Relativistic Kinematics of Particle Interactions

by W von Schlippe, March 2002

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Introduction

These notes are intended to provide a summary of the essentials of relativistic kinematics of particle reactions. A basic familiarity with the special theory of relativity is assumed. Most derivations are omitted: it is assumed that the interested reader will be able to verify the results, which usually requires no more than elementary algebra. Only the phase space calculations are done in some detail since we recognise that they are frequently a bit of a struggle. For a deeper study of this subject the reader should consult the monograph on particle kinematics by Byckling and Kajantie.

Section 1 sets the scene with an introduction of the notation used here. Although other notations and conventions are used elsewhere, I present only one version which I believe to be the one most frequently encountered in the literature on particle physics, notably in such widely used textbooks as Relativistic Quantum Mechanics by Bjorken and Drell and in the books listed in the bibliography.

This is followed in section 2 by a brief discussion of the Lorentz transformation. This can be dealt with fairly briefly because Lorentz transformations play no major part in the kind of calculations characteristic for the rest of this book: central to most relativistic calculations is the notion of Lorentz invariants. This is so because all observables can be expressed in terms of invariants. The transformations between different reference frames are usually straight forward for the invariants of the problem without involving the use of the Lorentz transformation formulæ.

In the following sections the kinematics of particle reactions is developed beginning with the simplest case, that of two-body decays of unstable particles, and culminating in multiparticle kinematics which is of paramount importance for the study of particle reactions at modern high energy accelerators and colliders. The notes conclude with a fairly detailed discussion of phase space calculations.

The most exhaustive treatment of the subject of relativistic kinematics is given in the monograph by Byckling and Kajantie [1]. The notation and other conventions used here are those of [2]. A summary of formulas of relativistic kinematics can be found in the Particle Data Tables [3],

1. Notation and units

The covariant components of 4-vectors are denoted by superscript Greek letters[1] μ, ν *etc.* Thus we write x^μ, $\mu = 0, 1, 2, 3$, such that

$$x^0 = ct,\ x^1 = x,\ x^2 = y,\ x^3 = z$$

The contravariant components of 4-vectors are labeled by subscripts:

$$x_0 = x^0,\ x_1 = -x,\ x_2 = -y,\ x_3 = -z$$

The scalar product of two 4-vectors is given by

$$x \cdot y = x^\mu y_\mu = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3$$

where the notation $x^\mu y_\mu$ implies summation over μ.

The minus signs in the scalar product signify that the 4-vector space is not Euclidian. This is also expressed by introducing the metric tensor $g^{\mu\nu}$ with diagonal elements $(1, -1, -1, -1)$ and zero off-diagonal elements:

$$g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$$

The contravariant elements of the metric tensor coincide with its covariant elements, *i.e.* we have also $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$

Using the summation convention one can check by explicit calculation that

$$x_\mu = g_{\mu\nu} x^\nu \qquad \text{and} \qquad x^\mu = g^{\mu\nu} x_\nu$$

This procedure is referred to as the lowering and raising of the indices.

Using the metric tensor one can express the scalar product in the following equivalent forms:

$$x \cdot y = g^{\mu\nu} x_\mu y_\nu = g_{\mu\nu} x^\mu y^\nu$$

The invariant square of a 4-vector is given by

$$x^2 = x^\mu x_\mu$$

If $x^2 > 0$ the 4-vector is said to be *time-like*, if $x^2 = 0$ it is *light-like*, and if $x^2 < 0$ it is *space-like*.

In the following we shall use units defined by $c = 1$, where c is the speed of light. This is convenient in the kind of calculations characteristic of relativistic kinematics, because all expressions must then be homogeneous in energies, momenta and masses, which all have the same dimension. Velocities do not occur very often in these calculations, but one must remember that particle velocities are dimensionless and do not exceed 1. Thus the relativistic γ factor of a particle of velocity v is written as

$$\gamma = \left(1 - v^2\right)^{-1/2} \tag{1}$$

The units of energy, momentum and mass most frequently used in elementary particle physics are the Mega-electron Volt (1 MeV = 10^6 eV), Giga-electron Volt (1 GeV = 10^9 eV)

[1] c.f. P.A.M. Dirac, The Principles of Quantum Mechanics, 4th edition, OUP, 1958, p. 254

and Tera-electron Volt (1 TeV $= 10^{12}$ eV). One should remember that the electron mass is approximately 0.5 MeV, the proton mass is nearly 1 GeV (a more accurate value is 0.94 GeV), and the highest energy accelerators in operation accelerate protons to about 1 TeV.

2. Lorentz transformation.

The Lorentz transformation between two inertial frames is of the form

$$x^\mu \to x'^\mu = a^\mu_\nu x^\nu \tag{2}$$

The invariance of the scalar product of two 4-vectors implies that

$$a^\mu_\nu a^\lambda_\mu = \delta^\lambda_\nu \tag{3}$$

where δ^λ_ν is the Kronecker δ symbol.

If the axes of the two frames coincide at time $t = t' = 0$ and the frames move with relative velocity $\vec{v}$, then

$$\begin{aligned} x^{0\prime} &= \gamma(x^0 + v x_{||}) \\ x'_{||} &= \gamma(x_{||} + v x^0) \\ \vec{x}'_\perp &= \vec{x}_\perp \end{aligned} \tag{4}$$

where $v = |\vec{v}|$, $x_{||}$ is the component of x in the direction of $\vec{v}$, *i.e.* the modulus of $\vec{x}_{||} = (\vec{x}\cdot\vec{v})\vec{v}/v^2$, $\vec{x}_\perp = \vec{x} - \vec{x}_{||}$ and $\gamma = (1 - v^2)^{-1/2}$. Thus in this particular case the Lorentz transformation is defined by three parameters, namely the three components of the velocity $\vec{v}$.

If in addition the two frames are rotated with respect to each other, then the transformation matrix depends also on the three Euler angles. Therefore the most general Lorentz transformation depends on six parameters.

The inverse Lorentz transformation from the primed to the unprimed frame is obtained by replacing in Eq. (4) the primed by the unprimed coordinates and vice versa, and changing the sign of v, *i.e.*

$$x^0 = \gamma(x^{0\prime} - v x'_{||}) \quad x_{||} = \gamma(x'_{||} - v x^{0\prime}) \quad \vec{x}_\perp = \vec{x}'_\perp \tag{5}$$

Energy-momentum 4-vector.

Like time, so also energy is not invariant in relativity, but rather transforms under Lorentz transformation like the zeroth component of a 4-vector, the other three components being the ordinary 3-vector momentum.

Let us denote the energy-momentum 4-vector of a particle of mass m by p, *i.e.*

$$\mathrm{p} = (p^0, \vec{p}) \tag{6}$$

Frequently we shall denote the zeroth component p^0, *i.e.* the energy, by E. Thus the invariant square of p is

$$\mathrm{p}^2 = p^\mu p_\mu = E^2 - \vec{p}^2 = m^2 \tag{7}$$

The velocity $\vec{v}$ of the particle is defined by

$$\vec{p} = \gamma m \vec{v} \tag{8}$$

From Eqs. (1), (7) and (8) we find the useful relations

$$\gamma = E/m \tag{9}$$

and

$$\vec{v} = \vec{p}/E \tag{10}$$

The energy-momentum 4-vector p^μ transforms like the space-time 4-vector x^μ (c.f. Eq. (4)):

$$\begin{aligned} p^{0\prime} &= \gamma(p^0 + vp_{||}) \\ p'_{||} &= \gamma(p_{||} + vp^0) \\ \vec{p}'_\perp &= \vec{p}_\perp \end{aligned} \tag{11}$$

Minkowski metric; rapidity.

An alternative description of the Lorentz transformation (4) is based on the Minkowski trick of introducing an imaginary time $x_4 = it$. There is now no need to have two kinds of vectors, such as the contravariant and covariant vectors considered above. Instead we denote the 4-vector components of space-time by

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = it.$$

and the Lorentz invariant square becomes

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = \text{invariant}.$$

The transformation, which leaves this expression invariant, is a rotation. Thus, for instance, a rotation in the (x_1, x_4) plane, leaving x_2 and x_3 unchanged, is of the form

$$x_1' = x_1 \cos\alpha - x_4 \sin\alpha, \qquad x_4' = x_1 \sin\alpha + x_4 \cos\alpha$$

Of course, no mathematical trick gets us off the hook: nature demands space-time to be non-Euclidian. Going back from the world of an imaginary time to the real world we must write our rotation as

$$x' = x\cos\alpha - it\sin\alpha, \qquad it' = x\sin\alpha + it\cos\alpha$$

and using the identities

$$i\sin\alpha = \sinh i\alpha, \qquad \cos\alpha = \cosh i\alpha$$

and setting $y = i\alpha$ we recover a completely real form of the Lorentz transformation, viz

$$x' = x\cosh y + t\sinh y, \qquad t' = x\sinh y + t\cosh y$$

For this to be equivalent with Eq. (4) we must demand that y be real, and hence α is imaginary. Thus the price to pay for a familiar Euclidian form of the rotation in the (x_1, x_4) plane is an imaginary angle of rotation. The real quantity y defined above is called the *rapidity* of the transformation; it is related to the relative velocity v between the two frames and to the relativistic γ factor by

$$\cosh y = \gamma, \qquad \sinh y = v\gamma$$

hence $v = \tanh y$.

We can similarly write the Lorentz transformation of the 4-momentum as

$$p' = p\cosh y + E\sinh y, \qquad E' = p\sinh y + E\cosh y$$

In the ultra relativistic case, when the particle mass can be neglected and we have $E = p$, we get the useful formula

$$y_{\rm ur} = \ln(E'/E) \tag{12}$$

3. Two-body decays of unstable particles.

The simplest kind of particle reaction is the two-body decay of unstable particles. A well known example from nuclear physics is the alpha decay of heavy nuclei. In particle physics one observes, for instance, decays of charged pions or kaons into muons and neutrinos, or decays of neutral kaons into pairs of pions, *etc.* The unstable particle is the **mother particle** and its decay products are the **daughter particles**.

Consider the decay of a particle of mass M which is initially at rest. Then its 4-momentum is $\mathrm{P} = (M, 0, 0, 0)$. This reference frame is called the **centre-of mass frame** (CMS). Denote the 4-momenta of the two daughter particles by p_1 and p_2: $\mathrm{p}_1 = (E_1, \vec{p}_1)$, $\mathrm{p}_2 = (E_2, \vec{p}_2)$. 4-momentum conservation requires that

$$\mathrm{P} = \mathrm{p}_1 + \mathrm{p}_2 \tag{13}$$

and hence $\vec{p}_2 = -\vec{p}_1$. We can therefore omit the subscript on the particle momenta and hence energy conservation takes on the form

$$E_1 + E_2 = \sqrt{m_1^2 + p^2} + \sqrt{m_2^2 + p^2} = M \tag{14}$$

Solving this equation for p we get

$$p = \frac{1}{2M}\sqrt{[M^2 - (m_1 - m_2)^2]\,[M^2 - (m_1 + m_2)^2]} \tag{15}$$

An immediate consequence of Eq. (14) is that

$$M \geq m_1 + m_2 \tag{16}$$

i.e. a particle can decay only if its mass exceeds the sum of the masses of its decay products. Conversely, if some particle has a mass that exceeds the masses of two other particles, then this particle is unstable and decays, unless the decay is forbidden by some conservation law, such as conservation of charge or of angular momentum *etc.*

Another point to note is that the momenta of the daughter particles and hence also their energies are fixed by the masses of the three particles. In the next section we shall see that there is no equivalent statement in the case of three-body decays: the momenta of the daughter particles in three-body decays can take on any value from zero to some maximum, and it is only the maximum momentum that is fixed by the masses of the particles.

Let us complete our calculation by deriving the formulæ for the energies of the daughter particles. This is straightforward if we begin from the energy conservation formula (14) and express E_2 in terms of E_1, viz. $E_2 = \sqrt{E_1^2 - m_1^2 + m_2^2}$, and then solve for E_1 to get

$$E_1 = \frac{1}{2M}\left(M^2 + m_1^2 - m_2^2\right) \tag{17}$$

and similarly

$$E_2 = \frac{1}{2M}\left(M^2 + m_2^2 - m_1^2\right) \tag{18}$$

We also note that there is no preferred direction in which the daughter particles travel (the decay is said to be *isotropic*), but if the direction of one of the particles is chosen (e.g. by the positioning of a detector), then the direction of the second particle is fixed by momentum conservation: the daughter particles are traveling *back-to-back* in the rest frame of the mother particle.

Frequently one observes that the masses of the two daughter particles are equal, for instance in the decay of a neutral kaon into a pair of pions. In this case the previous formulæ simplify: the energies of the daughter particles are $E_1 = E_2 = \frac{1}{2}M$ and the momenta are $p = \frac{1}{2}\sqrt{M^2 - 4m^2}$, if we denote the common mass of the daughter particles by m.

Of interest is also the two-body decay of unstable particles in flight. For instance, high energy beams of muons or of neutrinos are produced in accelerators by letting the internal beam of protons impinge on a target of metal (thin foils or wires are used in practice) to produce pions and kaons, which are then steered in a vacuum tube in which they decay into muons and neutrinos. Other cases of great interest are the decays of very short-lived reaction products of high energy collisions, such as, for instance, the decays of B mesons or of D mesons, which are copiously produced in modern high energy colliders. To illustrate the importance of a proper discussion of such in-flight decays suffice it to say that this is frequently the only way to measure the mass of a neutral particle.

Thus we now have the following 4-momenta of the three particles: for the mother particles we write $\mathrm{P} = (E, 0, 0, p)$, and for the daughter particles we have $\mathrm{p}_1 = (E_1, \vec{p}_{1\perp}, p_{1z})$ and $\mathrm{p}_2 = (E_2, \vec{p}_{2\perp}, p_{2z})$. This means that we have chosen the z axis along the direction of flight of the mother particle. The immediate result of this is that by momentum conservation the (two-dimensional) transverse momentum vectors are equal in magnitude and opposite in sign:

$$\vec{p}_\perp \equiv \vec{p}_{1\perp} = -\vec{p}_{2\perp} \tag{19}$$

The energies and the z components of the particle momenta are related to those in the CMS by a Lorentz boost with a boost velocity equal to the speed of the mother particle. We label the kinematical variables in the CMS with asterisks and write the Lorentz transformation of particle 1 in the form of

$$\begin{aligned} E_1 &= \gamma(E_1^* + v p_{1z}^*) \\ p_{1z} &= \gamma(p_{1z}^* + v E_1^*) \\ \vec{p}_{1\perp} &= \vec{p}_{1\perp}^{\,*} \end{aligned} \tag{20}$$

and similarly for particle 2. Here $v = p/E$ and $\gamma = E/M$. This completely solves the problem in principle. We can now, for instance, find the angles which the two daughter particles make with the z axis and with each other as functions of the momentum of the mother particle (Exercise 1). But it is also of interest to approach the problem in a different way, without using the Lorentz transformation, starting from energy-momentum conservation:

$$E = E_1 + E_2 = \sqrt{m_1^2 + p_1^2} + \sqrt{m_2^2 + p_2^2} \tag{21}$$

$$\vec{p} = \vec{p}_1 + \vec{p}_2 \tag{22}$$

Thus, replacing in the energy conservation equation, p_2^2 by $(\vec{p}-\vec{p}_1)^2$ we get an equation with unknown momentum p_1 and angle θ_1 between $\vec{p}_1$ and the z axis. Solving for p_1 is a straightforward if lengthy calculation (Exercise 2). In the end we get

$$p_1 = \frac{(M^2 + m_1^2 - m_2^2)p\cos\theta_1 \pm 2E\sqrt{M^2 p^{*2} - m_1^2 p^2 \sin^2\theta_1}}{2(M^2 + p^2\sin^2\theta_1)} \tag{23}$$

This result is of interest in the following sense. Reality of p_1 demands that $(M^2p^{*2} - m_1^2p^2\sin^2\theta_1) \geq 0$. This condition is satisfied for all angles θ_1 if $Mp^*/m_1p > 1$. In this case the lower sign must be rejected since otherwise we would get unphysical negative values of p_1 for $\theta_1 > \pi/2$. On the other hand, if $Mp^*/m_1p < 1$, then there is a maximum value of θ_1, given by $\sin\theta_{1\,\mathrm{max}} = Mp^*/m_1p$. Now both signs must be kept: for each value of $\theta_1 < \theta_{1\,\mathrm{max}}$ there are two values of p_1, and correspondingly also two values of p_2. Several examples illustrating this are given in Exercise 3.

Exercise 1: Show that the LAB angle θ_1 that daughter particle 1 makes with the direction of flight of the mother particle in a two-body decay is related to the CMS angle θ_1^* by the following equation:

$$\tan\theta_1 = \frac{\sin\theta_1^*}{\gamma(v/v_1^* + \cos\theta_1^*)} \tag{24}$$

where v is the the LAB velocity of the mother particle and v_1^* is the CMS velocity of the daughter particle.

Exercise 2: Derive Eq. (23).

Exercise 3: Consider the following decay processes:

(i) $K_S^0 \to \pi^+\pi^-$,

(ii) $\Lambda^0 \to p\pi^-$

(iii) $\pi^0 \to 2\gamma$.

In each case assume that the mother particle has a LAB energy of 1 GeV. Find the maximum Lab angles θ and the corresponding LAB momenta for all decay products. Show that the minimum opening angle between the decay products corresponds to a CMS angle of 90° with the line of flight of the mother particle and calculate the corresponding LAB momenta. Assuming that one of the daughter particles makes an angle half the maximum angle, find the corresponding two momenta and the two momenta and LAB angles of the other daughter particle.

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4. Three-body decays; Dalitz plot.

Consider the decay of a mother particle of mass M into three particles of masses m_1, m_2 and m_3. Denote their 4-momenta by P, p_1, p_2 and p_3, respectively. Energy-momentum conservation is expressed by

$$\mathrm{P} = \mathrm{p}_1 + \mathrm{p}_2 + \mathrm{p}_3 \tag{25}$$

Define the following invariants:

$$\begin{aligned} s &= \mathrm{P}^2 = M^2 \\ s_1 &= (\mathrm{P}-\mathrm{p}_1)^2 = (\mathrm{p}_2+\mathrm{p}_3)^2 \\ s_2 &= (\mathrm{P}-\mathrm{p}_2)^2 = (\mathrm{p}_3+\mathrm{p}_1)^2 \\ s_3 &= (\mathrm{P}-\mathrm{p}_3)^2 = (\mathrm{p}_1+\mathrm{p}_2)^2 \end{aligned} \tag{26}$$

Invariant s is a *trivial* invariant, trivial in the sense that it is constant and thus has no dynamical significance. To understand the physical significance of invariant s_1 consider the second part of its definition: $s_1 = (\mathrm{p}_2+\mathrm{p}_3)^2$. This shows that $\sqrt{s_1}$ is the invariant mass of the subsystem of particles 2 and 3.[2] Similarly $\sqrt{s_2}$ and $\sqrt{s_3}$ are the invariant masses of subsystems (3,1) and (1,2), respectively.

The three invariants s_1, s_2 and s_3 are not independent: it follows from their definitions together with 4-momentum conservation that

$$s_1 + s_2 + s_3 = M^2 + m_1^2 + m_2^2 + m_3^2 \tag{27}$$

[2] therefore we could alternatively write s_{23} instead of s_1, with similar notation for s_2 and s_3.

Kinematical limits

In the case of three-particle decay an important question arises as to the limits of the kinematical variables. We shall see that this question also arises in collision problems. Therefore the present derivation is of more general use. It is for this reason that we discuss it in greater detail than most other derivations in these notes.

The space spanned by any set of independent kinematical variables is called *phase space*. Therefore we can alternatively say that we are deriving the boundaries of phase space.

Consider the decay process in the rest frame of the mother particle (CMS). Here we have

$$s_1 = M^2 + m_1^2 - 2ME_1 \tag{28}$$

with $E_1 = \sqrt{m_1^2 + p_1^2}$, where p_1 is the CMS momentum of particle 1. Thus $E_1 \geq m_1$, and hence

$$\max\, s_1 = (M - m_1)^2 \tag{29}$$

To find min s_1 we evaluate s_1 in the rest frame of subsystem (2,3).[3] Let us denote all frame dependent kinematical variables in this frame by a little zero above the symbol. Thus we get

$$s_1 = (\mathrm{p}_2 + \mathrm{p}_3)^2 = (\mathring{E}_2 + \mathring{E}_3)^2 \geq (m_2 + m_3)^2 \tag{30}$$

with similar formulæ for s_2 and s_3. In summary, we get the following limits of the invariants s_1, s_2, and s_3:

$$\begin{aligned} s_1 &\in \left[(m_2+m_3)^2, (M-m_1)^2\right] \\ s_2 &\in \left[(m_3+m_1)^2, (M-m_2)^2\right] \\ s_3 &\in \left[(m_1+m_2)^2, (M-m_3)^2\right] \end{aligned} \tag{31}$$

However not the entire cube defined by Eq. (31) is kinematically accessible. To find the limits of s_2, say, for fixed $s_1 \in [(m_2+m_3)^2, (M-m_1)^2]$ it is convenient to consider the Jackson frame S23. This frame is defined by $\vec{\mathring{p}}_3 = -\vec{\mathring{p}}_2$. Momentum conservation then implies that $\vec{\mathring{p}}_1 = \vec{\mathring{P}}$, and it follows that

$$s_1 = (\mathring{E} - \mathring{E}_1)^2 = \left(\sqrt{M^2 + \mathring{p}_1^2} - \sqrt{m_1^2 + \mathring{p}_1^2}\right)^2 \tag{32}$$

Solving for $\mathring{p}_1^{\,2}$ we get[4]

$$\mathring{p}_1^2 = \frac{1}{4s_1}\left[s_1 - (M-m_1)^2\right]\left[s_1 - (M+m_1)^2\right] \equiv \frac{1}{4s_1}\lambda\left(s_1, M^2, m_1^2\right) \tag{33}$$

Similarly we get from

$$s_1 = (\mathrm{p}_2 + \mathrm{p}_3)^2 = \left(\mathring{E}_2 + \mathring{E}_3\right)^2 \tag{34}$$

the corresponding expression for $\mathring{p}_2^2$ and $\mathring{p}_3^2$:

$$\mathring{p}_2^2 = \mathring{p}_3^2 = \frac{1}{4s_1}\lambda\left(s_1, m_2^2, m_3^2\right) \tag{35}$$

Now consider the invariant s_2:

$$s_2 = (\mathrm{p}_1 + \mathrm{p}_3)^2 = m_1^2 + m_3^2 + 2\left(\mathring{E}_1\mathring{E}_3 - \mathring{p}_1\mathring{p}_3\cos\alpha\right) \tag{36}$$

[3] such a frame is called *Jackson* frame; this particular Jackson frame will be denoted S23.

[4] The kinematical function $\lambda(x,y,z)$ is defined by $\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$

where α is the angle between $\vec{\mathring{p}}_1$ and $\vec{\mathring{p}}_3$. Taking into account Eqs. (33) and (35), we see that s_2 depends only on α if s_1 is fixed. It follows that $s_{2+} \equiv \max s_2$ and $s_{2-} \equiv \min s_2$ correspond to $\alpha = \pi$ and $\alpha = 0$, respectively, *i.e.*

$$s_{2\pm} = m_1^2 + m_3^2 + 2\left(\mathring{E}_1\mathring{E}_3 \pm \mathring{p}_1\mathring{p}_3\right) \tag{37}$$

If we express $\mathring{E}_1$ and $\mathring{E}_3$ in terms of s_1, *i.e.* if we write

$$\mathring{E}_1 = \frac{1}{2\sqrt{s_1}}\left(s - s_1 - m_1^2\right), \qquad \mathring{E}_3 = \frac{1}{2\sqrt{s_1}}\left(s_1 + m_3^2 - m_2^2\right) \tag{38}$$

then we get $s_{2\pm}$ explicitly as a function of s_1:

$$s_{2\pm} = m_1^2 + m_3^2 + \frac{1}{2s_1}\left[\left(s - s_1 - m_1^2\right)\left(s_1 - m_2^2 + m_3^2\right) \pm \lambda^{1/2}(s_1, s, m_1^2)\lambda^{1/2}(s_1, m_2^2, m_3^2)\right] \tag{39}$$

The curve defined by Eq. (39) is the boundary of the *Dalitz plot* in the (s_1, s_2) plane.

Of interest are also the maximum values of the three-momenta of the daughter particles in the rest frame of the mother particle. From Eq. (28) we see that $\max E_1$, and hence also $\max p_1$, corresponds to $\min s_1$, and a straight forward calculation gives the result

$$p_{1max} = \frac{1}{2M}\sqrt{[M^2 - (m_1 + m_2 + m_3)^2][M^2 - (m_2 + m_3 - m_1)^2]} \tag{40}$$

and we get the similar expressions for p_{2max} and p_{3max} by the cyclic replacement of the subscripts: $1 \to 2$, $2 \to 3$ and $3 \to 1$.

In the particular case when one of the daughter particles is massless, such as for instance in $K^+_{\mu 3}$ decay: $K^+ \to \pi^0\mu^+\nu_\mu$, we find that the maximum momenta of the two massive daughter particles are equal and greater than the maximum momentum of the massless daughter particle. If two daughter particles are massless, such as in muon decay: $\mu \to e\nu\bar{\nu}$, then the maximum momenta of all daughter particles coincide.

5. Particle collisions

Centre-of-mass frame and Laboratory frame.

The total 4-momentum p^μ of a system of n particles with 4-momenta p_1^μ, p_2^μ, $\ldots, \mathrm{p}_n^\mu$ is given by

$$\mathrm{p}^\mu = \mathrm{p}_1^\mu + \mathrm{p}_2^\mu + \ldots + \mathrm{p}_n^\mu$$

The centre-of-mass frame (CMS) of the system is defined as the frame in which the total three-momentum of the system is equal to nought. Labeling CMS variables by asterisks, we express this definition by the following equation:

$$\vec{p}^{\,*} = \vec{p}_1^{\,*} + \vec{p}_2^{\,*} + \ldots + \vec{p}_n^{\,*} = 0 \tag{41}$$

In particle physics the laboratory frame (LAB frame) of the system is defined as that reference frame in which one of the initial particles is at rest. This particle is called the target particle, the other being the beam particle or incident particle. The CMS and LAB frames are illustrated for the reaction $a + b \to c + d$ by the kinematical diagrams in Fig. 1.

We use the convention that the beam particle is incident along the positive z axis. Its momentum and energy are denoted p_{LAB} and E_{LAB}, respectively. Then its 4-momentum is given by

$$\mathrm{p}_1 = (E_{LAB}, 0, 0, p_{LAB}) \tag{42}$$

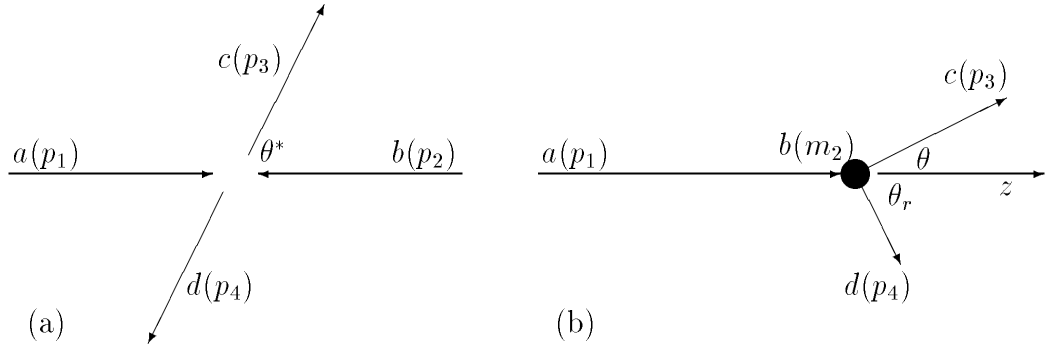

Figure 1: Kinematical diagram: (a) CMS; (b) LAB.

and the 4-momentum of the target particle is given by

$$\mathrm{p}_2 = (m_2, 0, 0, 0)$$

where we have arbitrarily denoted the incident particle as particle 1 and the target particle as particle 2. θ is the scattering angle and θ_r is the recoil angle. In the CMS the scattering angle is θ^*.

The total 4-momentum of the initial particles is

$$\mathrm{p} = \mathrm{p}_1 + \mathrm{p}_2 \tag{43}$$

The invariant square of p is usually denoted by s.[5] In the CMS frame we have $\vec{p}_1^{\,*} + \vec{p}_2^{\,*} = 0$ and hence

$$s = (E_1^* + E_2^*)^2 \tag{44}$$

i.e. $\sqrt{s}$ is the total CMS energy of the system.

We also note that $\vec{p}_1^{\,*} + \vec{p}_2^{\,*} = 0$ implies that we can drop the subscripts on the magnitudes of the three-momenta and write

$$E_1^* = \sqrt{m_1^2 + p^{*2}}, \qquad E_2^* = \sqrt{m_2^2 + p^{*2}} \tag{45}$$

Substituting these expressions in Eq. (44) and solving for p^* we get

$$p^* = \frac{1}{2\sqrt{s}} \left\{ \left[s - (m_1 - m_2)^2 \right] \left[s - (m_1 + m_2)^2 \right] \right\}^{1/2} \tag{46}$$

If particles c and d are different from a and b, then we must distinguish between the *initial* CMS momentum p_i^*, given by Eq. (46), and the *final* CMS momentum p_f^*, given by

$$p_f^* = \frac{1}{2\sqrt{s}} \left\{ \left[s - (m_3 - m_4)^2 \right] \left[s - (m_3 + m_4)^2 \right] \right\}^{1/2} \tag{47}$$

In the LAB frame s is given by

$$s = \mathrm{p}^\mu \mathrm{p}_\mu = m_1^2 + m_2^2 + 2 m_2 E_{LAB} \tag{48}$$

[5] This is one of the Mandelstam variables; in section 6 we shall define the full set of three Mandelstam variables.

from which we get the useful relation

$$E_{LAB} = (s - m_1^2 - m_2^2)/2m_2 \tag{49}$$

and hence, substituting $E_{LAB} = \sqrt{p_{LAB}^2 + m_1^2}$ and solving for p_{LAB}, we get

$$p_{LAB} = \frac{1}{2m_2}\left\{\left[s - (m_1 - m_2)^2\right]\left[s - (m_1 + m_2)^2\right]\right\}^{1/2} = \frac{1}{2m_2}\lambda^{\frac{1}{2}}(s, m_1^2, m_2^2) \tag{50}$$

Comparing Eq. (50) with (46) we find that

$$p^* = p_{LAB}\frac{m_2}{\sqrt{s}} \tag{51}$$

and hence

$$E_{1,2}^* = (m_{1,2}^2 + m_2 E_{LAB})/\sqrt{s} \tag{52}$$

It is interesting to note that for ultra-relativistic particles, *i.e.* when we can neglect all masses in comparison with the particle energies, we get $E_1^* = E_2^*$, which means that we can drop the subscript of the CMS energies, and hence we get from Eqs. (48) and (52) the simple relation

$$E^* \approx \sqrt{\frac{1}{2}m_2 E_{LAB}} \tag{53}$$

thus the CMS energy grows only like the square-root of the LAB energy. In particle collisions only the centre-of-mass energy can be converted into the reaction products of the final state. Therefore we see from Eq. (53) that in fixed-target experiments at high energies most of the beam energy is lost in kinetic energy of the particle system. This limitation is overcome by the construction of colliders, *i.e.* accelerators with two beams moving in opposite directions: in the case of particles of equal masses in the two beams, such an arrangement ensures that the combined beam energy is the centre-of-mass energy and no energy is lost in the form of overall kinetic energy of the reaction products. This can be illustrated strikingly by the example of the creation of Z bosons in electron-positron annihilations. In an e^+e^- collider we need two beams of half the Z boson mass, *i.e.* of approximately 45.5 GeV each. The equivalent LAB energy of a positron beam impinging on target electrons is 8.3×10^6 GeV, *i.e.* an energy which is unattainable in the foreseeable future.

Let us use our results to find the LAB velocity of the centre of mass of the initial particles. This is also the boost velocity of the Lorentz transformation from the CMS to the LAB frame. We note that the Lorentz transformation Eq. (20) applies to any particle. Therefore, if we add the first of Eqs. (20) to the corresponding equation for particle 2, *i.e.* for the target particle, then we get

$$E_1 + E_2 \equiv E_{LAB} + m_2 = \gamma_{cm}[E_1^* + E_2^* + v_{cm}(p_{1z}^* + p_{2z}^*)]$$

and hence with (44) and recalling that $p_{1z}^* + p_{2z}^* = 0$, we get

$$\gamma_{cm} = (E_{LAB} + m_2)/\sqrt{s}$$

and finally also the LAB velocity of the centre of mass

$$v_{cm} = p_{LAB}/(E_{LAB} + m_2)$$

Consider the particular case of equal masses. Then in the nonrelativistic limit we expect intuitively that the centre of mass is moving with half the speed of the incident particle. The relativistic formula simplifies to

$$v_{cm} = \sqrt{\frac{E_{LAB} - m}{E_{LAB} + m}}$$

where we have set $m = m_1 = m_2$. Then, with $p_{LAB} \ll m$, we get

$$v_{cm} = \frac{p_{LAB}}{2m}\left(1 - \frac{p_{LAB}^2}{4m^2}\right)$$

or, since the LAB velocity v_{LAB} of the incident particle is $v_{LAB} = p_{LAB}/\gamma_{LAB}m$ and $\gamma_{LAB} = 1/(1 - v_{LAB}^2)^{-1/2}$ we get, neglecting terms of order v_{LAB}^4,

$$v_{cm} = \frac{1}{2}v_{LAB}\left(1 + \frac{1}{4}v_{LAB}^2\right)$$

and we see that the second term in the brackets can be neglected in the nonrelativistic limit. Thus we have found the expected result.

We can use the expressions for v_{cm} and γ_{cm} to write down the Lorentz transformation from the LAB frame to the centre-of-mass frame for the energy of particle 1 using Eq. (20):

$$E_1^* = \gamma_{cm}(E_{LAB} - v_{cm}p_{LAB}) = (m_1^2 + m_2 E_{LAB})/\sqrt{s}$$

which reproduces the result Eq. (52). The corresponding transformation formula for the momentum reproduces Eq. (51).

To carry out the Lorentz transformation of the target particle from the LAB frame to the CMS we substitute its LAB energy $E_{2\,LAB} = m_2$ and LAB momentum $p_{2\,LAB} = 0$ in Eq. (20), hence

$$E_2^* = \gamma_{cm} m_2 = m_2(m_2 + E_{LAB})/\sqrt{s}$$

in agreement with Eq. (52).

6. Elastic Collisions

In the case of elastic collisions we denote the 4-momenta of the incident particles p_1 and p_2, and the 4-momenta of the scattered particles p_3 and p_4, such that

$$\mathrm{p}_3^2 = \mathrm{p}_1^2 = m_1^2 \qquad \text{and} \qquad \mathrm{p}_4^2 = \mathrm{p}_2^2 = m_2^2 \tag{54}$$

4-momentum conservation is expressed by

$$\mathrm{p}_1 + \mathrm{p}_2 = \mathrm{p}_3 + \mathrm{p}_4 \tag{55}$$

Consider the invariants which can be constructed from the 4-momenta $\mathrm{p}_1, \ldots, \mathrm{p}_4$. Such invariants are of the form $\mathrm{p}_i \cdot \mathrm{p}_j$, $i, j = 1, 2, 3, 4$, and hence there are sixteen invariants. Four of these are of the form $\mathrm{p}_i^2 = m_i^2$, *i.e.* they are constants without any dynamical contents. They are therefore referred to as *trivial* invariants. Of the remaining twelve invariants there are only six different ones as a result of the symmetry $\mathrm{p}_i \cdot \mathrm{p}_j = \mathrm{p}_j \cdot \mathrm{p}_i$. This leaves us with six invariants, which are further constrained by the 4-momentum conservation (55). Thus there remain two independent invariants. However rather than working with two invariants it is frequently convenient to use three invariants with one constraint. The most widely used choice of such invariants are the Mandelstam variables, which are defined by the following equations:

$$\begin{aligned} s &= (\mathrm{p}_1 + \mathrm{p}_2)^2 = (\mathrm{p}_3 + \mathrm{p}_4)^2 \\ t &= (\mathrm{p}_1 - \mathrm{p}_3)^2 = (\mathrm{p}_2 - \mathrm{p}_4)^2 \\ u &= (\mathrm{p}_1 - \mathrm{p}_4)^2 = (\mathrm{p}_2 - \mathrm{p}_3)^2 \end{aligned} \tag{56}$$

Since only two of the Mandelstam variables are independent there exists one relation between them, namely

$$s + t + u = 2m_1^2 + 2m_2^2 \tag{57}$$

The variable t has a simple meaning in the CMS where one has $E_1^* = E_3^*$, and hence

$$t = -(\vec{p}_1 - \vec{p}_3)^2 = -2p^{*2}(1 - \cos\theta^*) \tag{58}$$

Thus up to a sign, t is the squared momentum transfer in the CMS. For this reason t is generally and not very accurately referred to as the 4-momentum transfer. An important result of Eq. (58) is that in elastic scattering t is always negative except at $\theta^* = 0$ (forward scattering) where $t = 0$.

Interesting is also the significance of t in the LAB frame. Here we have

$$t = (\mathrm{p}_2 - \mathrm{p}_4)^2 = 2m_2(m_2 - E_4) = -2m_2T_4 \tag{59}$$

where we have denoted the LAB kinetic energy of particle 4 - the recoil particle - by T_4, *i.e.* $T_4 = E_4 - m_2$.

7. Inelastic collisions

If the particles in the final state are different from the particles in the initial state then the collision is said to be inelastic. Examples of inelastic collisions are the creation of additional pions in pion-proton collisions,

$$\pi^+ + p \to \pi^+ + p + \pi^+ + \pi^- \tag{60}$$

or the annihilation of an electron-positron pair into a muon pair,

$$e^+ + e^- \to \mu^+ + \mu^- \tag{61}$$

At very high energies where many particles are created one frequently measures only one of the final state particles, for instance a fast outgoing electron in electron-proton collisions, with all other particles remaining unobserved. This is referred to as *inclusive collision*, and the corresponding reaction equation is written in the form

$$e^- + p \to e^- + X \tag{62}$$

where X denotes any system of final state particles.

Let us denote the 4-momenta of the initial particles p_1 and p_2, and the 4-momenta of the final state particles $\mathrm{p}_3, \mathrm{p}_4, \ldots, \mathrm{p}_n$. Energy and momentum conservation are expressed by

$$\mathrm{p}_1 + \mathrm{p}_2 = \mathrm{p}_3 + \mathrm{p}_4 + \ldots + \mathrm{p}_n \tag{63}$$

Let us calculate the minimum LAB energy needed for this reaction. This energy is called the threshold energy. Let particle 1 be the incident particle and particle 2 the target particle, then in the LAB frame

$$\mathrm{p}_1 = (E_{LAB}, 0, 0, p_{LAB}) \qquad \mathrm{p}_2 = (m_2, 0, 0, 0) \tag{64}$$

and hence

$$s = (\mathrm{p}_1 + \mathrm{p}_2)^2 = m_1^2 + m_2^2 + 2m_2E_{LAB} \tag{65}$$

By 4-momentum conservation we have also $s = (\mathrm{p}_3 + \mathrm{p}_4 + \ldots + \mathrm{p}_n)^2$. We make use of the invariance of s and evaluate the latter expression in the CMS; then, by Eq. (63), we have

$$s = (E_3^* + E_4^* + \ldots + E_n^*)^2 \geq (m_3 + m_4 + \ldots + m_n)^2 \tag{66}$$

where in the last step we have used $E_i^* = \sqrt{m_i^2 + \vec{p}_i^{*2}} \geq m_i$.

Thus the threshold CMS energy is

$$E^*_{thr} = \sqrt{s_{min}} = m_3 + m_4 + \ldots + m_n \tag{67}$$

and hence with Eq. (65) and setting $M = m_3 + m_4 + \ldots + m_n$

$$E^{thr}_{LAB} = \frac{1}{2m_2}\left[M^2 - m_1^2 - m_2^2\right] \tag{68}$$

or, if we introduce the LAB kinetic energy $T_{LAB} = E_{LAB} - m_1$,

$$T^{thr}_{LAB} = \left[M^2 - (m_1 + m_2)^2\right]/2m_2 \tag{69}$$

Equation (67) has an important physical interpretation: at threshold the final state particles are at rest relative to each other. Expressed differently this means that there is no internal motion in the system of final state particles. This implies that they move together with equal velocity in the LAB frame.[6]

Let us calculate the threshold kinetic energy for the reaction $\pi^+ + p \to \pi^+ + p + \pi^+ + \pi^-$ using the approximate mass values $m_p = 940$ MeV, $m_\pi = m_{\pi^+} = m_{\pi^-} = 140$ MeV:

$$\begin{aligned} T^{thr}_{LAB} &= \frac{1}{2m_p}\left[(m_p + 3m_\pi)^2 - (m_p + m_\pi)^2\right] \\ &= \frac{1}{2m_p}(2m_p + 4m_\pi)(2m_\pi) = 2m_\pi\left(1 + \frac{2m_\pi}{m_p}\right) \\ &= 2 \times 140 \times (1 + 2 \times 140/940) = 363.4 \text{ MeV} \end{aligned} \tag{70}$$

The kinetic energy of the incident pion is the only source of energy to create the additional mass of the final state, which is $2m_\pi$. But our result shows clearly that we must supply more kinetic energy than the additional mass. Qualitatively this result can be understood from our discussion of Eq. (67): part of the LAB kinetic energy is converted into the rest energy of the additional particles, and more kinetic energy is needed to impart LAB kinetic energy to the created particles.

Quasi-elastic collisions.
A particular case of inelastic collisions are the quasi-elastic collisions, *i.e.* $2 \to 2$ body reactions, in which the two particles in the final state are different from the initial particles. The reaction $e^+ + e^- \to \mu^+ + \mu^-$ is a typical example of such reactions. Elastic collisions are a particular case of quasi-elastic collisions.

To describe $2 \to 2$ body reactions we shall again use the Mandelstam variables s, t and u. In this case Eq. (57) must be replaced by the more general relation

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 \tag{71}$$

As before we have in the CMS $\vec{p}_1^* + \vec{p}_2^* = \vec{p}_3^* + \vec{p}_4^*$, but

$$|\vec{p}_1^*| \neq |\vec{p}_3^*| \tag{72}$$

Let us therefore distinguish the magnitudes of the CMS momenta by a prime on the CMS momentum in the final state. Thus

$$p^* = |\vec{p}_1^*| = |\vec{p}_2^*| \qquad \text{and} \qquad p^{*\prime} = |\vec{p}_3^*| = |\vec{p}_4^*| \tag{73}$$

[6] This statement has to be modified if there are massless particles in the final state.

Equation (46) remains valid for p^*; to express $p^{*\prime}$ in terms of s we must replace in Eq. (46) m_1 by m_3 and m_2 by m_4, thus

$$p^{*\prime} = \frac{1}{2\sqrt{s}} \left\{ \left[s - (m_3 - m_4)^2 \right] \left[s - (m_3 + m_4)^2 \right] \right\}^{1/2}) \tag{74}$$

Consider the reaction

$$\pi^- + p \to K^- + \Sigma^+$$

with the pion incident on the target proton. The threshold energy for producing the kaon ($m_K = 0.494$ GeV) and the Σ^+ ($m_{\Sigma^+} = 1.189$ GeV) is 1.03 GeV by Eq. (68). Assuming a LAB energy of the incident pion of 1.5 GeV we get

$$s = m_\pi^2 + m_p^2 + 2m_p E_{LAB} = 0.14^2 + 0.94^2 + 2 \times 0.94 \times 1.5 = 3.71 \text{ GeV}^2$$

and hence by Eqs. (46) and (74) we have

$$p^* = 0.727 \text{ GeV} \qquad \text{and} \qquad p^{*\prime} = 0.438 \text{ GeV}$$

i.e. the CMS momenta of the final state particles are less than the CMS momenta of the initial particles. Qualitatively this result is obvious: the reaction products are heavier than the initial particles, and therefore part of the initial kinetic energy must be converted into mass.

Conversely one expects the CMS momentum of the final state to be greater than the CMS momentum of the initial state if the final state particles are lighter than the initial state particles. This can be checked for the example of the annihilation of a proton-antiproton pair into a pair of pions. It is interesting to note that this reaction can take place at zero relative momentum of the proton and antiproton; even under this condition the pions are created with highly relativistic momenta. Experimentally the annihilation at rest of proton-antiproton pairs can be realized for instance in the low-energy antiproton ring LEAR at CERN, where antiproton beams of extremely low momenta can be produced. These low-energy antiprotons can lose practically their entire kinetic energy in a liquid hydrogen target before being captured by a hydrogen atom to form a protonium atom and finally annihilate with the proton.

Without derivation we collect here some useful formulas relating CMS and LAB variables to the Mandelstam invariants.

<u>CMS variables</u>:

$$E_1^* = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}, \qquad E_2^* = \frac{s + m_2^2 - m_1^2}{2\sqrt{s}}, \tag{75}$$

$$E_3^* = \frac{s + m_3^2 - m_4^2}{2\sqrt{s}}, \qquad E_4^* = \frac{s + m_4^2 - m_3^2}{2\sqrt{s}}, \tag{76}$$

$$p_1^* = p_2^* = \frac{1}{2\sqrt{s}} \sqrt{\left[s - (m_1 - m_2)^2 \right] \left[s - (m_1 + m_2)^2 \right]}, \tag{77}$$

$$p_3^* = p_4^* = \frac{1}{2\sqrt{s}} \sqrt{\left[s - (m_3 - m_4)^2 \right] \left[s - (m_3 + m_4)^2 \right]}, \tag{78}$$

Denoting the CMS angle between $\vec{p}_3^*$ and $\vec{p}_1^*$ by θ^*, we have

$$\cos\theta^* = 1 - (t_0 - t)/2p_1^* p_3^* = 1 - (u - u_0)/2p_1^* p_4^* \tag{79}$$

where

$$t_0 = m_1^2 + m_3^2 - 2(E_1^* E_3^* - p_1^* p_3^*) = m_2^2 + m_4^2 - 2(E_2^* E_4^* - p_2^* p_4^*)$$

and

$$u_0 = m_1^2 + m_4^2 - 2(E_1^* E_4^* + p_1^* p_4^*) = m_2^2 + m_3^2 - 2(E_2^* E_3^* + p_2^* p_3^*)$$

LAB variables: by definition we have in the LAB $\vec{p}_2 = 0$, hence

$$E_1 = \frac{s - m_1^2 - m_2^2}{2m_2}, \qquad E_2 = m_2 \tag{80}$$

$$E_3 = \frac{m_2^2 + m_3^2 - u}{2m_2}, \qquad E_4 = \frac{m_2^2 + m_4^2 - t}{2m_2} \tag{81}$$

the production angle of particle 3, *i.e.* the angle made by $\vec{p}_3$ with the z axis, is given by

$$\cos\theta_3 = \frac{t - m_1^2 - m_3^2 + 2E_1E_3}{2p_1p_3} \tag{82}$$

and similarly

$$\cos\theta_4 = \frac{u - m_1^2 - m_4^2 + 2E_1E_4}{2p_1p_4} \tag{83}$$

8. Deep inelastic scattering.

In lepton-hadron scattering at sufficiently high energies one finds a large number of hadrons in the final state: this is *deep inelastic scattering* (DIS). The multiplicity of the hadronic system varies event by event. The reaction equation for electron-proton DIS is written as

$$e^- + p \to e^- + X \tag{84}$$

where X stands for the hadronic system with an arbitrary number of particles. A generic diagram depicting the DIS process is shown in Fig. 2.

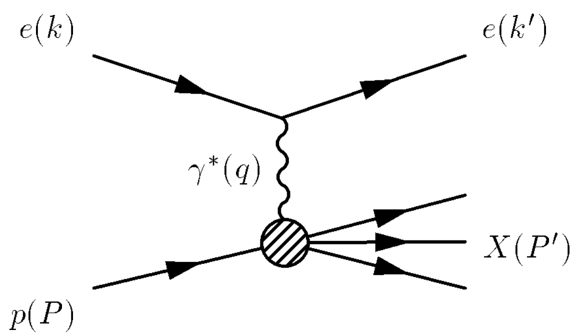

Figure 2: Generic diagram of deep inelastic scattering.

To describe the DIS reaction kinematics we denote the 4-momentum of the incoming electron by $\mathrm{k} = (E, 0, 0, k)$, that of the target proton by P and those of the scattered electron and of the hadronic system by k' and P', respectively. The exchanged virtual photon γ^* has 4-momentum $\mathrm{q} = \mathrm{k} - \mathrm{k}'$. 4-momentum conservation demands

$$\mathrm{k} + \mathrm{P} = \mathrm{k}' + \mathrm{P}' \tag{85}$$

and we have the mass-shell conditions $\mathrm{k}^2 = \mathrm{k}'^2 = m_e^2$ and $\mathrm{P}^2 = m_p^2$. Since energies characteristic of DIS are at least of several GeV, the electron mass can be safely set equal to zero. Then we get for the square of the 4-momentum transfer $\mathrm{q}^2 = (\mathrm{k} - \mathrm{k}')^2 = -2EE'(1 - \cos\theta)$, and we see that $\mathrm{q}^2 \leq 0$, *i.e.* the exchanged photon is space-like.

The invariant $W^2 = P'^2$ is variable because of the variable multiplicity of particles in the hadronic system, each of which can have an arbitrary kinetic energy up to some maximum value. Therefore the complete kinematics of DIS is determined by three independent invariants rather than two as we are used to in elastic collisions. A natural choice of one of these invariants is the square of the total CMS energy S,

$$S = (\mathrm{k} + \mathrm{P})^2 = m_p^2 + 2\mathrm{k} \cdot \mathrm{P} \tag{86}$$

which is defined by the beam energy.
The second invariant is usually chosen to be the negative square of 4-momentum transfer:

$$Q^2 = -\mathrm{q}^2 = -(\mathrm{k} - \mathrm{k}')^2 = 4EE' \sin^2 \frac{\theta}{2} \tag{87}$$

The third independent invariant can be taken to be W or alternatively one of the dimensionless variables

$$x = \frac{Q^2}{2\mathrm{P} \cdot \mathrm{q}} \tag{88}$$

or

$$y = \frac{\mathrm{P} \cdot \mathrm{q}}{\mathrm{k} \cdot \mathrm{P}} \tag{89}$$

where $\mathrm{q} = \mathrm{k} - \mathrm{k}'$.

The variable y has a simple physical meaning in the target rest frame where $\mathrm{P} = (m_p, 0, 0, 0)$, $\mathrm{k} = (E_{\mathrm{LAB}}, 0, 0, E_{\mathrm{LAB}})$, and $\mathrm{k}' = (E'_{\mathrm{LAB}}, \vec{p}_3')$, hence $y = 1 - E'_{\mathrm{LAB}}/E_{\mathrm{LAB}}$, *i.e.* y is the relative energy loss of the electron in the LAB frame.

The invariant x is the Bjorken scaling variable or simply Bjorken-x. It was first recognised as an important variable of DIS by J.D. Bjorken who predicted the property of scaling in DIS which was subsequently confirmed experimentally.

Interesting is the expression of S in terms of the beam energies. In fixed target DIS we have the electron or muon beam with 4-momentum $\mathrm{k} = (E, 0, 0, E)$ and the proton target with $\mathrm{P} = (m_p, 0, 0, 0)$, hence

$$S = m_p^2 + 2m_p E$$

whereas in an electron-proton collider like HERA we have 4-momenta $\mathrm{P} = (E_p, 0, 0, E_p)$ and $\mathrm{k} = (E_e, 0, 0, -E_e)$ and hence

$$S = 4E_e E_p$$

Other useful relations between the various kinematical variables are the following:

$$Q^2 = xyS \tag{90}$$

and

$$W^2 = m_p^2 + Q^2(1/x - 1) \tag{91}$$

where in the latter formula we have kept the proton mass in order to indicate that the threshold of W corresponds to elastic scattering.

Within the framework of the parton model, DIS proceeds by the exchange of a photon or intermediate vector boson with only one of the quarks in the proton. This is shown in the diagram in Fig. 3.

The electron-quark collision is elastic. As a result of this collision the struck quark acquires a sufficient momentum to break away from the rest of the proton as far as the colour force allows it to travel. At this stage some of the binding energy is converted into the creation of a quark-antiquark pair from the vacuum; the antiquark combines with the original quark into a meson, leaving behind a quark which can give rise to the creation of another quark-antiquark

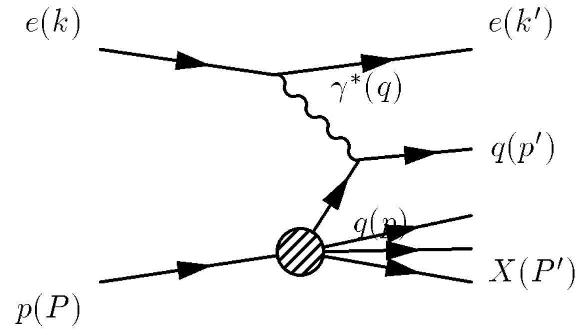

Figure 3: Parton model diagram of deep inelastic scattering.

pair. This process, called *fragmentation*, continues until the remaining energy drops below the threshold for the creation of another pair. Thus, as a result of fragmentation, several mesons are created which travel roughly in the direction of the struck quark. Such a system of mesons, or more generally of hadrons, is called a *jet*. The residue of the proton is a highly unstable system: it has lost a quark, absorbed a quark presumably of the wrong sort that is left over from the fragmentation, and has absorbed a fraction of the energy transferred from the electron. Therefore it breaks up into several hadrons.

The elastic electron-quark collision is the *hard subprocess* of DIS. If we think of the incoming electron and proton as travelling in opposite directions, then the quark carries a fraction of the proton momentum. At a sufficiently high momentum, where the proton mass is negligible, the energy of the quark is the same fraction of the proton energy. It turns out that this fraction is identical with the Bjorken-x defined above. Denoting the 4-momentum of the incoming quark by p we have therefore

$$p = xP$$

Denoting the invariant $(k+p)^2$ by s, which is the squared CMS energy of the subprocess, we have therefore also

$$s = xS \tag{92}$$

This, together with the definition of Q^2, shows that the two independent invariants that control the kinematics of the subprocess are x and Q^2.

The first DIS experiments were carried out in 1967 at the Stanford 2-mile linear electron accelerator with electron beams of up to 20 GeV and hydrogen targets at rest, giving a CMS energy of about 6 GeV. Subsequent fixed target experiments were done in other laboratories, notably at the CERN SPS with muon beams of up to nearly 300 GeV and hence of CMS energies up to about 25 GeV. The range of energies available for DIS was extended by an order of magnitude when in 1992 the electron-proton collider HERA came into operation at the DESY laboratory in Hamburg. In this collider the electrons are accelerated up to nearly 30 GeV and the protons up to 820 GeV, giving a CMS energy of 314 GeV. Theoretically the corresponding values of Q^2 go up to about 10^5 GeV^2.

An important tool to study the structure of the nucleon is also deep inelastic scattering with neutrinos as beam particles. The kinematics is identical with the one described above, but one must bare in mind that the exchanged particle in neutrino-DIS is an intermediate vector boson, either the W or the Z.

9. Phase Space Integrals.

One step in the evaluation of a decay rate or of a cross section is the evaluation of phase space integrals. This step is logically part of kinematics, the dynamics of the process being contained in the matrix element.

The differential decay rate of a particle of 4-momentum $\mathrm{P} = (E, \vec{P})$ into n particles of 4-momenta p_1, p_2, ..., p_n is given by

$$d\Gamma = \frac{(2\pi)^4}{2E} |\mathcal{M}|^2 d^{3n}\Phi_2(\mathrm{P}; \mathrm{p}_1, \mathrm{p}_2, \ldots, \mathrm{p}_n) \tag{93}$$

where $|\mathcal{M}|^2$ is the Lorentz invariant square of the matrix element and $d^{3n}\Phi_2(\mathrm{P}; \mathrm{p}_1, \mathrm{p}_2, \ldots, \mathrm{p}_n)$ is the Lorentz invariant differential phase space factor. The usual task of phase space calculations is to integrate over all unobserved degrees of freedom.

Consider the simplest case, namely two-body decay. Then the Lorentz invariant differential phase space factor is

$$d^6\Phi_2(\mathrm{P}; \mathrm{p}_1, \mathrm{p}_2) = \delta^4(\mathrm{P} - \mathrm{p}_1 - \mathrm{p}_2) \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} \tag{94}$$

where $\mathrm{P} = (E, \vec{P})$, $\mathrm{p}_1 = (E_1, \vec{p}_1)$ and $\mathrm{p}_2 = (E_2, \vec{p}_2)$ are the 4-momenta of the mother particle and the two daughter particles, respectively, and the four dimensional delta function ensures energy-momentum conservation.

Thus, in the present case, if we want to find the total decay rate, then we must integrate the differential decay rate

$$d\Gamma = \frac{(2\pi)^4}{2E} \overline{|\mathcal{M}|^2} d^6\Phi_2(\mathrm{P}; \mathrm{p}_1, \mathrm{p}_2) \tag{95}$$

over all components of p_1 and p_2. Here $\overline{|\mathcal{M}|^2}$ is the spin-averaged square of the matrix element, which in the case of two-body decay is a constant. Therefore the integrations can be carried out exactly. Integrating over one of the three-momenta removes the three dimensional delta function $\delta^3(\vec{P} - \vec{p}_1 - \vec{p}_2)$, hence

$$d^3\Phi_2(\mathrm{P}; \mathrm{p}_1, \mathrm{p}_2) = (2\pi)^{-6} \delta(E - E_1 - E_2) \frac{d^3p_1}{2E_1}$$

where $E_1 = \sqrt{p_1^2 + m_1^2}$ and $E_2 = \sqrt{(\vec{P} - \vec{p}_1)^2 + m_2^2}$. Now we exploit the Lorentz invariance of Φ_2 by finishing the integration in the "*frame of convenience*", which in this case is the rest frame of the mother particle (CMS), where $\vec{P} = 0$ and $E = m$, hence

$$d^3\Phi_2(\text{CMS}) = (2\pi)^{-6} \delta(m - E_1 - E_2) \frac{d^3p_1}{2E_1}$$

Here E_1 retains its previous form and $E_2 = \sqrt{p_1^2 + m_2^2}$. Since the integrand has no angular dependence, the integration over the full solid angle gives a factor of 4π, leaving us with the integration of the modulus of $\vec{p}_1$. This is conveniently converted into an integration over E_1, using $E_1^2 = p_1^2 + m_1^2$, hence $p_1\, dp_1 = E_1\, dE_1$. The final integration over E_1 is accomplished by setting the argument of the delta function equal to $f(E_1)$ and using the identity

$$\delta(f(E_1)) = |f'(E_{10})|^{-1} \delta(E_1 - E_{10})$$

where E_{10} is the root of $f(E_1) = 0$. Thus finally we get

$$\Phi_2 = \frac{2p_{1\,\text{CMS}}}{(2\pi)^5 m}$$

where the CMS momentum is $p_{1\,\text{CMS}} = \{[m^2 - (m_1 - m_2)^2][m^2 - (m_1 + m_2)^2]\}^{1/2}/2m$.

Example: pion decay $\pi \to \mu\nu$.
The spin averaged modulus squared of the matrix element of pion decay is in the lowest order of perturbation theory given by

$$\overline{|\mathcal{M}|^2} = 4G^2 f_\pi^2 m_\mu^2 (\mathrm{p}\cdot\mathrm{k}) \tag{96}$$

where G is the Fermi constant, f_π is the pion decay constant, m_μ is the muon mass, p and k are the four-momenta of the muon and neutrino, respectively, and we assume the neutrino to have zero mass. The differential decay rate of this process is given by

$$d\Gamma = \frac{(2\pi)^4}{2m_\pi}\overline{|\mathcal{M}|^2}\, d^6\Phi_2(\mathrm{P},\mathrm{p},\mathrm{k})$$

where P is the four-momentum of the pion and

$$d^6\Phi_2(\mathrm{P};\mathrm{p},\mathrm{k}) = \delta^4(\mathrm{P}-\mathrm{p}-\mathrm{k})\frac{d^3p}{(2\pi)^3 2E}\frac{d^3k}{(2\pi)^3 2\omega}$$

where E and ω are the energies of the muon and the neutrino, respectively. We get the total decay rate Γ by integrating over the momenta $\vec{p}$ and $\vec{k}$. Let us do this in two steps. First we integrate over the neutrino momentum:

$$\begin{aligned}
d^3\Gamma &= \frac{(2\pi)^4}{2m_\pi}\int \overline{|\mathcal{M}|^2}\,\delta(E_\pi - E - \omega)\delta^3(\vec{P}-\vec{p}-\vec{k})\frac{d^3p}{(2\pi)^3 2E}\frac{d^3k}{(2\pi)^3 2\omega} \\
&= \frac{1}{32\pi^2 m_\pi}\int \overline{|\mathcal{M}|^2}\,\delta(E_\pi - E - \omega)\frac{d^3p}{E\omega}
\end{aligned}$$

then we integrate over the polar angle θ and azimuth ϕ of the muon. To do this we represent d^3p as $p^2\,dp\,d\Omega$, where $d\Omega = d\phi\,d\cos\theta$, and we use $E\,dE = p\,dp$, thus

$$d^3\Gamma == \frac{1}{32\pi^2 m_\pi}\int \overline{|\mathcal{M}|^2}\,\delta(E_\pi - E - \omega)\frac{pdE\,d\Omega}{\omega}$$

Now we note that the matrix element is constant. Indeed, we have $\mathrm{P}^2 = (\mathrm{p}+\mathrm{k})^2 = m_\mu^2 + 2\mathrm{p}\cdot\mathrm{k}$, but also $\mathrm{P}^2 = m_\pi^2$, hence $\mathrm{p}\cdot\mathrm{k} = (1/2)(m_\pi^2 - m_\mu^2)$, and the matrix element is given by

$$\overline{|\mathcal{M}|^2} = 2G^2 f_\pi^2 m_\mu^2 (m_\pi^2 - m_\mu^2)$$

and therefore the integrand has no angular dependence and integration over Ω gives just a factor of 4π. Thus

$$d\Gamma = \frac{1}{8\pi m_\pi}\overline{|\mathcal{M}|^2}\,\delta(E_\pi - E - \omega)\frac{pdE}{\omega}$$

Now, since momentum conservation has already been imposed by integration over $\vec{k}$, the energies E and ω are no longer independent: we have $\omega = k = p = \sqrt{E^2 - m_\mu^2}$. Let us denote the argument of the remaining δ function by $f(E)$: $f(E) = E_\pi - E - \omega$, or, working from now on in the pion rest frame, $f(E) = m_\pi - E - \omega$. Then, using the identity

$$\delta(f(E)) = |f'(E_0)|^{-1}\delta(E - E_0)$$

we get

$$\Gamma = \frac{1}{8\pi m_\pi^2}\overline{|\mathcal{M}|^2}\,p_0$$

where $p_0 = \sqrt{E_0^2 - m_\mu^2}$. Using once more momentum conservation, we also have $p_0 = k_0 = \omega_0 = (m_\pi^2 - m_\mu^2)/2m_\pi$. Finally we get for the pion decay rate

$$\Gamma = \frac{G^2 f_\pi^2}{8\pi} m_\pi m_\mu^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 \tag{97}$$

Now the decay rate is related to the mean life τ by $\Gamma = \hbar/tau$. Therefore we can find the pion decay constant f_π from Eq. (97) by substituting the known values of the Planck constant $hbar$, the Fermi constant G and the muon and pion masses.

Example: Muon decay $\mu \to e\nu_\mu\nu_e$.

In this example we have three final state particles, two of which are massless. Each of the outgoing particles can have an energy of up to one half of the muon rest energy, *i.e.* approximetely up to 50 MeV, which is an ultrarelativistic energy for the electron. At momenta close to zero, the electron is of course nonrelativistic, but at such energies the approximation which we are going to use, where the matrix element is calculated in the lowest order of perturbation theory, is not very accurate and one must include radiative corrections. This is outside the scope of these notes. We shall therefore carry out the calculations with an ultrarelativistic electron over the entire phase space, and hence set the electron mass equal to zero.

The reaction equation of muon decay is

$$\mu(\mathrm{p}) \to e(\mathrm{p}') + \nu_\mu(\mathrm{k}) + \nu_e(\mathrm{k}')$$

with the following four-momenta in the muon rest frame:

$$\mathrm{p} = (m, \vec{0}), \qquad \mathrm{p}' = (E', \vec{p'}), \qquad \mathrm{k} = (\omega, \vec{k}), \qquad \mathrm{k}' = (\omega', \vec{k'})$$

The differential decay rate is given by

$$d\Gamma = \frac{1}{2m}\overline{|\mathcal{M}|^2}\, d\Phi_3(\mathrm{p}; \mathrm{p}', \mathrm{k}, \mathrm{k}')$$

where in lowest order perturbation theory we have

$$\overline{|\mathcal{M}|^2} = 64G^2(\mathrm{p}\cdot\mathrm{k}')(\mathrm{k}\cdot\mathrm{p}')$$

Now, $\mathrm{p}\cdot\mathrm{k}' = m\omega'$, but $\mathrm{k}\cdot\mathrm{p}'$ involves the energies ω and E' and the angle between $\vec{k}$ and $\vec{p'}$. This is inconvenient. Therefore we use four-momentum conservation to transform this scalar product:

$$2\mathrm{k}\cdot\mathrm{p}' = (\mathrm{k} + \mathrm{p}')^2 = (\mathrm{p} - \mathrm{k}')^2 = m^2 - 2m\omega'$$

hence

$$\overline{|\mathcal{M}|^2} = 64G^2 m^2 \omega'(m/2 - \omega') \tag{98}$$

and in this form $\overline{|\mathcal{M}|^2}$ depends only on the energy ω'; therefore we need not consider it until we get to the integration over ω'.

Writing the Lorentz invariant phase space factor $d\Phi_3(\mathrm{p}; \mathrm{p}', \mathrm{k}, \mathrm{k}')$ in detail, we have therefore for the differential decay rate

$$d\Gamma = \frac{1}{2m}\overline{|\mathcal{M}|^2}\,(2\pi)^4\delta^4(\mathrm{p} - \mathrm{p}' - \mathrm{k} - \mathrm{k}')\frac{d^3p'}{(2\pi)^3 2E'}\frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}$$

and we begin the calculation by integrating over the muon-neutrino three-momentum $\vec{k}$; we get

$$\int \delta^4(\mathrm{p}-\mathrm{p}'-\mathrm{k}-\mathrm{k}')\frac{d^3k}{\omega} = \frac{1}{\omega}\delta(m-E'-\omega-\omega')$$

with $\omega = |\vec{k}| = |\vec{k'}+\vec{p'}| = \sqrt{\omega'^2+E'^2+2\omega'E'\cos\theta}$, where θ is the angle between $\vec{p'}$ and $\vec{k'}$.

Next we integrate over $\vec{k'}$. To do this we represent d^3k' in polar coordinates with the polar axis along $\vec{p'}$, which is kept fixed at this stage, *i.e.* $d^3k' = 2\pi\omega'^2 d\omega'\, d\cos\theta$, where the factor 2π comes from the integration over the azimuth. First the integration over $\cos\theta$:

$$d\Gamma = \frac{1}{(4\pi)^4 m}\overline{|\mathcal{M}|^2}\frac{d^3p'}{E'}\omega' d\omega' \int_{-1}^{1}\delta(m-\omega-\omega'-E')\frac{d\cos\theta}{\omega}$$

Let us put $x=\cos\theta$ and denote the argument of the δ function by $f(x)$, *i.e.*

$$f(x) = m-\omega'-E'-\sqrt{\omega'^2+E'^2+2\omega'E'x}$$

and proceed by using the standard formula

$$\delta(f(x)) = |f'(x_0)|^{-1}\delta(x-x_0)$$

where x_0 is defined by $f(x_0)=0$ and $f'(x_0)$ is the derivative of $f(x)$ at $x=x_0$:

$$f'(x_0) = -(E'\omega'/\omega_0)$$

with $\omega_0 = \omega(x_0)$, and hence

$$\delta(f(x))\frac{dx}{\omega} = \delta(x-x_0)\frac{dx}{E'\omega'}$$

which gives us immediately

$$d\Gamma = \frac{1}{(4\pi)^4 m}\overline{|\mathcal{M}|^2}\frac{d^3p'}{E'^2}d\omega'$$

and we must remember that the δ function gives a nonzero value only if $-1\leq x_0\leq 1$ or, equivalently, if $\omega_-\leq\omega\leq\omega_+$, where

$$\omega_\pm = \sqrt{\omega'^2+E'^2\pm 2\omega'E'} = |\omega'\pm E'|$$

and since by virtue of the δ function we have $\omega = m-\omega'-E'$, this condition can be cast in the form of

$$|\omega'-E'| \leq m-\omega'-E' \leq \omega'+E'$$

or, if we add $(\omega'+E')$ and divide by 2,

$$\frac{1}{2}(|\omega'-E'|+\omega'+E') \leq \frac{m}{2} \leq \omega'+E' \tag{99}$$

The left-hand side of this inequality is the greater of ω' and E', and hence the left-hand inequality is equivalent to two inequalities:

$$E'\leq m/2 \qquad \text{and} \qquad \omega'\leq m/2$$

which, together with the right-hand inequality, defines a triangle in the $(E',\,\omega')$ plane over which the integration over ω' and E' has to be taken, see Fig. 4.

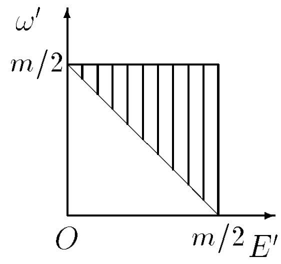

Figure 4: The shaded region is the domain of integration (99)

The integral over ω' goes from $(m/2 - E')$ to $m/2$, and we must now remember that the matrix element depends on ω'. Thus

$$d\Gamma = \frac{64G^2\,m}{(4\pi)^4}\,\frac{d^3p'}{E'^2}\int_{m/2-E'}^{m/2}\omega'\left(\frac{m}{2}-\omega'\right)\,d\omega' = \frac{16G^2\,m^2}{(4\pi)^4}\left(1-\frac{4E'}{3m}\right)\,4\pi\,E'^2\,dE'$$

where in the last step we have also represented d^3p' in polar coordinates and done the angular integrations, which have yielded a factor of 4π. We can rewrite our result in the form of the energy distribution of the emitted electron:

$$\frac{d\Gamma}{dE'} = \frac{16\Gamma^2 m^2}{(4\pi)^3}\,E'^2\left(1-\frac{4E'}{3m}\right) \tag{100}$$

This result is very close to the experimental data.[7]

Now we can also get the total decay rate by carrying out the final integration over E' from zero to $m/2$, hence

$$\Gamma = \frac{G^2 m^5}{192\pi^3} \tag{101}$$

and hence the muon lifetime:

$$\tau = \frac{\hbar}{\Gamma} = \frac{192\hbar\pi^3}{G^2 m^5}$$

This formula can be used to get a value for the Fermi constant G from the accurately known values of $\hbar$ and of the muon mass and lifetime. The result is remarkably close to the result obtained from similar calculations for nuclear β decay. The small differences are accounted for by radiative corrections. However, we shall not enter into these considerations, remembering that the purpose of this exercise was an illustration of the phase space integration in the case of three-body decay.

The derivation of the three-particle phase space given here followed the most direct route. One can find other, more ingenious methods in the literature. For instance, in Part 2 of the "*Relativistic Quantum Mechanics*" by E.M. Lifshits and L.P. Pitaevsky,[8] the following alternative derivation is presented.

[7] M. Bardon et al., Phys. Rev. Lett. **14** (1965) 449; the deviation is partly due to radiative corrections and partly due to experimental acceptance.

[8] this is Vol. IV of the monograph on theoretical physics by Landau and Lifshits; the derivation mentioned here is given in Chapter XVI, section 146 of the Russian 1971 edition

Without spin summation, the modulus squared matrix element is

$$|\mathcal{M}|^2 = 32G^2(p_e - m_e a_e)_\alpha (p_\mu - m_\mu a_\mu)_\beta k_e^\alpha k_\mu^\beta$$

where p_e and p_μ are the four-momenta of the electron and muon, respectively, m_e and m_μ are their masses, a_e and a_μ are their polarization vectors, and k_e and k_μ are the four-momenta of the electron and muon neutrino, respectively. The 4-vector components are labeled by α and β. The differential decay rate is given by

$$d\Gamma = (2\pi)^4 \delta^4(p_e + k_e + k_\mu - p_\mu) \frac{|\mathcal{M}|^2}{2m_\mu} \frac{d^3p_e}{(2\pi)^3 2E_e} \frac{d^3k_e}{(2\pi)^3 2\omega_e} \frac{d^3k_\mu}{(2\pi)^3 2\omega_\mu}$$

where E_e, ω_e and ω_μ are the energies of the electron, electron neutrino and muon neutrino, respectively.

Integration over the neutrino momenta is accomplished by taking the following integral:

$$I^{\alpha\beta} = \int k_e^\alpha k_\mu^\beta \delta^4(k_e + k_\mu - q) \frac{d^3k_e}{\omega_e} \frac{d^3k_\mu}{\omega_\mu}$$

where $q = p_\mu - p_e$. The only dynamical variable, on which this symmetric tensor depends, is q; therefore the integral must be of the form of

$$I^{\alpha\beta} = Aq^2 g^{\alpha\beta} + Bq^\alpha q^\beta$$

where A and B are numerical factors. Contracting $I^{\alpha\beta}$ with $g_{\alpha\beta}$ we get

$$(4A + B)q^2 = \frac{1}{2} I \, q^2$$

and contracting with $q_\alpha q_\beta$ we get

$$(A + B)(q^2)^2 = \frac{1}{4} I \, (q^2)^2$$

where

$$I = \int \delta^4(k_e + k_\mu - q) \frac{d^3k_e}{\omega_e} \frac{d^3k_\mu}{\omega_\mu}$$

Integration over d^3k_μ removes the three-dimensional δ function $\delta^3(\vec{k}_e + \vec{k}_\mu - q)$, leaving us with the integral

$$I = \int \delta(\omega_e + \omega_\mu - \omega) \frac{d^3k_e}{\omega_e \omega_\mu}$$

which is done in the CMS of the two neutrinos, where $\vec{k}_e + \vec{k}_\mu = 0$ and $\omega_e = \omega_\mu$, and there is no angular dependence, so the angular integration gives a factor of 4π, leaving us with the integral

$$I = 4\pi \int \delta(2\omega_e - \omega) \, d\omega_e = 2\pi$$

and hence

$$I^{\alpha\beta} = \frac{\pi}{6}(q^2 g^{\alpha\beta} + 2q^\alpha q^\beta)$$

and the final result is found by straight forward calculations.

For n particle decays the phase space integrals can be done recursively in the CMS:

$$\Phi_n(E) = \frac{1}{(2\pi)^3} \int \frac{d^3p_n}{2E_n} \Phi_{n-1}([E^2 - 2EE_n + m_n^2]^{1/2})$$

As we have seen, the three-particle phase space integral can be evaluated analytically. For $n > 3$ one has to resort to numerical integration which is usually done by Monte Carlo methods.

Phase Space Calculations in the Case of Collision Processes.

Phase space calculations are equally important in collision processes. Here one must bear in mind that the matrix element is not constant even in the simplest case of $2 \to 2$ collisions. However the structure of the spin-averaged square of the matrix element of a two-body collision is sufficiently simple for analytical calculation to be feasible. For inelastic $2 \to n$ particle collisions with $n \geq 3$ one resorts to numerical integration.

For $2 \to 2$ body collisions, *i.e.* elastic in quasielastic collisions, the observed quantity is the differential cross section; it is given by the formula

$$d\sigma = (2\pi)^4 \frac{\overline{|\mathcal{M}|^2}}{F}\, d^6\Phi_2(\mathrm{p}_1 + \mathrm{p}_2; \mathrm{p}_3, \mathrm{p}_4) \tag{102}$$

where $F = 4\sqrt{(\mathrm{p}_1 \cdot \mathrm{p}_2)^2 - m_1^2 m_2^2}$ is the Lorentz invariant flux of the incoming particles of masses m_1 and m_2 and 4-momenta p_1 and p_2, and the 4-momenta of the outgoing particles are denoted p_3 and p_4. In the LAB frame the flux factor takes the form of

$$F = 4\, m_2\, p_{\mathrm{LAB}}$$

where, as before, m_2 is the mass of the target partcle. In the CMS we have

$$F = 4\, p_{\mathrm{CMS}}\, \sqrt{s}$$

Consider the particular example of electron-muon elastic scattering in the LAB frame. Then the spin-averaged matrix element is given by

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{\mathrm{q}^4} 2M^2 E E' \left[\cos^2\frac{\theta}{2} - \frac{\mathrm{q}^2}{2M^2}\sin^2\frac{\theta}{2}\right]$$

where M is the muon mass, E and E' are the initial and final electron energies, θ is the scattering angle, $\mathrm{q}^2 = -4EE'\sin^2\frac{\theta}{2}$ is the 4-momentum transfer, and we have neglected the electron mass.

Note that $\overline{|\mathcal{M}|^2}$ depends on the observable scattering angle θ. Therefore, bearing in mind that we want to integrate over all *unobserved* degrees of freedom, we must from the beginning plan the phase space integrations such as *not* to integrate over θ, except possibly as a very last step if we want to find the total cross section.

The Lorentz invariant two-body phase space factor is

$$d^6\Phi_2 = \delta^{(4)}\left(\mathrm{p}_3 + \mathrm{p}_4 - \mathrm{p}_1 - \mathrm{p}_2\right) \frac{d^3 p_3}{(2\pi)^3 2E_3} \frac{d^3 p_4}{(2\pi)^3 2E_4}$$

and we have to carry out four of the six integrations to get rid of the four-dimensional δ function.

We begin by integrating over the 3-momentum $\vec{p}_4$; this removes the three-dimensional δ function $\delta^{(3)}\left(\vec{p}_3 + \vec{p}_4 - \vec{p}_1 - \vec{p}_2\right)$ and leaves us with

$$d^3\Phi_2 = \delta\left(E_3 + E_4 - E_1 - E_2\right) \frac{p_3^2 dp_3 d\Omega}{[2(2\pi)^3]^2 E_3\, E_4}$$

where we have expressed the three-dimensional differential d^3p_3 in polar coordinates with $d\Omega = \sin\theta d\theta d\phi$, where θ and ϕ are the polar angle and azimuth, respectively.

At this point we change the notation to that used previously for the LAB frame by setting

$$E_1 = E = \sqrt{\vec{k}^2}, \qquad E_3 = E' = \sqrt{\vec{k}'^2}, \qquad E_2 = M, \qquad \text{and} \qquad E_4 = \sqrt{\vec{p}'^2 + M^2}$$

hence

$$d^3\Phi_2 = \frac{1}{4(2\pi)^6}\,\delta\left(E' + E_4 - E - M\right)\,\frac{E'dE'}{E_4}\,d\Omega$$

We note that our integration over $\vec{p}_4$ has already enforced momentum conservation. Therefore we have

$$\vec{p}' = \vec{k} - \vec{k}', \qquad \text{hence} \qquad \vec{p}'^2 = E^2 + E'^2 - 2EE' \cos\theta$$

Now consider the argument of the δ function. Let us denote it by $f(E')$, *i.e.*

$$f(E') = E' + \sqrt{\vec{p}'^2 + M^2} - E - M$$

Its zero corresponds to energy conservation. To evaluate the integral over E' we rewrite the δ function in the form of

$$\delta(f(E') = \left|\frac{df(E_0')}{dE'}\right|^{-1} \delta(E' - E_0')$$

where E_0' is the zero of $f(E')$. Differentiating $f(E')$ with respect to E' we get

$$\frac{df(E')}{dE'} = 1 + \frac{E' - E\cos\theta}{E_4} = \frac{E_4 + E' - E\cos\theta}{E_4}$$

or, applying energy conservation, $E_4 + E' = E + M$,

$$\frac{df(E')}{dE'} = \frac{M + E(1-\cos\theta)}{E_4} = \frac{ME}{E'E_4}$$

where in the last step we have used $\mathrm{q}^2 = -2EE'\,(1-\cos\theta) = -2M(E - E')$. Thus finally we have

$$\delta(f(E')) = \frac{E'E_4}{ME}\,\delta(E' - E_0')$$

Now we can do the integral over E' and get

$$d^2\Phi_2 = \frac{1}{4(2\pi)^6}\,\frac{1}{(4\pi)^2}\,\frac{E'^2}{ME}\,d\Omega$$

where in the final expression we have dropped the subscript of E_0', which is now redundant.

Putting our results for the matrix element, the phase space factor and the flux factor together, we get for the differential cross section of elastic electron-muon scattering the following expression:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{LAB}} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\,\frac{E'}{E}\left(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right) \tag{103}$$

where $\alpha = e^2/4\pi$ is the fine structure constant.

References

[1] E. Byckling and K. Kajantie, "*Particle Kinematics*", Wiley, New York, 1973.

[2] J.D. Bjorken and S.D. Drell, "*Relativistic Quantum Mechanics*", McGraw-Hill, New York, 1964.

[3] "*Review of Particle Physics*", Particle Data Group, European Physical Journal C 15 (2000) 1-878.
Popularly known as the Particle Data Tables, this review is published every other year; it is available on the website http://pdg.lbl.gov