cross section formulas

The reaction cross section for $p_a + p_b \rightarrow p_1 + p_2$ is:

$$\sigma(s) = \frac{1}{16\pi^2 \sqrt{(p_a \cdot p_b)^2 - m_a^2 m_b^2}} \int \frac{d^3\overline{p}_1}{2E_1} \frac{d^3\overline{p}_2}{2E_2} \delta^4(p_a + p_b - p_1 - p_2) |\mathrm{M}|^2$$

The matrix element M depends here on two independent variables (e.g. an invariant and an angle). If a differential cross section $d\sigma/dx$ computed, no further integration over M necessary (since one independent variable still left).

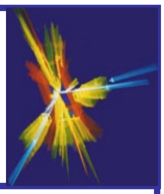
Doing the integration over phase space partly, one gets:

$$\frac{d\sigma}{d\Omega_1^*} = \frac{|\mathrm{M}|^2 \sqrt{\lambda(s, m_1^2, m_2^2)}}{64\pi^2 s \sqrt{\lambda(s, m_a^2, m_b^2)}} \quad \text{(in CMF)}$$

Similar formulas for $d\sigma/d\Omega_2^*$ obtained by inter-changing $1 \leftrightarrow 2$. In many cases, more convient with an invariant cross section like $d\sigma/dt$ than the expression above:

$$\frac{d\sigma}{dt} = \frac{|\mathrm{M}|^2}{16\pi\lambda(s, m_a^2, m_b^2)} \quad \text{(in any frame)}$$

Optical theorem



The **optical theorem** relates the **total cross section** for the process $p_a + p_b \to$ anything with the forward scattering amplitude of the corresponding elastic process (see e.g. G. Källen: Elementary particle physics, Addison–Wesley, 1964):

$$\mathrm{Im}\{\mathrm{M}_{\mathrm{elastic}}(s,t=0)\} = \sqrt{\lambda(s,m_a^2,m_b^2)}\,\sigma_{tot}(s)$$

plus $\quad d\sigma/dt = |\mathrm{M}|^2 / 16\pi\lambda(s,m_a^2,m_b^2) \quad \Rightarrow$

$$(\mathrm{Re}\{\mathrm{M}_{\mathrm{elastic}}(s,t=0)\})^2 = \lambda(s,m_a^2,m_b^2)\left(16\pi \left.\frac{d\sigma}{dt}\right|_{t=0} - \sigma_{tot}^2(s)\right)$$

Implies that the "optical point" is related to the total cross section: $\left.\dfrac{d\sigma_{elastic}}{dt}\right|_{t=0} \geq \dfrac{1}{16\pi}\sigma_{tot}^2(s)$

NB! QM predicts: $Im\{\mathrm{M}_{\mathrm{elastic}}\} \gg Re\{\mathrm{M}_{\mathrm{elastic}}\}$ @ t = 0

The TOTEM experiment at LHC uses the optical theorem to determine the total *pp* cross section using the elastic differential distribution or cross section dN_{el}/dt or $d\sigma_{el}/dt$ at t = 0 (very forward scattering). The dN_{el}/dt at t = 0 is determined from extrapolating the measured distribution at small t (~ 10^{-3} GeV^2) to t = 0. TOTEM does it in several ways, relying or not relying on luminosity (L) measuement

$$\sigma_{tot}^2 = \frac{16\pi}{1+\rho^2} \times \left.\frac{dN_{el}}{dt}\right|_{t=0} / L$$

$$\sigma_{tot} = (N_{el} + N_{inel})/L$$

$$\Rightarrow \quad \sigma_{tot} = \frac{16\pi}{1+\rho^2} \times \frac{(dN_{el}/dt)|_{t=0}}{N_{el}+N_{inel}}$$

$$\rho \equiv \mathrm{Re}\{\mathrm{M}_{\mathrm{elastic}}(s,t=0)\} / \mathrm{Im}\{\mathrm{M}_{\mathrm{elastic}}(s,t=0)\}$$

$$\approx 0.10 \text{ at LHC energies (13 TeV)}$$

Optical theorem

$$\left.\begin{aligned} \operatorname{Im} M_{el}\big|_{t=0} &= \sqrt{\lambda}\,\sigma_{tot} \\ &\text{and} \\ \frac{d\sigma}{dt} &= |M|^2/16\pi\lambda \end{aligned}\right\} \Rightarrow$$

$$\operatorname{Re} M_{el}\big|^2_{t=0} = |M|^2_{t=0} - \operatorname{Im} M_{el}\big|^2_{t=0}$$

$$= 16\pi\lambda \frac{d\sigma_{el}}{dt}\Big|_{t=0} - \lambda\sigma_{tot}^2$$

$$= \lambda\left(16\pi \frac{d\sigma_{el}}{dt}\Big|_{t=0} - \sigma_{tot}^2\right) \geq 0$$

optical point (OP): $\displaystyle \frac{d\sigma_{el}}{dt}\Big|_{t=0} \geq \frac{\sigma_{tot}^2}{16\pi}$

define: $\displaystyle \rho = \frac{\operatorname{Re} M_{el}}{\operatorname{Im} M_{el}}\Big|_{t=0} \Rightarrow$

$$\rho^2 = \frac{\operatorname{Re} M_{el}\big|^2_{t=0}}{\operatorname{Im} M_{el}\big|^2_{t=0}} = \frac{\cancel{\lambda}\left(16\pi \frac{d\sigma_{el}}{dt}\Big|_{t=0} - \sigma_{tot}^2\right)}{\cancel{\lambda}\,\sigma_{tot}^2}$$

$$\Rightarrow \boxed{\sigma_{tot}^2 = \frac{16\pi}{1+\rho^2}\underbrace{\frac{d\sigma_{el}}{dt}\Big|_{t=0}}_{\frac{dN_{el}}{dt}\big|_{t=0}/\mathcal{L}}} \leftarrow \text{OP}$$

$\mathcal{L}$ = luminosity

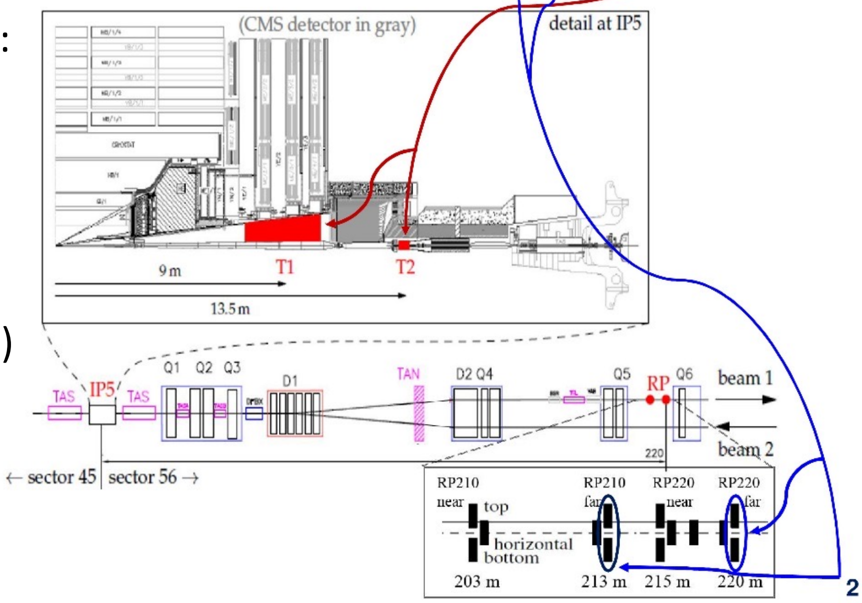
σ_{tot} measurement @ √s = 13 TeV

Luminosity independent method:

$$\sigma_{tot} = \frac{16\pi}{(1+\rho^2)} \frac{(dN_{el}/dt)_{t=0}}{(N_{el}+N_{inel})}$$

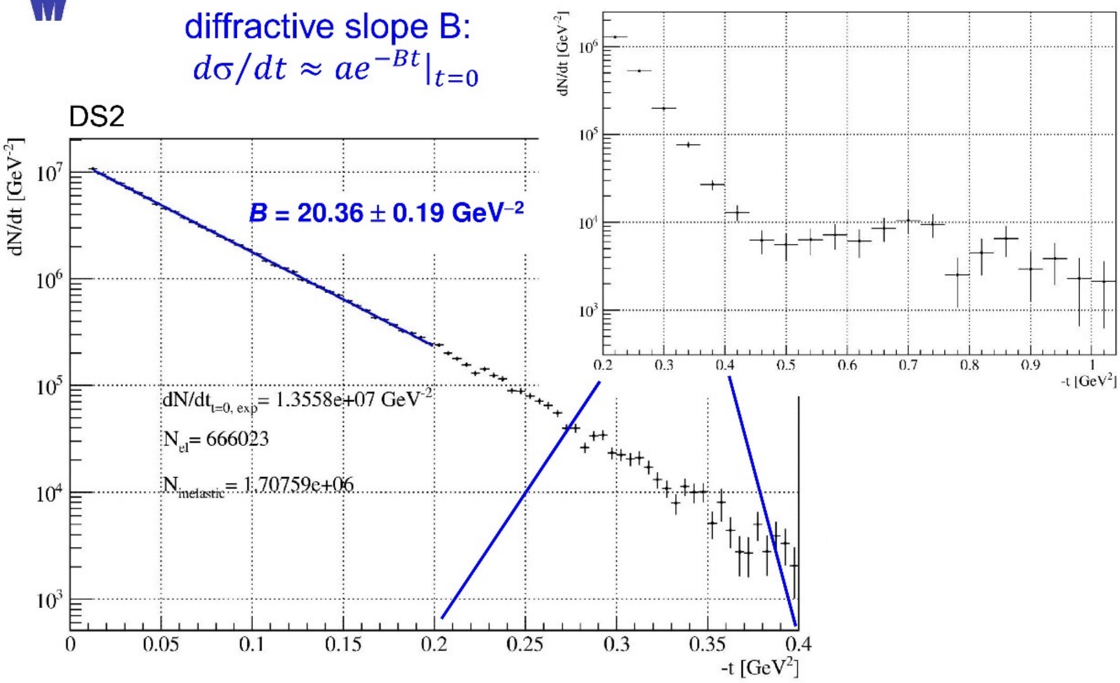
Inelastic acceptance:
- T1: 3.1 < |η| < 4.7
- T2: 5.3 < |η| < 6.5

with low transverse momentum (P_T) threshold (40 MeV (T2) & 100 MeV (T1))

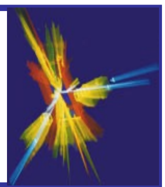


β* = 90 m dN_{el}/dt @ √ s = 13 TeV

diffractive slope B:
$d\sigma/dt \approx ae^{-Bt}|_{t=0}$



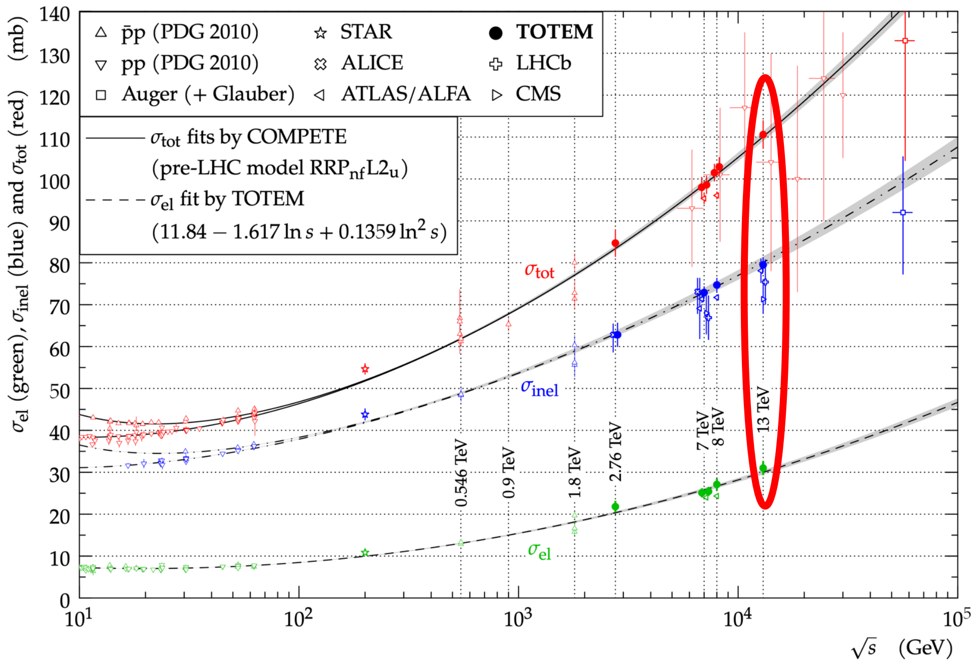
Total cross-section & elastic scattering



σ_{tot}, σ_{inel} & σ_{el} vs $\sqrt{s}$

TOTEM @ $\sqrt{s}$ = 13 TeV (ρ = 0.10): *EPJC 79 (2019) 103*

σ_{tot} = 110.6 ± 3.4 mb, σ_{inel} = 79.5 ± 1.8 mb, σ_{el} = 31.0 ± 1.7 mb



5

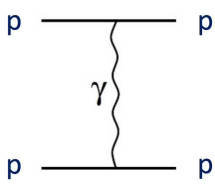
Elastic scattering: t-channel exchange

Elastic proton (anti)proton scattering at TeV scale: gluonic exchange

Experimental variable: $t \approx -P^2\theta^2$, four-momentum transfer squared

Strong interaction (non-pertutative QCD)

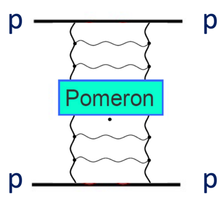
Electromagnetism (QED): $J^{PC} = 1^{--}$



Photon exchange

dominates at very low |t| (< ≈ 10^{-3})

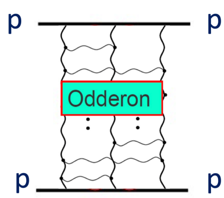
Crossing even C = +



"Pomeron" exchange: system of 2 (or more number of) gluons

dominates at low |t|, ≈ imaginary part of A_{el}^{nucl}
same for pp & pp̄

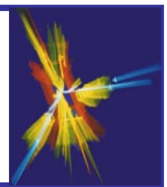
Crossing odd C = –



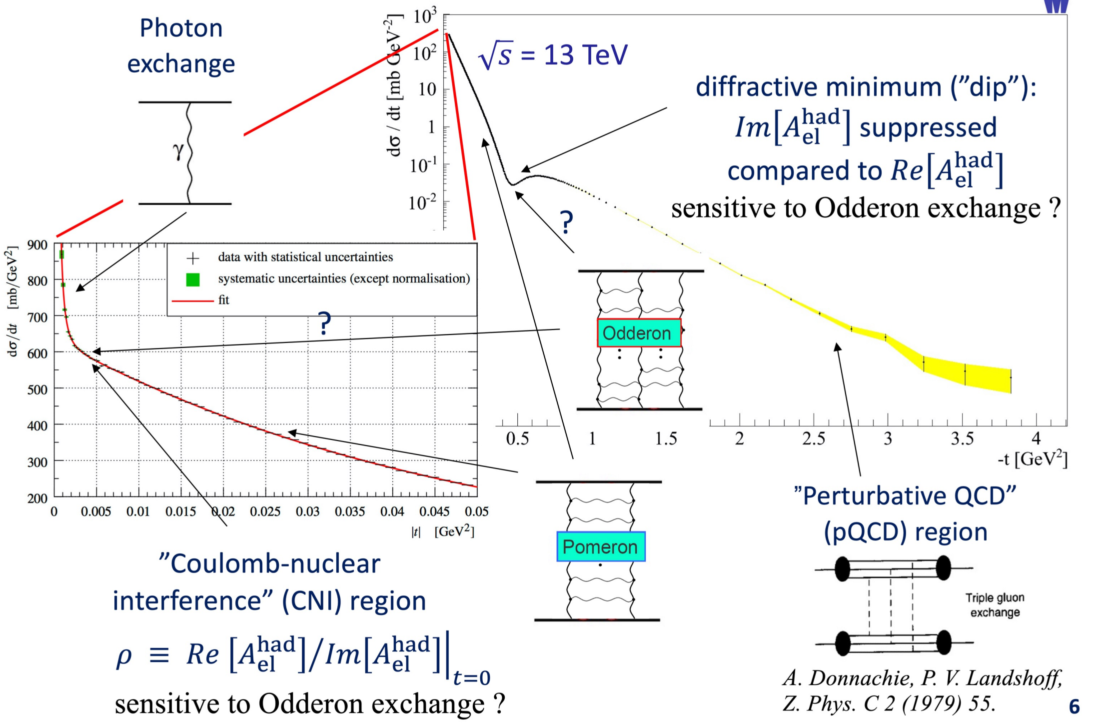
"Odderon" exchange: system of 3 (or more number of) gluons

mostly suppressed, mainly real part of A_{el}^{nucl}
different sign for pp & pp̄

3



Elastic pp differential cross-section



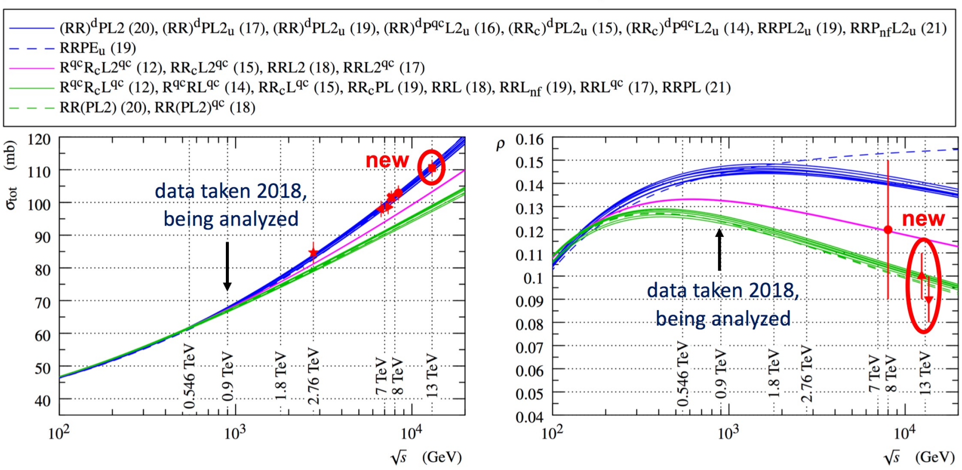
A. Donnachie, P. V. Landshoff, Z. Phys. C 2 (1979) 55.

6

σ_{tot} & ρ measurement in pp @ $\sqrt{s}$ = 13 TeV

TOTEM @ $\sqrt{s}$ = 13 TeV: σ_{tot} = 110.5 ± 2.4 mb, ρ = 0.09/0.10 ± 0.01
EPJC 79 (2019) 785

Comparison to conventional (no-Odderon) model predictions *(PRL 89 (2002) 201801)*:



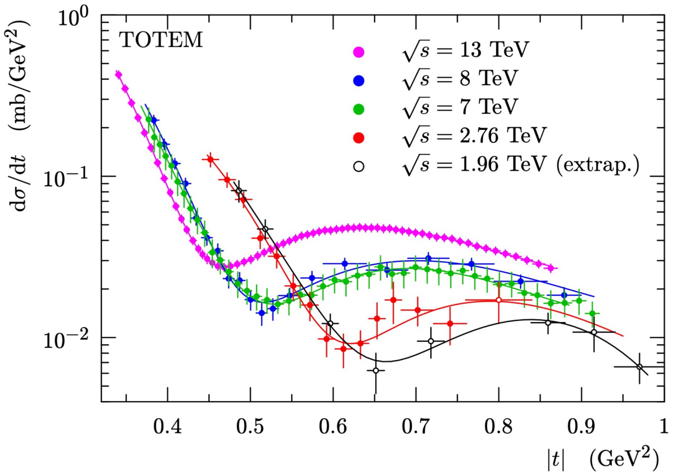
no conventional (no-Odderon) model able to describe simultaneously TOTEM σ_{tot} & ρ measurements ⇒ adding t-channel exchange of a ”Odderon” improves model descriptions

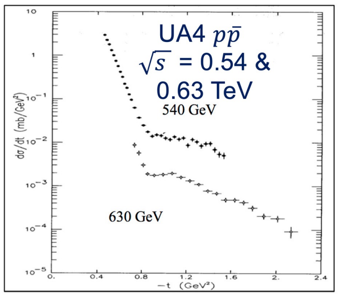
4

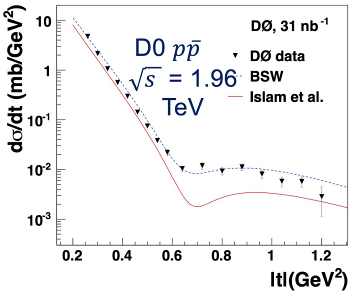
Elastic scattering & more Odderon evidence

Elastic $pp/p\bar{p}$ cross-section characteristics

At TeV-scale, pp elastic $d\sigma/dt$ characterized by a diffractive minimum ("dip") & a secondary maximum ("bump"), wheras $p\bar{p}$ $d\sigma/dt$ characterized only by a "kink".







@TeV scale: persistancy of dip & bump for pp, absence of dip & bump for $p\bar{p}$

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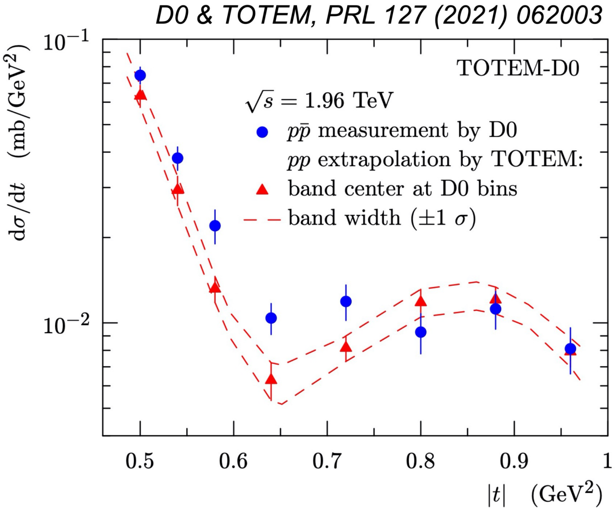
Comparison of pp & $p\bar{p}$ cross section

D0 & TOTEM, PRL 127 (2021) 062003



χ^2 test of pp & $p\bar{p}$ difference: 3.4σ significance for t-channel exchange of Odderon

- Combine independent evidence of odderon from TOTEM ρ & σ_{tot}^{pp} measurements in a completely different $|t|$-domain with evidence from the pp & $p\bar{p}$ comparison.
- Combination excludes all models without odderon exchange @ 5.2-5.7σ ⇒ first observation of odderon

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Physical region in *s,t,u*

When the reaction $p_a + p_b \to p_1 + p_2$ is described by E_1^* & θ_{a1}^*, then the physical region for the s–channel can be defined by $E_1^* \geq m_1$, $-1 \leq \cos\theta_{a1}^* \leq 1$. In other words, the reaction $p_a + p_b \to p_1 + p_2$ can experimentally have any values satisfying these conditions. To extract the physical region for 2 → 2 scattering, it is most convient to fix the boundary from the condition $\sin\theta_{a1}^* = 0 \Leftrightarrow \cos\theta_{a1}^* = \pm 1$.

$$\Delta_3(p_a, p_b, p_1) \equiv \begin{vmatrix} p_a^2 & p_a \cdot p_b & p_a \cdot p_1 \\ p_b \cdot p_a & p_b^2 & p_b \cdot p_1 \\ p_1 \cdot p_a & p_1 \cdot p_b & p_1^2 \end{vmatrix} = s(P_a^*)^2 (P_1^*)^2 \sin^2\theta_{a1}^*$$

From the determinant one obtains **the basic four-particle kinematic function** $G(x, y, z, u, v, w)$, which corresponds to Δ_3 in the same way as $\lambda(x, y, z)$ corresponds to Δ_2 :

$$\Delta_3(p_a, p_b, p_1) =$$

$$-\tfrac{1}{4} G\{(p_a + p_b)^2, (p_a - p_1)^2, (p_a + p_b - p_1)^2, p_a^2, p_b^2, p_1^2\} =$$

$$-\tfrac{1}{4} G(s, t, m_2^2, m_a^2, m_b^2, m_1^2)$$

So then $$\sin^2\theta_{a1}^* = -4s \frac{G(s,t,m_2^2,m_a^2,m_b^2,m_1^2)}{\lambda(s,m_a^2,m_b^2)\lambda(s,m_1^2,m_2^2)}$$

Physical region for 2 → 2 scattering in the st plane has to satisfy the requirement: $\Delta_3 \geq 0$ where arguments may be any three linearly independent combinations of p_a, p_b, p_1 and p_2.

Physical region in *s,t,u*

An equivalent requirement based on G–function is:

$$G(s,t,m_2^2,m_a^2,m_b^2,m_1^2) \le 0$$

(E.Byckling & K.Kajantie)

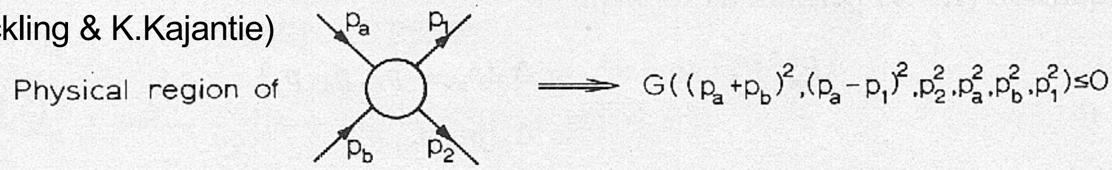


Figure IV.5.3

NB! Valid even if some of the p_i's are groups of particles.

The algebraic expression for $G(x, y, z, u, v, w)$ is:

$$\begin{aligned} G(x,y,z,u,v,w) = {} & x^2y + xy^2 + z^2u + zu^2 + v^2w + vw^2 + xzw \\ & + xuv + yzv + yuw - xy(z+u+v+w) \\ & - zu(x+y+v+w) - vw(x+y+z+u) \end{aligned}$$

NB! The physical region of symmetric Gram determinants depends on the order, e.g. $\Delta_2 \le 0$ and $\Delta_3 \ge 0$. The rule is that odd orders ≥ 0 and even ≤ 0. Implies e.g. that for 2 $\to$ 3 scattering, the physical region is described by $\Delta_4 \le 0$.

Considering a decay $p \to p_i + p_j$, it can be shown that

$$\Delta_2(p_i,p_j) = p_i^2 p_j^2 - (p_i \cdot p_j)^2 \le 0 \quad \Leftrightarrow$$

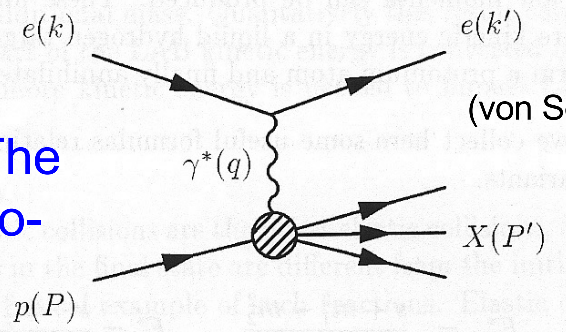
$$(p_i + p_j)^2 \ge (m_i + m_j)^2$$

Logically then in $p_a + p_b \to p_1 + p_2$, $s = (p_1+p_2)^2 = (p_a+p_b)^2$ has to be larger than both $(m_1+m_2)^2$ & $(m_a+m_b)^2$ or smaller than both $(m_1-m_2)^2$ & $(m_a-m_b)^2$. Same will be valid for other invariants so physical s, t & u in scattering have to satisfy:

$$s \ge \max\{(m_a+m_b)^2,(m_1+m_2)^2\}$$

DIS kinematics

Lepton-hadron scattering at sufficiently high energies create a large number of final state hadrons & such reactions are called **Deep Inelastic Scattering** (DIS). The reaction can generally be written as $a + N \to b + X$, where X stands for a hadronic system with an arbitrary number of particles, N a nucleon and a & b are leptons. An example is electro-magnetic electron–proton DIS shown in the figure. The probe can either be electro-magnetic (γ), neutral (Z^0) or charged ($W^{\pm}$) current.



(von Schlippe)

Figure 2: Generic diagram of deep inelastic scattering.

To describe DIS reactions, the 4–momentum of incoming lepton is denoted $k = (E,\bar{k})$, that of target nucleon P and those of the scattered lepton & hadronic system by $k' = (E',\bar{k}')$, and P', respectively. The exchanged boson has the 4–momentum $q = k - k'$. 4–momentum conservation gives $k + P = k' + P'$. Since DIS energies are at least a few GeV, the lepton masses can safely be set to zero. Then the 4–momentum transfer squared $q^2 = (k-k')^2 \approx -2EE'(1-\cos\theta) \leq 0$, i.e. the exchanged boson spacelike.

Invariant $W^2 = P'^2$ is a DIS–variable since the hadronic system can have variable multiplicity. The kinematics of a DIS reaction is therefore determined by 3 independent invariants rather than 2 as in the case in $2 \to 2$ scattering. The reaction is called ”deep” since $-q^2 \gg m_N^2$ & inelastic since final state X not just a nucleon (& usually $W^2 \gg m_N^2$).

DIS kinematics

A natural choice of one of these invariants is $S = (k + P)^2 = m_N^2 + 2k \cdot P$, which defines the experimental setup. The 2nd invariant is usually chosen to be the negative (4–momentum transfer)^2, $Q^2 = -q^2 = 4EE'\sin^2(\theta/2)$. The 3rd independent invariant can be taken to be W or one of the dimensionless variables $x = Q^2/(2P \cdot q)$ or $y = (P \cdot q)/(k \cdot P)$. The invariant x is called Bjorken–x and gives the fraction of the nucleon momentum carried by the involved parton. The variable y has a simple physical meaning in the TF: $y = 1 - E^{T\prime}/E^T$ i.e. the relative energy loss of the lepton.

In fixed target DIS, $S = m_N^2 + 2m_N E_a$, whereas in a lepton–proton collider like HERA $S = 4\,E_a E_p$. Note following useful DIS variable relations $Q^2 \approx xyS$ & $W^2 \approx m_N^2 + Q^2(1/x - 1)$. Within the parton model framework, the process can be viewed as below with the lepton–quark collision as the **hard subprocess**. If we think of the incoming lepton and nucleon as travelling in opposite directions, then at sufficiently high momenta, the energy of the quark is the same fraction of the nucleon energy i.e. x. Then the subprocess invariant $s = (k + p)^2 = xS$, where p (= xP) is the 4–momentum of the incoming quark.

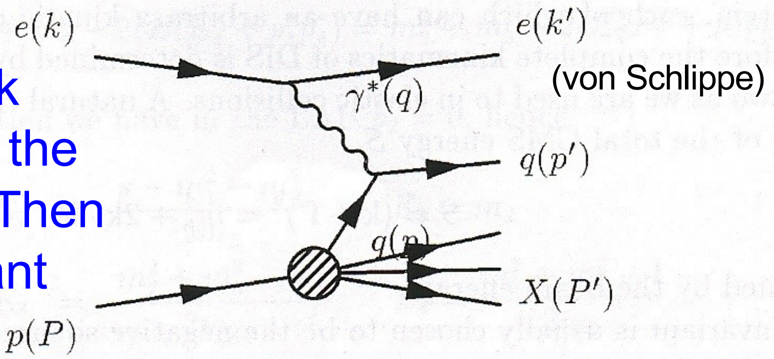


Figure 3: Parton model diagram of deep inelastic scattering.

One is naturally interested in the parton content of the proton (to be able to describe e.g. proton–proton processes).

Parton distribution functions

The proton content is described by the parton distribution functions (**pdf's**) that gives the momentum distribution in the proton separately for each partons species (e.g. for a valence quark flavour or gluons). The kinematics of the parton process is controlled by 2 variables, x & Q^2 so then also the proton pdf's $q_i(x, Q^2)$ are functions of both x & Q^2. Below is shown the underlying physics reasons for the experimentally observed shape of the valence quark pdf.

(F.Halzen & A.Martin)

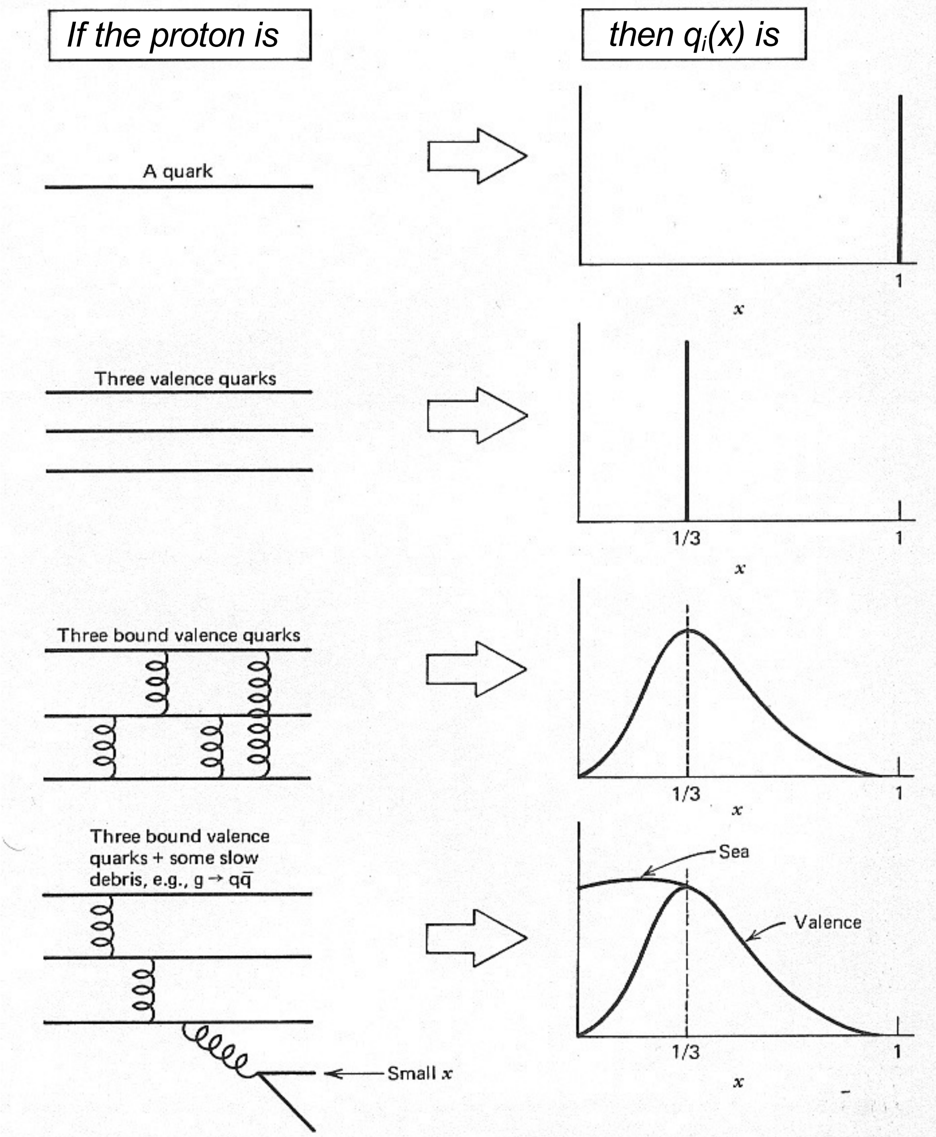
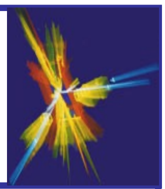


Fig. 9.7 The structure function pictured corresponding to different compositions assumed for the proton.



Evolution of pdfs

Parton model: pdfs converge at fixed x when $Q^2 \to \infty$

i.e. $q_i(x, Q^2) \to q_i(x)$, when $Q^2 \to \infty$

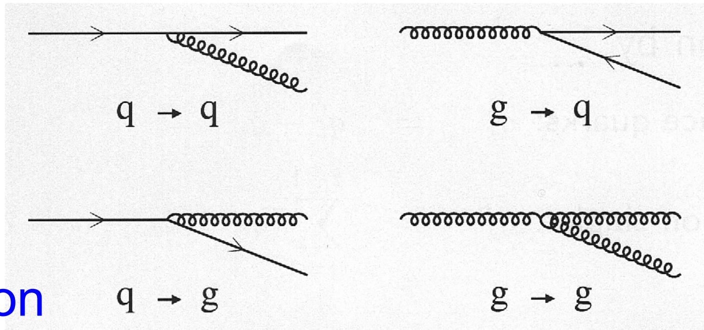
Violated due to hard gluon radiation, perturbative QCD predicts dependence on scale μ via "DGLAP"-equations:

$$\frac{\partial (q_i - \bar{q}_i)}{\partial \ln \mu^2} = \frac{\alpha_s(\mu^2)}{2\pi} \int_x^1 \frac{dz}{z} P_{qq}(z)(q_i(x/z, Q^2) - \bar{q}_i(x/z, Q^2))$$

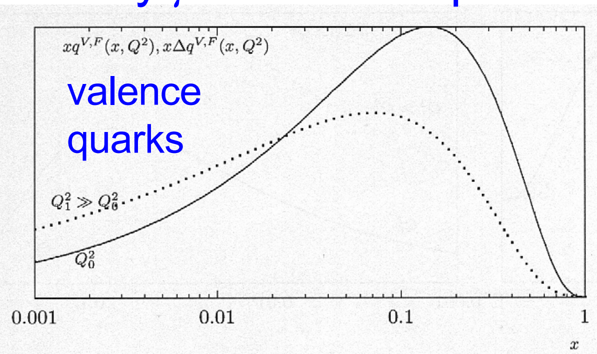
$$\frac{\partial}{\partial \ln \mu^2} \begin{pmatrix} (q_i + \bar{q}_i) \\ g \end{pmatrix} = \frac{\alpha_s(\mu^2)}{2\pi} \begin{pmatrix} P_{qq} & 2N_F P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} (q_i + \bar{q}_i) \\ g \end{pmatrix}$$

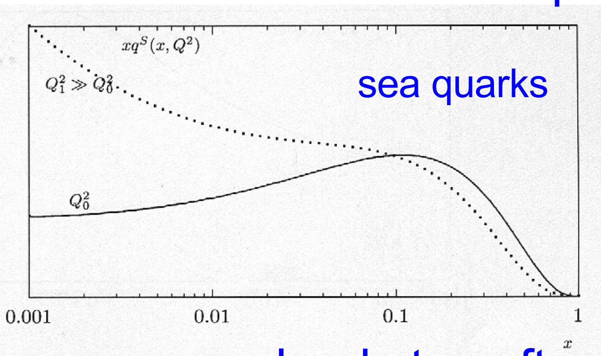
(Dokshitzer–Gribov–Lipatov–Altarelli–Parisi)

where $P_{ij}(z)$ gives the probability for a i $\to$ j splitting with momentum fraction z. N_F = number of flavours. NB! even if QCD predicts the evolution

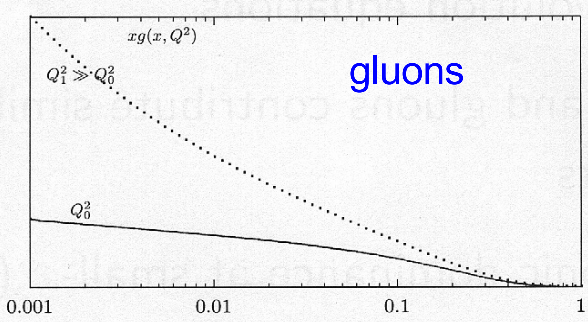


of pdf's from a particular scale, μ_0, it cannot predict them at any μ without experimental measurements as input.

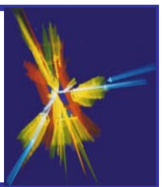




qualitative effects of DGLAP:

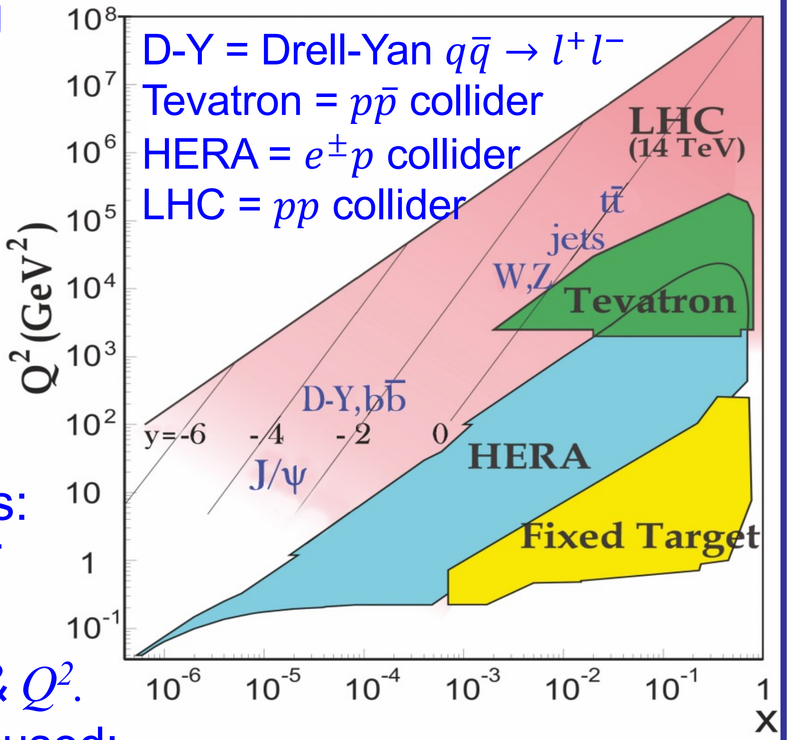


leads to softer pdf's & larger number of partons. large–x: valence quarks dominate, small–x: gluons.



Determination of pdfs

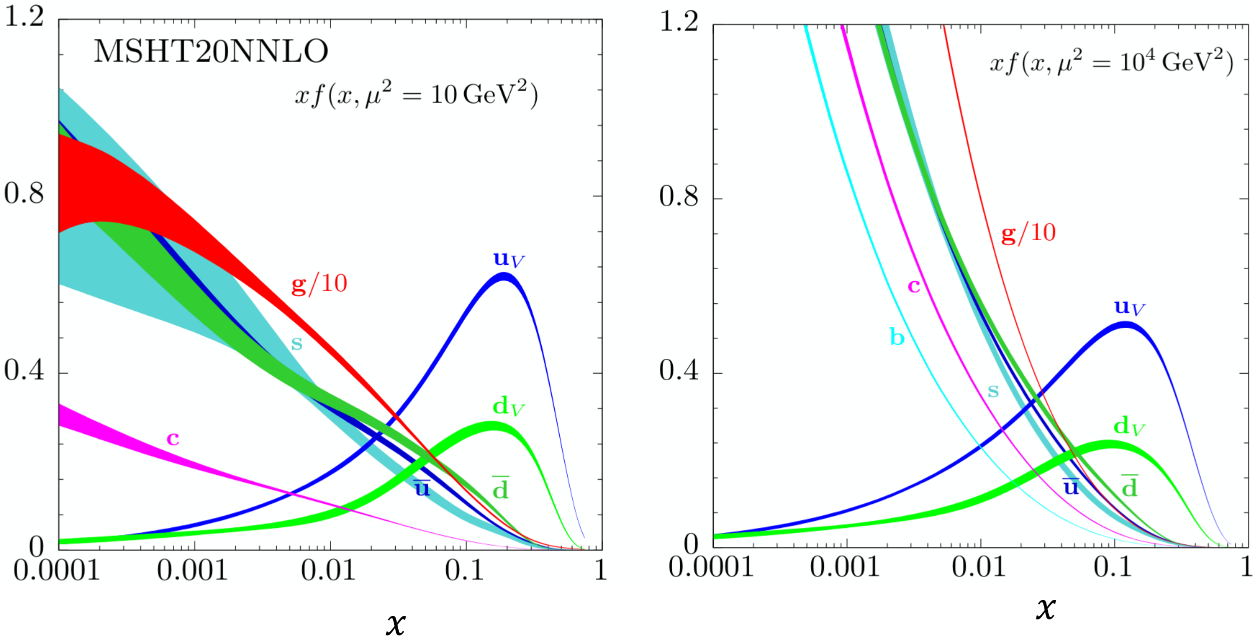
pdf's determined using an iterative procedure based on a functional ansatz combined with DGLAP evolution & experimental data. Some recent next-to-next leading order (NNLO) determinations: CTEQ, MSHT, NNPDF and ABMP. All use a wide range of data, x & Q^2. typical list of processes used:



Process	Subprocess	Partons	x range	
$\ell^{\pm}\{p,n\} \to \ell^{\pm} X$	$\gamma^* q \to q$	$q,\bar{q},g$	$x \gtrsim 0.01$	Fixed target
$\ell^{\pm} n/p \to \ell^{\pm} X$	$\gamma^* d/u \to d/u$	d/u	$x \gtrsim 0.01$	
$pp \to \mu^+\mu^- X$	$u\bar{u}, d\bar{d} \to \gamma^*$	$\bar{q}$	$0.015 \lesssim x \lesssim 0.35$	
$pn/pp \to \mu^+\mu^- X$	$(u\bar{d})/(u\bar{u}) \to \gamma^*$	$\bar{d}/\bar{u}$	$0.015 \lesssim x \lesssim 0.35$	
$\nu(\bar{\nu}) N \to \mu^-(\mu^+) X$	$W^* q \to q'$	$q,\bar{q}$	$0.01 \lesssim x \lesssim 0.5$	
$\nu N \to \mu^-\mu^+ X$	$W^* s \to c$	s	$0.01 \lesssim x \lesssim 0.2$	
$\bar{\nu} N \to \mu^+\mu^- X$	$W^* \bar{s} \to \bar{c}$	$\bar{s}$	$0.01 \lesssim x \lesssim 0.2$	
$e^{\pm} p \to e^{\pm} X$	$\gamma^* q \to q$	$g,q,\bar{q}$	$10^{-4} \lesssim x \lesssim 0.1$	HERA
$e^+ p \to \bar{\nu} X$	$W^+\{d,s\} \to \{u,c\}$	d,s	$x \gtrsim 0.01$	
$e^{\pm}p \to e^{\pm} c\bar{c}X, e^{\pm} b\bar{b}X$	$\gamma^* c \to c,\ \gamma^* g \to c\bar{c}$	c,b,g	$10^{-4} \lesssim x \lesssim 0.01$	
$e^{\pm}p \to$ jet$+X$	$\gamma^* g \to q\bar{q}$	g	$0.01 \lesssim x \lesssim 0.1$	
$p\bar{p}, pp \to$ jet(dijet)$+X$	$gg, qg, qq \to 2j$	g,q	$0.00005 \lesssim x \lesssim 0.5$	LHC & Tevatron
$p\bar{p} \to (W^{\pm} \to \ell^{\pm}\nu) X$	$ud \to W^+, \bar{u}\bar{d} \to W^-$	$u,d,s,\bar{u},\bar{d},\bar{s}$	$x \gtrsim 0.05$	
$pp \to (W^{\pm} \to \ell^{\pm}\nu) X$	$u\bar{d} \to W^+, d\bar{u} \to W^-$	$u,d,s,\bar{u},\bar{d},\bar{s},g$	$x \gtrsim 0.001$	
$p\bar{p}(pp) \to (Z \to \ell^+\ell^-)X$	$uu, dd, ..(u\bar{u}, ..) \to Z$	$u,d,s,..(g)$	$x \gtrsim 0.001$	
$pp \to W^- c,\ W^+\bar{c}$	$gs \to W^- c$	$s,\bar{s}$	$x \sim 0.01$	
$pp \to (\gamma^* \to \ell^+\ell^-)X$	$u\bar{u}, d\bar{d}, .. \to \gamma^*$	$\bar{q},g$	$x \gtrsim 10^{-5}$	
$pp \to (\gamma^* \to \ell^+\ell^-)X$	$u\gamma, d\gamma, .. \to \gamma^*$	γ	$x \gtrsim 10^{-2}$	
$pp \to b\bar{b}X,\ t\bar{t}X$	$gg \to b\bar{b},\ t\bar{t}$	g	$x \gtrsim 10^{-5}, 10^{-2}$	
$pp \to t(\bar{t})X,$	$bu(\bar{b}d) \to td(\bar{t}u)$	$b, d/u$	$x \gtrsim 10^{-2}$	
$pp \to$ exclusive $J/\psi, \Upsilon$	$\gamma^*(gg) \to J/\psi, \Upsilon$	g	$x \gtrsim 10^{-5}, 10^{-4}$	
$pp \to \gamma X$	$gq \to \gamma q,\ g\bar{q} \to \gamma\bar{q}$	g	$x \gtrsim 0.005$	

Determination of pdfs

The output are pdf's for different flavours of valence and sea quarks & gluons at specific x & Q^2 as shown below. Note that the gluon pdf is shown divided by a factor 10.



Even if exact values & shape differs for different Q^2 (or μ^2 as in Fig.) the general trends are the same: valence quarks dominate at $x > 0.1$, gluons dominate at $x < 0.1$. The higher Q^2, the faster gluon and sea quark distributions raise at low x

In hadron–hadron collisions, usually pdf's are one of the largest uncertainties in measurements of cross sections, etc …

At best pdfs are known to a few % but in certain corners of the Q^2–x plane the uncertainty can be much larger …

For more info see PDG structure function review

Decay of one particle into three

Since a decay process $p \rightarrow p_1 + p_2 + p_3$ is related by crossing to 2 → 2 scattering, the number of invariant variables must be the same, namely two. Let's consider first the invariant variables for the process 1 → 3.

(E.Byckling & K.Kajantie)

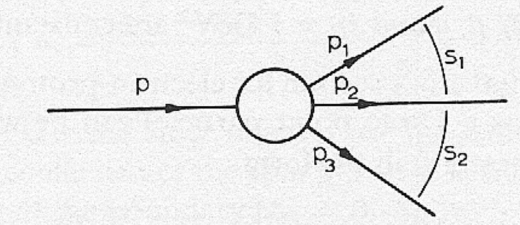


Figure V.1.1 Three-particle decay $p \rightarrow p_1 + p_2 + p_3$ with invariant variables s_1 and s_2

As invariant variables, it is convient to choose s, t & u as in 2 → 2 scattering. To avoid mixup let's change notation

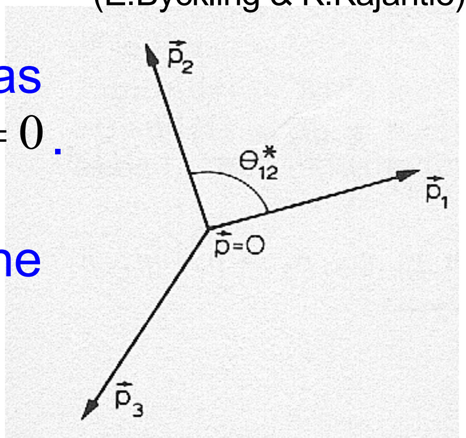
$$\text{new invariants}: \begin{cases} s_{12} \equiv s_1 = (p_1 + p_2)^2 = (p - p_3)^2 \\ s_{23} \equiv s_2 = (p_2 + p_3)^2 = (p - p_1)^2 \\ s_{31} \equiv s_3 = (p_3 + p_1)^2 = (p - p_2)^2 \end{cases}$$

$$\text{their common relation}: s_1 + s_2 + s_3 = s + m_1^2 + m_2^2 + m_3^2$$

Non-invariant variables are three-momenta & angles. To define them one has to specify a Lorentz frame. The most common one correspond to 2 → 2 scattering CMF.

The rest frame of the decaying particle or overall CMF is defined as the frame in which $\bar{p} = \bar{p}_1 + \bar{p}_2 + \bar{p}_3 = 0$. This is the analogue of TF in 2 → 2 scattering in the sense that one of the external momenta is taken to be at rest. Quantities in this frame are denoted by an asterisk.

(E.Byckling & K.Kajantie)



Expressions for energies and momenta can immediately be derived using s_i (i = 1...3) definitions in frame $p = (\sqrt{s}, \overline{0})$.

$$E_1^* = \frac{s + m_1^2 - s_2}{2\sqrt{s}} \quad E_2^* = \frac{s + m_2^2 - s_3}{2\sqrt{s}} \quad E_3^* = \frac{s + m_3^2 - s_1}{2\sqrt{s}}$$

$$P_1^* = \frac{\sqrt{\lambda(s, m_1^2, s_2)}}{2\sqrt{s}} \quad P_2^* = \frac{\sqrt{\lambda(s, m_2^2, s_3)}}{2\sqrt{s}} \quad P_3^* = \frac{\sqrt{\lambda(s, m_3^2, s_1)}}{2\sqrt{s}}$$

Note the logic: E_1* is obtained by considering the two-particle decay $p \to p_1$ + (p_2 + p_3) with final state masses m_1 & $\sqrt{s_2}$. The angles between the momentum vectors follow from expansions of the s_i (i = 1...3) definitions, e.g.

$$s_1 = (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2E_1^* E_2^* - 2P_1^* P_2^* \cos\theta_{12}^* \Rightarrow$$

$$\cos\theta_{12}^* = \overline{p}_1 \cdot \overline{p}_2 / P_1 P_2 \Big|_{\overline{p}=0} =$$

$$\frac{(s + m_1^2 - s_2)(s + m_2^2 - s_3) + 2s(m_1^2 + m_2^2 - s_1)}{\sqrt{\lambda(s, m_1^2, s_2)}\sqrt{\lambda(s, m_2^2, s_3)}}$$

$\sin\theta_{12}$* is related to corresponding symmetric Gram determinant Δ_3 in CMF: $\sin^2\theta_{12}^* = \left.\frac{|\overline{p}_1 \times \overline{p}_2|^2}{P_1^2 P_2^2}\right|_{\overline{p}=0} \Rightarrow$

$$\sin^2\theta_{12}^* = \frac{\Delta_3(-p_1, p, p_2)}{s(P_1^*)^2 (P_2^*)^2} = \frac{-4s\,G(s_2, s_1, m_3^2, m_1^2, s, m_2^2)}{\lambda(s, m_1^2, s_2)\lambda(s, m_2^2, s_3)}$$

Dalitz plot

The Dalitz plot is defined as the physical region of $p \rightarrow p_1 + p_2 + p_3$ in the $s_1 s_2$ plane. More generally, can be defined as the physical region in terms of any variables related to s_1 & s_2 by a linear transformation with constant Jacobian e.g. any pair s_i, s_j or any pair E_i^*, E_j^*, where i & j = 1...3.

The Dalitz plot is given by all points in the $s_1 s_2$ plane that satisfies the following equation: $G(s_2, s_1, m_3^2, m_1^2, s, m_2^2) \leq 0$

(E.Byckling & K.Kajantie)

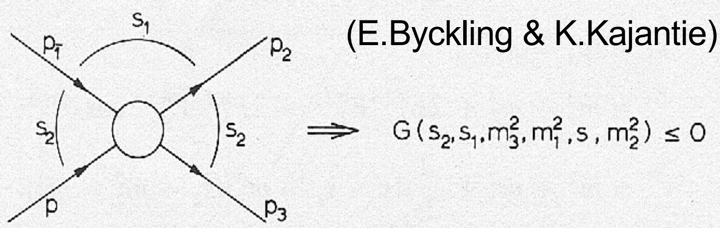


Figure V.2.1 A direct procedure giving the physical region in the $s_1 s_2$ plane

The G here is the same as in the expressions for $\sin \theta_{12}^*$. The equality gives the boundary in the Dalitz plot. This can be obtained e.g. by solving s_1 in terms of s_2.

$$s_1^{\pm} = m_1^2 + m_2^2 - \frac{\left\{(s_2 + m_1^2 - s)(s_2 + m_2^2 - m_3^2) - \pm\sqrt{\lambda(s_2, s, m_1^2)}\sqrt{\lambda(s_2, m_2^2, m_3^2)}\right\}}{2 s_2}$$

The equation giving s_2 in terms of s_1 is obtained from the above by the exchange $p_1 \leftrightarrow p_3$; p, p_2 unchanged. Both give, of course, the same curve. By requiring the √ to be real, one gets the physical region in s_2. Cyclic symmetry implies that also following conditions have to be satisfied:

$$(m_1 + m_2)^2 \leq s_1 = (p_1 + p_2)^2 \leq (\sqrt{s} - m_3)^2$$
$$(m_2 + m_3)^2 \leq s_2 = (p_2 + p_3)^2 \leq (\sqrt{s} - m_1)^2$$
$$(m_3 + m_1)^2 \leq s_3 = (p_3 + p_1)^2 \leq (\sqrt{s} - m_2)^2$$

Dalitz plot

To determine the phase space density & obtain as well the condition for the boundary of the Dalitz plot directly, let's consider the phase space integral:

$$R_3(s) = \int \prod_{i=1}^{3} \frac{d^3\bar{p}_i}{2E_i} \delta^3(\bar{p} - \bar{p}_1 - \bar{p}_2 - \bar{p}_3)\delta(\sqrt{s} - E_1 - E_2 - E_3)$$

Integrate first over $\bar{p}_2$ in the rest frame $\bar{p} = 0 \quad \Rightarrow$

$$R_3(s) = \int \frac{d^3\bar{p}_1^* d^3\bar{p}_3^*}{8E_1^* E_2^* E_3^*} \delta(\sqrt{s} - E_1^* - E_2^* - E_3^*), \quad \text{where}$$

$$E_2^{*2} = |\bar{p}_1^* + \bar{p}_3^*|^2 + m_2^2 = P_1^{*2} + P_3^{*2} + 2P_1^* P_3^* \cos\theta_{13}^* + m_2^2$$

Write further $\quad d^3\bar{p}_1^*\, d^3\bar{p}_3^* = P_1^* E_1^* dE_1^* d\Omega_1^* P_3^* E_3^* dE_3^* d\cos\theta_{13}^* d\phi_3^*$

The δ–function containing energies can be used to integrate over $\cos\theta_{13}^*$ ($d\cos\theta_{13}^* = E_2^* dE_2^* / (P_1^* P_3^*)$) giving:

$$R_3(s) = \int \frac{dE_1^* dE_3^* d\Omega_1^* d\phi_3^* P_1^* E_1^* P_3^* E_3^*}{8E_1^* E_2^* E_3^* (P_1^* P_3^* / E_2^*)} \Theta(1 - \cos^2\theta_{13}^*)$$

$$= \int \frac{dE_1^* dE_3^* d\Omega_1^* d\phi_3^*}{8} \Theta(1 - \cos^2\theta_{13}^*)$$

Here the Θ–function restricts $\cos\theta_{13}^*$ to physical values only.

The variables E_1^* & E_3^* are linearly connected to s_1 & s_2 with the Jacobian $\partial(E_1^*, E_3^*) / \partial(s_2, s_1) = 1 / 4s$, thus

$$R_3(s) = \frac{1}{32s} \int ds_1 ds_2 d\Omega_1^* d\phi_3^* \Theta\{-G(s_2, s_1, m_3^2, m_1^2, s, m_2^2)\}$$

NB! the $\cos\theta_{13}^*$–condition is exchanged to a G–function condition, obtained by algebra from the E_2^*–condition in the δ–function. Analogous forms of $R_3(s)$ with the pairs s_2, s_3 & s_3, s_1 obtained by cyclic permutations of indices.

Dalitz plot

The solid angle Ω_1* describes the $\overline{p}_1$–orientation in CMF & ϕ_3*, the rotation of the entire momentum configuration about some axis. Integrating over Ω_1* & ϕ_3*, we obtain:

$$\text{The phase space distribution}: \frac{d^2R_3}{ds_1 ds_2} = \frac{\pi^2}{4s} \quad (= \text{constant})$$

In other words, if data of a three-particle decay is shown as points in a Dalitz plot, the density of points ∝ (matrix element)^2. Any structure is thus easily evident. This is why the Dalitz plot is so famous & used often. Note that this result is strictly only valid for three-particle decays.

$$\text{Further} \quad \frac{\mathrm{d}R_3}{\mathrm{d}s_2} = \frac{\pi^2}{4s}\int_{s_1^-}^{s_1^+} ds_1 = \frac{\pi^2}{4ss_2}\sqrt{\lambda(s_2,s,m_1^2)}\sqrt{\lambda(s_2,m_2^2,m_3^2)} \quad \Rightarrow$$

For the total volume of the three-particle phase space:

$$\mathrm{R}_3(s) = \frac{\pi^2}{4s}\int_{(m_2+m_3)^2}^{(\sqrt{s}-m_1)^2} \frac{ds_2}{s_2}\sqrt{\lambda(s_2,s,m_1^2)}\sqrt{\lambda(s_2,m_2^2,m_3^2)}$$

The √–factor is 4$^{\text{th}}$ order in s_2 & lead to elliptic functions. Explicit solutions exists only for some special cases.

Especially interesting are the extremes, $s \to$ threshold = $(m_1+m_2+m_3)^2$ (decay products non-relativistic, NR) or $s \to \infty$ (ultra-relativistic, UR). Latter is obtained by setting all m_i = 0.

$$\mathrm{R}_3^{\mathrm{UR}}(s) = \frac{\pi^2 s}{8} \qquad \mathrm{R}_3^{\mathrm{NR}}(s) = \frac{\pi^3\sqrt{m_1m_2m_3}}{2(m_1+m_2+m_3)^{\frac{3}{2}}}(\sqrt{s}-m_1-m_2-m_3)^2$$

Multi-particle production

The process $p_a + p_b \to p_1 + \ldots + p_n$ depends on $3n-4$ essential variables and can be described in numerous ways. Let's here concentrate on two methods that with some modifictions are the descriptions used in modern HEP event generators. It is possible to visualize a multi-particle reaction as taking place via resonance formation & decay. In the intermediate state, there are unstable particles, which successively decays to others & eventually form the final state particles. The alternative is to use a multi-peripheral mechanism, implying the dominance of a diagram of the type exhibited in the 2nd figure. Regardless of the actual validity of such dynamical ideas, we shall show that kinematically an n–particle final state can always be subdivided into simpler processes. This means that the phase space integral R_n can be recursively expressed in terms of R_l's,

(E.Byckling & K.Kajantie)

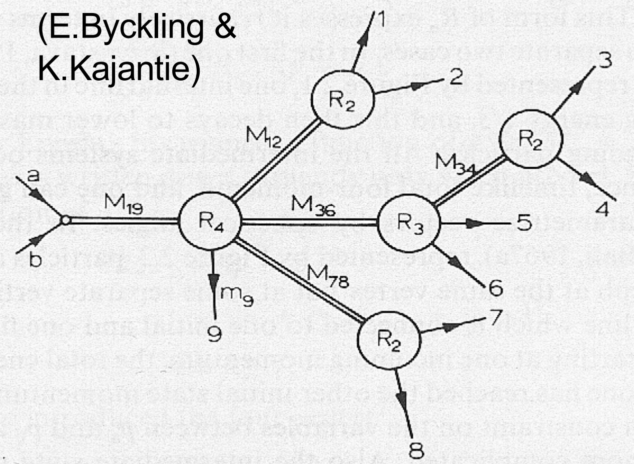
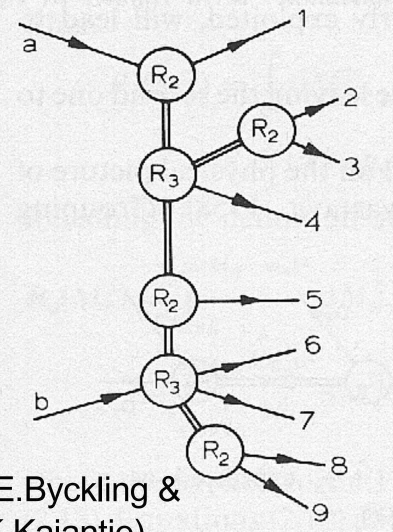


Figure VI.2.1 Example of cascade decay. Double lines denote systems of particles. Total energy is fixed. In this figure $M_{19}^2 = (p_1 + p_2 + \ldots + p_9)^2$, etc.



(E.Byckling & K.Kajantie)

where $l < n$. The 2 methods needs to be seperated since in the 1st one, all the intermediate systems occuring have timelike total four-momenta & hence one can go to their rest frames and parametrize vectors by spherical angles.

In the 2nd case, particles a & b do not join to the graph at the same vertex, and there is at least one line which is connected to one initial & one final state 4–momentum. Starting at one incoming momentum, the total energy s can only be fixed when one has reached the other initial state momentum in the graph, imposing a complicated constraint on the variables between p_a & p_b. Also intermediate state momenta may now be spacelike (see e.g. DIS) & some of the appropriate variables are now boosts.

The simplest possible recursion relation is based on the physical picture of sequential decay (see figure below):

$$R_n(p) = \int\int \frac{d^3\overline{p}_n}{2E_n} \prod_{i=1}^{n-1} \frac{d^3\overline{p}_i}{2E_i} \delta^4\left((p-p_n) - \sum_{i=1}^{n-1} p_i\right) = \int \frac{d^3\overline{p}_n}{2E_n} R_{n-1}(p-p_n)$$

R_{n-1} is only a function of $M_{n-1}^2 = (p-p_n)^2 = (p_1 + ... + p_{n-1})^2 \equiv k_{n-1}^2$

M_{n-1} is ofcourse the invariant mass of the system formed by particles 1, ..., $n-1$. Since R_{n-1} is a function of only 1 variable, it is most natural to take M_{n-1} as a variable of integration. When the following is inserted in integrand:

$$\int dM_{n-1}^2 \delta(M_{n-1}^2 - k_{n-1}^2) = 1, \int d^4k_{n-1} \delta^4(p - p_n - k_{n-1}) = 1 \quad \Rightarrow \quad R_n(M_n^2) =$$

$$\int dM_{n-1}^2 \int d^4k_{n-1} \int d^4p_n \delta(k_{n-1}^2 - M_{n-1}^2) \delta(p_n^2 - m_n^2) \delta^4(p - p_n - k_{n-1}) R_{n-1}(M_{n-1}^2)$$

(E.Byckling & K.Kajantie)

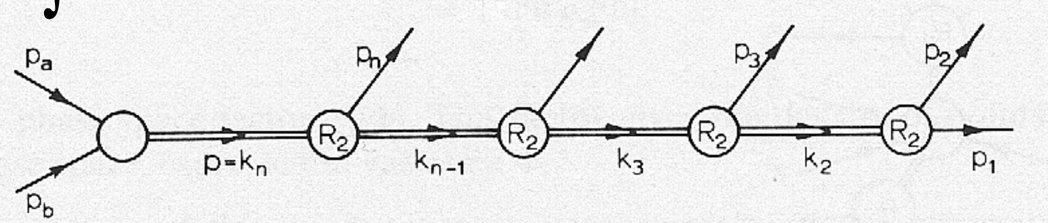


Figure VI.2.3 The reaction $p_a + p_b \rightarrow p_1 + \ldots + p_n$ expressed as a sequence of two-particle decays

$$R_n(M_n^2) = \int_{\mu_{n-1}^2}^{(M_n-m_n)^2} dM_{n-1}^2 R_2(k_n;k_{n-1}^2,p_n^2)R_{n-1}(M_{n-1}^2)$$

$$= \int_{\mu_{n-1}^2}^{(M_n-m_n)^2} dM_{n-1}^2 \int d\Omega_{n-1} R_{n-1}(M_{n-1}^2)\sqrt{\lambda(M_n^2,M_{n-1}^2,m_n^2)}\Big/8M_n^2,$$

where $\mu_i = m_1 + \ldots + m_i$. Thresholds give $\mu_{n\text{-}1} \le M_{n-1} \le M_n - m_n$

So R_n can be expressed as a product of R_2 describing the decay $p \to p_n + k_{n-1}$ & R_{n-1} describing the decay $k_{n-1} \to p_1 + \ldots. + p_{n-1}$, integrated over all possible values of the invariant mass M_{n-1}. To proceed, we iterate the above steps to obtain a relation corresponding to the entire chain. Let's use M_i instead of M_i^2 as variable. Then

$$R_n(M_n^2) = \frac{1}{2M_n}\int_{\mu_{n-1}}^{M_n-m_n} dM_{n-1}d\Omega_{n-1}\tfrac{1}{2}P_n \ldots \int_{\mu_2}^{M_3-m_3} dM_2 d\Omega_2 \tfrac{1}{2}P_3 \int d\Omega_1 \tfrac{1}{2}P_2,$$

where $P_i = \sqrt{\lambda(M_i^2,M_{i-1}^2,m_i^2)}\Big/2M_i.$

The $3n–4$ essential variables now consists of 2 types:

(i) $n–2$ invariant masses M_i, $M_i^2 = k_i^2$, $i = 2, .., n–1$, defined the masses of the intermediate states.

(ii) $2(n–1)$ angles θ_i, ϕ_i in $\Omega_i = (\cos\theta_i, \phi_i)$, $i = 1, ..., n–1$. These define the direction of $\bar{k}_i = -\bar{p}_{i+1}$ in the rest frame $\bar{k}_{i+1} = 0$ of the decay $k_{i+1} \to p_{i+1} + k_i$ (see figure).

(E.Byckling & K.Kajantie)

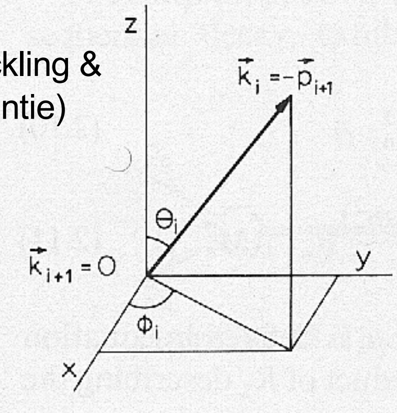


Figure VI.2.4 Definition of $\Omega_i = (\cos\theta_i, \phi_i)$. The orientation of coordinate axes can be chosen arbitrarily. To obtain the recursion relation (2.27) with multiperipheral momentum transfers, one chooses $\mathbf{p_a}$ as the z-axis and replaces $\cos\theta_i$ by the corresponding momentum transfer

The equation can be used as basis for a generator. Let's examine 2 special cases: (i) all m_i = 0, equivalent with the asymptotic limit, M_n (or s) → ∞ (the "ultrarelativistic" case):

$$R_n^{UR}(M_n^2) = R_n(s; m_i^2 = 0) = \left(\frac{\pi}{2}\right)^{n-1} \frac{s^{n-2}}{(n-1)!(n-2)!}$$

(ii) M_n (or s) → $\mu_n = \Sigma m_i$ (the "non-relativistic" case):

$$R_n^{NR}(M_n^2) = R_n(s; m_i) = \frac{(2\pi^3)^{(n-1)/2}\sqrt{\prod_{i=1}^n m_i}}{2\Gamma\left\{\frac{3}{2}(n-1)\right\}\left(\Sigma_{i=1}^n m_i\right)^{3/2}} \left(\sqrt{s} - \Sigma_{i=1}^n m_i\right)^{(3n-5)/2}$$

NB! $\Gamma(n) = (n-1)!$ for integer n $\quad \Gamma(x+1) = x\Gamma(x) \quad \Gamma(1/2) = \sqrt{\pi}$

A radically different relation for R_n is obtained exploiting the freedom of choosing the variables of intermediate R_i's in a ladder type diagram. If the direction of $\bar{p}_a$ is chosen as the z–axis, then the scattering angle, θ_{n-1}, of the process $p_a + p_b \to k_{n-1} + p_n$ can be replaced by the corresponding momentum transfer (see figure below).

$$t_{n-1} = (p_a - k_{n-1})^2 = m_a^2 + M_{n-1}^2 - 2E_a k_{n-1}^0 + 2P_a^{(n)} K_{n-1} \cos\theta_{n-1},$$

$$\text{where} \quad P_a^{(n)} = \sqrt{\lambda(M_n^2, m_a^2, m_b^2)}\Big/2M_n \quad \& \quad K_{n-1} = \sqrt{\lambda(M_n^2, M_{n-1}^2, m_n^2)}\Big/2M_n$$

(E.Byckling & K.Kajantie)

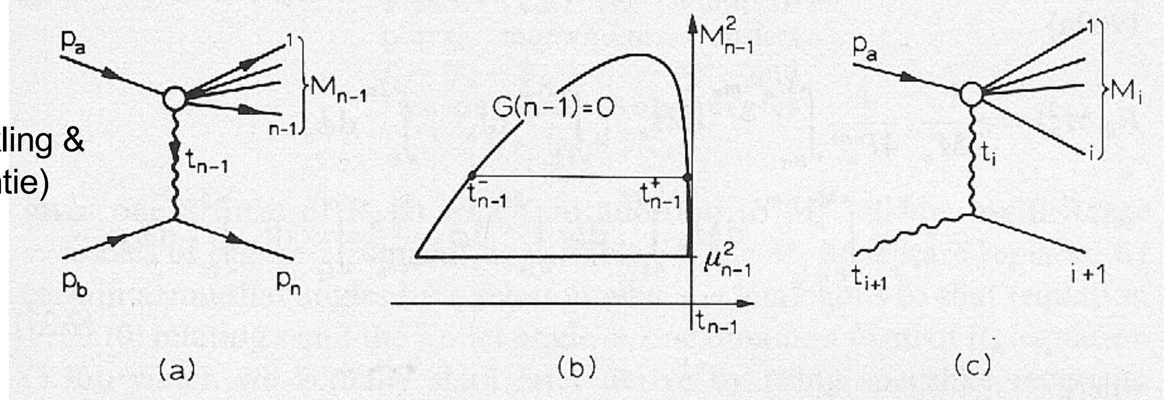


Figure VI.2.6 (a) The basic process when $\mathbf{p}_a$ is chosen as z axis in the frame $\mathbf{k}_n \equiv \mathbf{p}_a + \mathbf{p}_b = 0$; (b) the range of variation of t_{n-1} is given by the M_{n-1}^2, t_{n-1} Chew–Low plot; (c) the basic process at stage i of the iteration

When $\cos\theta_{n-1}$ is replaced by t_{n-1}, the range $-1 \le \cos\theta_{n-1} \le 1$ is transformed to a M_{n-1}–dependent range $t_{n-1}^{-} \le t_{n-1} \le t_{n-1}^{+}$, where the specific values of $t_{n-1}^{\pm}$ are solutions to $G(n-1) \equiv G(s, t_{n-1}, m_n^2, m_a^2, m_b^2, M_{n-1}^2) = 0$. In terms of t_{n-1},

$$R_n(M_n^2) = \int_{\mu_{n-1}^2}^{(M_n - m_n)^2} dM_{n-1}^2 \int_{t_{n-1}^-}^{t_{n-1}^+} dt_{n-1} \int_0^{2\pi} d\phi_{n-1} \frac{R_{n-1}(M_{n-1}^2; t_{n-1})}{4\sqrt{\lambda(M_n^2, m_a^2, m_b^2)}},$$

where R_{n-1} has to be regarded as a function of t_{n-1}, since t_{n-1} is the (mass)^2 of one of the initial particles leading to R_{n-1}. To iterate, we must apply the same equation for R_{n-1} remembering that $m_b^2 \equiv t_n$ now has to be replaced by t_{n-1}. If we also take M_n as a variable instead of M_n^2, we obtain

$$R_n(M_n^2) = \frac{1}{2M_n} \cdot \frac{1}{4P_a^{(n)}} \int_{\mu_{n-1}}^{(M_n - m_n)} dM_{n-1} \int_{t_{n-1}^-}^{t_{n-1}^+} dt_{n-1} \int_0^{2\pi} d\phi_{n-1} \ldots \frac{1}{4P_a^{(3)}} \int_{\mu_2}^{(M_3 - m_3)} dM_2 \int_{t_2^-}^{t_2^+} dt_2 \int_0^{2\pi} d\phi_2 \times$$

$$\frac{1}{4P_a^{(2)}} \int_{t_1^-}^{t_1^+} dt_1 \int_0^{2\pi} d\phi_1, \text{ where } P_a^{(i)} = \sqrt{\lambda(M_i^2, t_i, m_a^2)} \Big/ 2M_i \;\; \& \;\; t_i = q_i^2, q_i = p_a - k_i$$

(E.Byckling & K.Kajantie)

In this equation, R_n can takes a form in which multiperipheral momentum transfers t_i appear as variables. That might be a more convient starting point for Monte Carlos than previous expression. One may also further replace the azimuthal angles ϕ_i in the above expression by invariants, which turns out to be equivalent to the 2–particle subinvariant masses s_i.

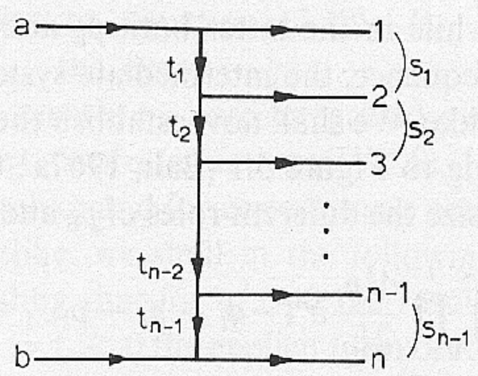


Figure VI.2.7 The multiperipheral momentum transfers $t_i = (p_a - p_1 - \ldots - p_i)^2$ and two-particle subenergies $s_i = (p_i + p_{i+1})^2$