Longitudinal beam motion:

- phase stability
- bunch & bucket
- adiabatic trapping
- dispersion & transition energy
- synchrotron motion

Electrons:

- synchrotron radiation & emitted power
- synchrotron radiation spectrum
- damping/excitation of synchrotron motion

Acceleration: particle momentum increased by voltage $V = V_0 \sin\phi_s$ to keep pace with magnetic field rise. Pre-programmed amplitude V_0 changed by voltage control circuitry. Synchronous phase lag ϕ_s set by comparing RF voltage phase with bunch passage. Beam monitor detectes bunch & gives feedback to control of V_0 & ϕ_s.

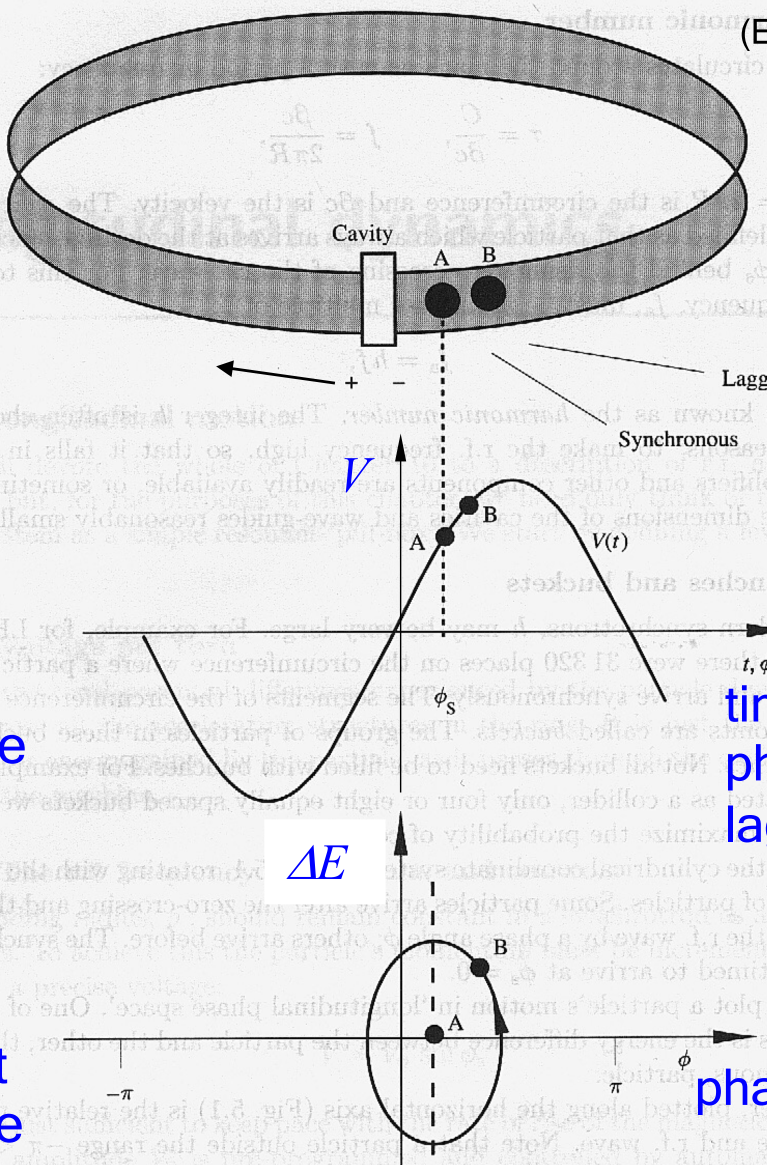
Assume $v \ll c$.

Particle B arriving late receives an extra energy increment that causes it to speed up & overtake synchronous particle A.

However particle B receive less energy than particle A next time etc...

Provided ΔE not too large particle will follow ellipse in phase space.

(E.Wilson)



time or phase lag

phase lag

Fig. 5.1 The cylindrical coordinate system which rotates with beam demonstrating the meaning of r.f. phase angle in longitudinal phase space.

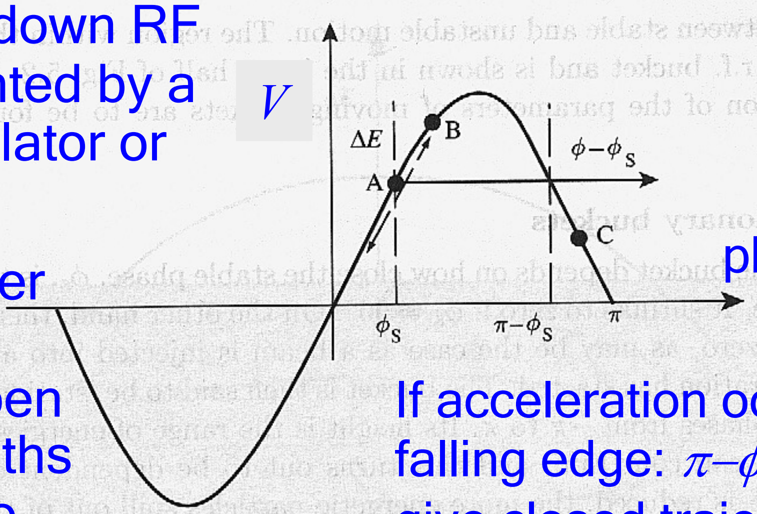
Particle circulate in accelerator with period & frequency:

$$\tau = 2\pi R/\beta c, \text{ or } f = \beta c/2\pi R$$

Synchronous particle arrives always at phase lag ϕ_s at rising RF wave. Require RF frequency f_{RF} to be integer multiple of f: $f_{RF} = h\,f$, where h = harmonic number. In modern synchrotrons h usually very large since high frequency amplifiers available & cavity dimensions can be kept small, i.e. LHC h = 35640 & f_{RF} = 400.79 MHz. Each beam segment centered around such a point is a ”bucket” (the beam has h buckets). A particle group in a bucket = ”bunch” (NB! Usually not all buckets filled).

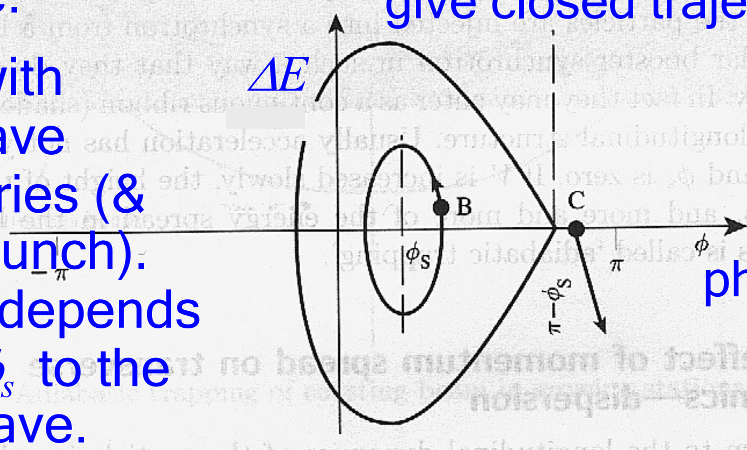
- Motion up & down RF wave represented by a harmonic oscillator or pendulum.
- Particles either follow closed (=stable) or open (=unstable) paths in phase space.
- All particles with $0 \le \phi \le \pi - \phi_s$ have closed trajectories (& belong to the bunch). Size of bucket depends on how close ϕ_s to the crest of sine wave.

(E.Wilson)



phase lag

If acceleration occur on falling edge: $\pi - \phi_s \le \phi \le \pi$ give closed trajectories



phase lag

Fig. 5.2 The limiting trajectory for a particle in a 'moving' or accelerating bucket when the stable phase is not zero.

(E.Wilson)

ϕ_s = 0 $\Rightarrow$ bucket ”stationary” & contains all particles $-\pi < \phi < \pi$. Range of energies $2\Delta E$ that RF constrains = bucket height, depends on $\sqrt{V}$ for given ϕ_s. If particles enter as continous beam (B constant, ϕ_s = 0), V increased so that bucket height grows $\Rightarrow$ more & more of energy spread in beam (”adiabatic trapping”). In modern colliders bunches injected according to scheme dependent on accelerator & pre-accelerators RF & bunching scheme

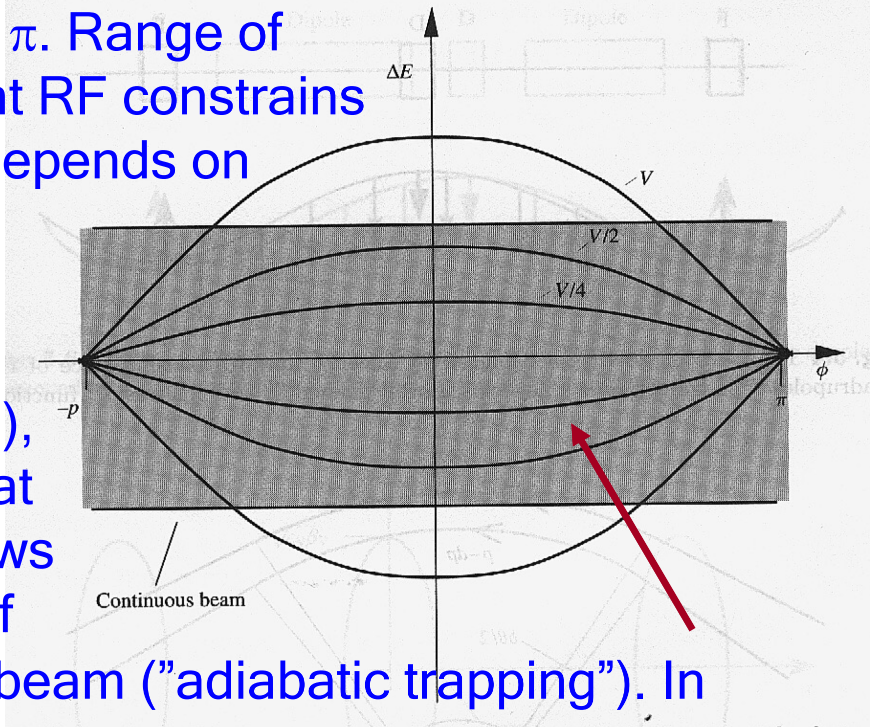


Fig. 5.3 Adiabatic trapping of coasting beam in growing stationary bucket.

Effect of momentum spread on transverse motion

Analogy: why gravity not make particles spiral down?

Larger vertical focusing at F quadrupoles with than without gravity; compensate saging due to gravity. Gives a very small correction to closed orbit.

(E.Wilson)

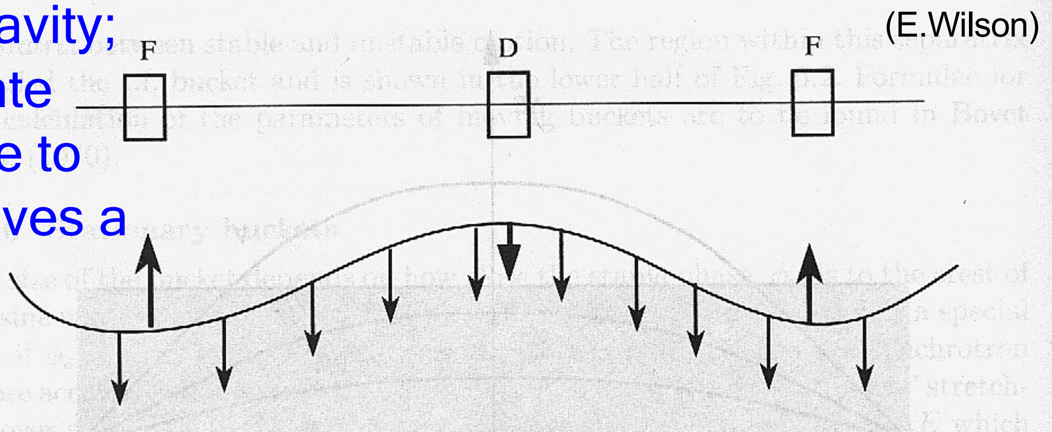


Fig. 5.4 The force of gravity is balanced by the net upward focusing force of the quadrupoles. The trajectory is a new closed orbit defined by the 'suspension' function.

Dispersion

- Lower momentum particles bent more horizontally
- Could spiral inwards but: have a (inwards) displaced closed orbit, closer to horizontal D quadrupole axis
- Extra (outward) force at horizontal F quadrupoles balances extra bending

(E.Wilson)

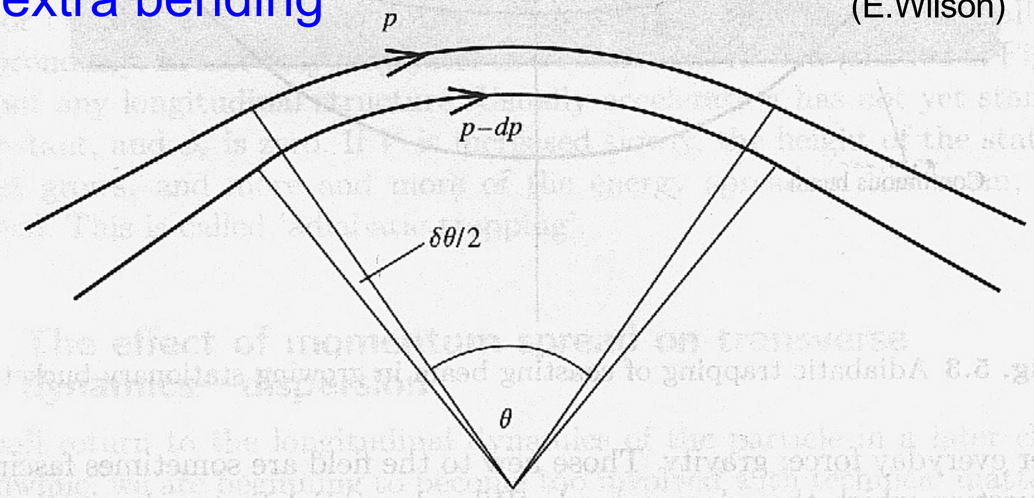


Fig. 5.5 Orbits in a bending magnet.

- $D(s)$ = "dispersion" $\quad x = D(s)\dfrac{\Delta p}{p}$

N.B. For F quadrupole: $\Delta x' = klx$, with $k < 0$

(E.Wilson)

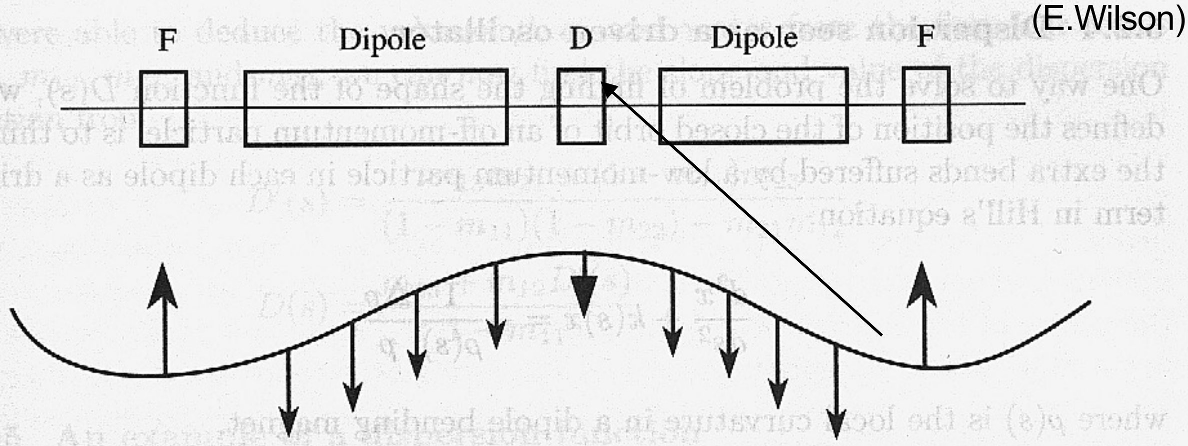


Fig. 5.6 The extra inward force given to a low-momentum particle by the dipoles is balanced by the quadrupoles and defines a dispersion function.

(E.Wilson)

Closed orbit ellipses for 3 cases: $\Delta p/p < 0$, $\Delta p/p = 0$ & $\Delta p/p > 0$

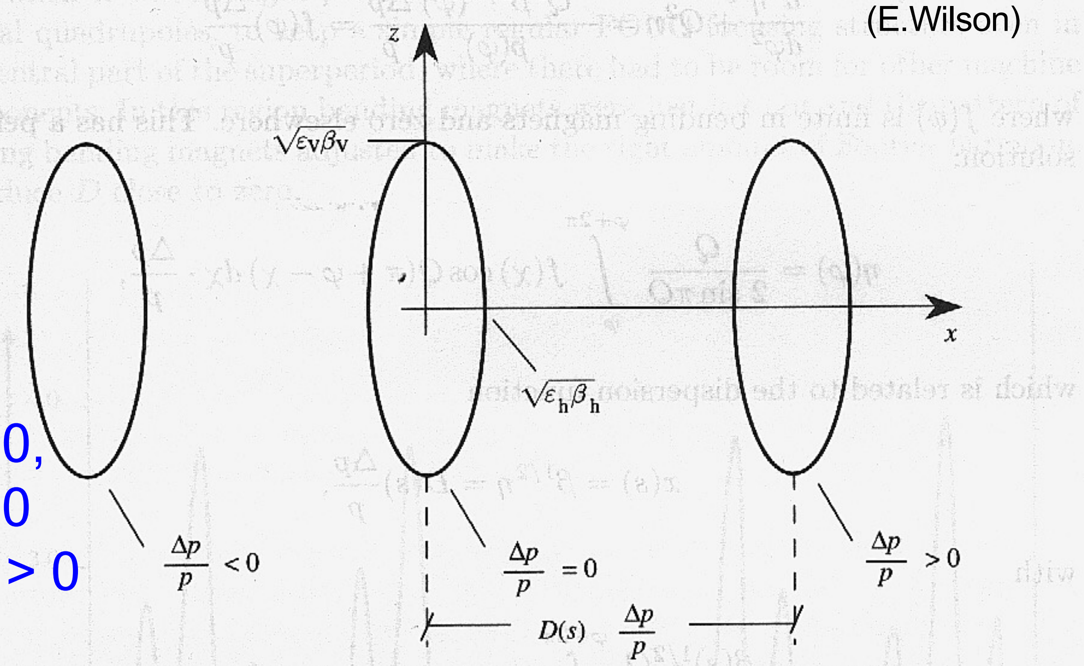


Fig. 5.7 The beam cross sections in real space for beams of three different momenta at a point where the dispersion function is large.

- Horizontal beam size grows due to momentum spread of beam particles. Vacuum chamber must be able accommodate full momentum spread of beam.
- Vertical and horizontal beam widths are:

$$\sigma_V = \sqrt{\beta_V \varepsilon_V}\ , \sigma_H = \sqrt{\beta_H \varepsilon_H} \oplus \left| D(s)\frac{\Delta p}{p} \right|$$

How does this influence phase stability ?

Get's more complicated, particle lagging behind ϕ_s gets accelerated more now also gets a longer path through accelerator (bent less in dipoles). Average dispersion ($\overline{D}$) must be included into mean radius of closed orbit. More on the implications soon.

$$R = R_0 + \overline{D}\frac{\Delta p}{p}$$

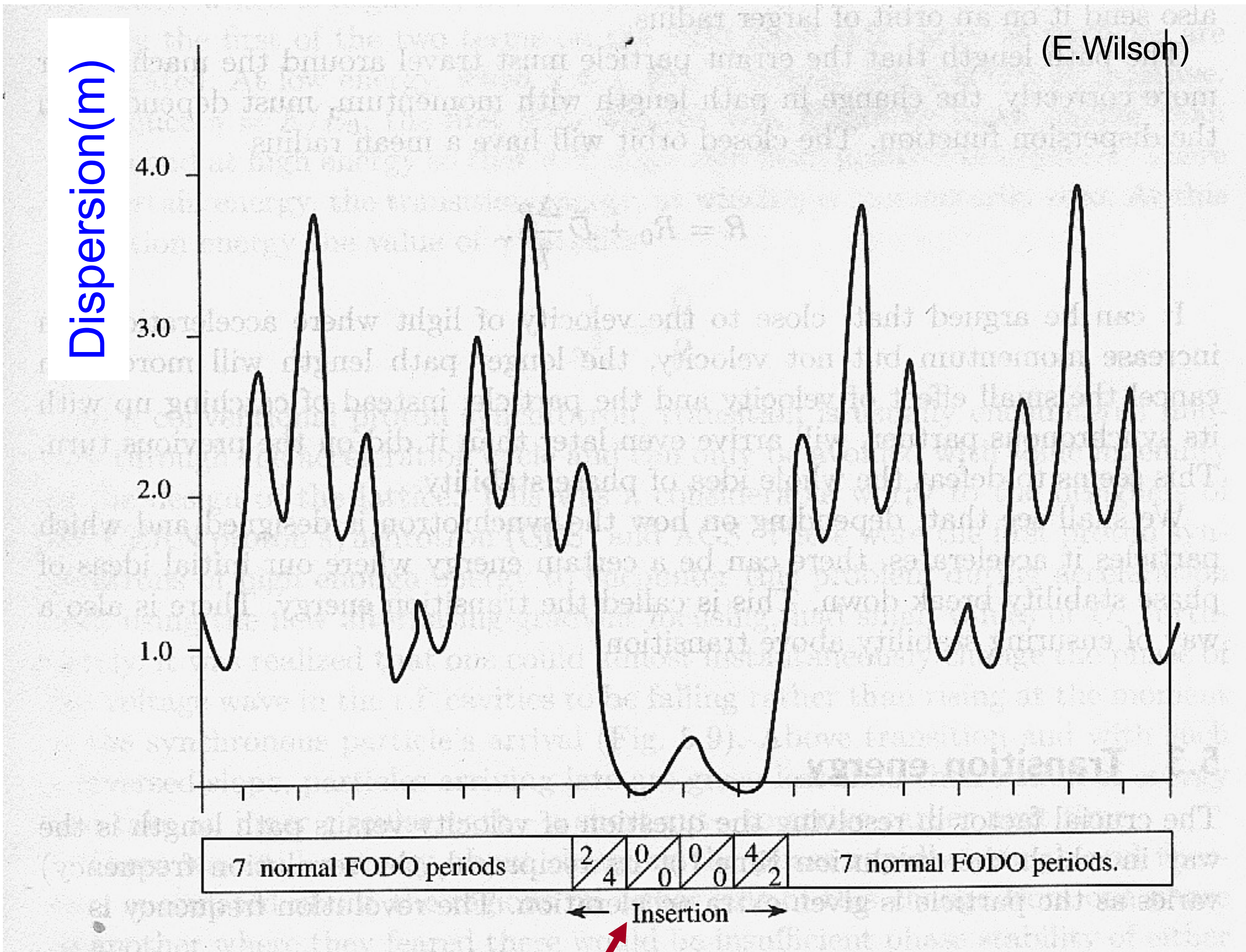


Fig. 5.8 The variation of the dispersion function in one sextant of the SPS centred on the long straight section.

- Long straight section where dipoles omitted leave room for other equipment e.g. RF cavities; Injection; Extraction; Experiments etc...
- Pattern of missing dipoles magnets in region indicated by “0” adjusted to reduce $D(s) \approx 0$.
- Big dispersion in arcs not catastrophic at SPS since $\Delta p/p$ small & horizontal quadrupole aperture large.
- Modern synchrotron has special designed focusing insertions to avoid such dispersion oscillations in arcs.

Dispersion D & its slope D' obtainable from dispersion-less "sine" (S, $\phi_0 = \pi/2$) & "cosine" (C, $\phi_0 = 0$) solutions.

- Combination of displacement x, divergence x' and dispersion D + its slope D' gives:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}$$

- Expressed as a matrix

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_{s_0}$$

- It can be shown that:

$$\boxed{D(s) = S(s) \int_{s_0}^{s} \frac{1}{\rho(t)} C(t)\,dt - C(s) \int_{s_0}^{s} \frac{1}{\rho(t)} S(t)\,dt}$$

- Fulfils Hill's equation if a "driving" term is added:

$$D''(s) + k(s)D(s) = -\frac{1}{\rho(s)} \quad \text{or} \quad \frac{d^2x}{ds^2} + k(s)x = -\frac{1}{\rho(s)}\frac{\Delta p}{p}$$

Revolution frequency $f(\beta, R)$ (both β & R depend on p).

$$f = \beta c / 2\pi R; \quad p = \frac{m\beta}{\sqrt{1-\beta^2}}; \quad R = R_0 + \overline{D}\frac{\Delta p}{p}$$

Use partial differentials to define a "slip factor" η_{RF}:

$$\frac{df}{dp} = \frac{df}{d\beta}\frac{d\beta}{dp} + \frac{df}{dR}\frac{dR}{dp} \qquad R_0 \equiv R(\Delta p / p = 0)$$

$$\gamma = 1/\sqrt{1-\beta^2}$$

$$\boxed{\eta_{RF} = \frac{\Delta f / f}{\Delta p / p} = \frac{p}{\beta}\frac{d\beta}{dp} - \frac{p}{R}\frac{dR}{dp} = \frac{1}{\gamma^2} - \frac{\overline{D}}{R_0}}$$

η_{RF} changes from positive to negative at ' transition':

"gamma" transition $\boxed{\frac{1}{\gamma_{tr}^2} = \frac{\overline{D}}{R_0}}$

(E.Wilson)

Problematic at early proton synchrotrons (electrons no problem - large γ initial).

Solution simple: at transition energy change phase; acceleration occurs on falling edge instead of rising. Particles lagging behind take shorter path around accelerator & catch up.

phase lag or time

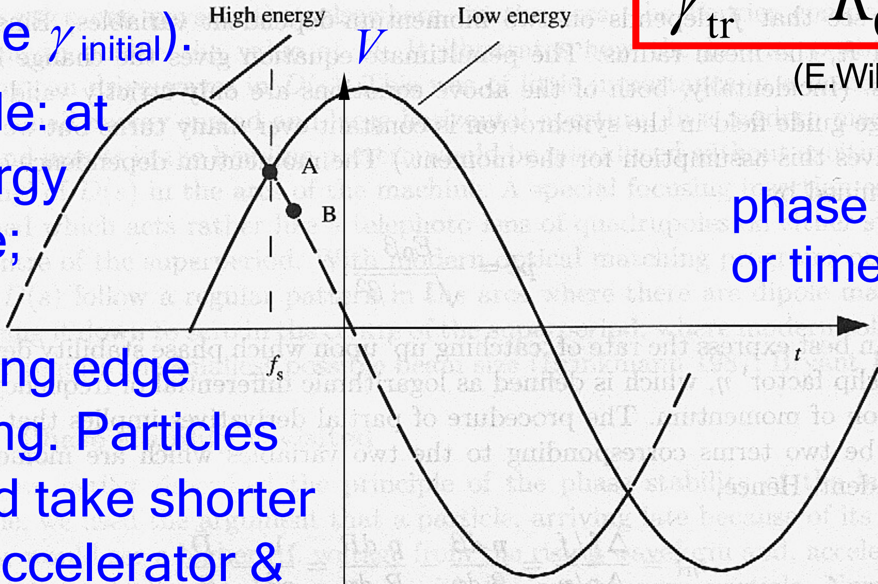


Fig. 5.9 Shows how changing the phase of the r.f. voltage waveform can give the lagging particle, B, less energy rather than more and can lead to stability above transition.

- recall $p = mc\beta\gamma$
- elliptical trajectory for small amplitudes ($\Delta E \propto \Delta(\beta\gamma)$)

$$\Delta(\beta\gamma) = \widehat{\Delta(\beta\gamma)} \sin 2\pi f_s t$$
$$\phi = \hat{\phi} \cos 2\pi f_s t$$

- f_s = synchrotron frequency i.e. execution frequency of oscillations in the ΔE–phase lag space
- change rate in arrival phase ($\leftrightarrow$ angular frequency of oscillator) w.r.t. synchronous particle written as

$$d\phi / dt = \dot{\phi} = -2\pi h\left[f(\Delta\beta\gamma) - f(0)\right] = -2\pi h \Delta f$$

- introducing slip factor η_{RF}: h = harmonic number

$$\Delta f = \eta_{RF} f \frac{\Delta p}{p} = \eta_{RF} f \frac{\Delta(\beta\gamma)}{\beta\gamma} = \frac{\eta_{RF} f}{\beta^2} \frac{\Delta\gamma}{\gamma} = \frac{\eta_{RF} f}{m\beta^2\gamma} \Delta E$$

- differentiating again: $\ddot{\phi} = -\dfrac{2\pi h \eta_{RF} f}{m\beta^2\gamma}(\Delta\dot{E})$
- utilizing that $\Delta E = eV_0(\sin\phi - \sin\phi_s)$
- identify energy change rate as $\Delta E \cdot f$

$$\boxed{\ddot{\phi} = -\frac{2\pi e V_0 \eta_{RF} h f^2}{m\beta^2\gamma}(\sin\phi - \sin\phi_s)}$$

exact description of synchrotron motion when parameters change slowly

- described by a biased rigid pendulum

$$\boxed{\ddot{\phi} = -\frac{2\pi e V_0 \eta_{RF} h f^2}{m\beta^2\gamma}(\sin\phi - \sin\phi_s)}$$

- to see analytic solution for small amplitudes ($\sin\phi \approx \phi$), set ϕ_s = 0:

$$\boxed{\ddot{\phi} + \frac{2\pi e V_0 \eta_{RF} h f^2}{m\beta^2\gamma}\phi = 0} \Rightarrow$$

$f_s = \omega_s/2\pi$, where $\omega_s =$ the angular frequency of the solution to the equation.

- synchrotron frequency f_s ($f_{RF} = hf$)

$$\boxed{f_s = \sqrt{\frac{|\eta_{RF}| h e V_0}{2\pi m\beta^2\gamma}} f = \sqrt{\frac{|\eta_{RF}| e V_0}{2\pi m\beta^2\gamma h}} f_{RF}}$$

- synchrotron “tune” Q_s , i.e. number of synchrotron oscillations a particle does per turn around accelerator

$$\boxed{Q_s = \frac{f_s}{f} = \sqrt{\frac{|\eta_{RF}| h e V_0}{2\pi m\beta^2\gamma}}}$$

In most synchrotrons Q_s order of 0.1 or less except at gamma transition where Q_s = 0. In 0-100 Hz range in large proton machine (can cause trouble when equal harmonics of 50 Hz related to wall plug power).

- particles with $\gamma \gg 1$ emit synchrotron radiation when their direction changed by external force.
- happens already at a few MeV for electrons but only at a few TeV for protons since $m_p \approx 2000 m_e$.
- synchrotron radiation related to bremsstrahlung – radiated power proportional to deceleration – however both bending & acceleration force now normal to trajectory (in case of bremsstrahlung force parallel to trajectory) - Larmor's formula:

$\rightarrow \bullet \leftarrow$ Force F

$$P = \frac{e^2}{6\pi\varepsilon_0 c^3} \cdot (\ddot{s})^2$$

radiated power $\propto F^2$ or $(\ddot{s})^2$

- $\ddot{s}$ not Lorentz invariant but radiated power must be, need an extra γ^2 from the Lorentz transformation.
- detailed derivation found in classical electrodynamics books like Jackson, also derived in Appendix of Wilson.

$$P = \frac{e^2}{6\pi\varepsilon_0 c^3} \cdot (\ddot{s}\gamma^2)^2$$

for motion on a circle:

$$\ddot{s} = v^2/\rho \approx c^2/\rho$$

$$\Rightarrow \quad P = \frac{1}{6\pi\varepsilon_0} \frac{e^2 c}{\rho^2} \cdot \gamma^4$$

Above valid for particles on a synchronous orbit !!

going to energy ($\gamma = E/mc^2$): $$P_\gamma = \frac{1}{6\pi\varepsilon_0}\frac{e^2}{m^4c^7}\frac{E^4}{\rho^2}$$

gives mass ($\propto 1/m^4$ so $1/m_e^4 \gg 1/m_p^4$), energy ($\propto E^4$) & bending radius ($\propto 1/\rho^2$) behaviour of radiated power

introducing classical electron "radius" r_e into formula:

$r_e \equiv \alpha/m_ec^2 = e^2/4\pi\varepsilon_0 m_e c^2 = 2.82 \cdot 10^{-15}$ m $$\Rightarrow P_\gamma = \frac{2}{3}\frac{r_e c}{(m_e c^2)^3}\frac{E^4}{\rho^2}$$

"energy loss per turn" U_o important for RF design (divide P by revolution frequency + correct for time fraction in bending magnets):

$f \approx c/2\pi R$

$f_{\text{bend}} = fR/\rho$

$$\Rightarrow U_0 = \frac{P_\gamma}{f_{bend}} = \frac{4\pi}{3}\frac{r_e}{(m_e c^2)^3}\frac{E^4}{\rho}$$

$U_0 \propto E^4 \Rightarrow$ factor 2 in energy gives factor 16 in energy loss / turn $\Rightarrow$ RF power limits electron synchrotrons !!

Synchrotron radiation at large electron synchrotrons:

	U_0	$P_{total\ dissipated} = P_\gamma \cdot N_{beam}$
LEP at 45.6 GeV	**125 MeV/turn**	**1.6 MW (14 MW)**
LEP at 100 GeV	**2.9 GeV/turn**	**18 MW (224 MW)**
182.5 GeV (91 km ring)	**10.4 GeV/turn**	**> 100 MW**

conclusion: in practice energy limited to ~350-400 GeV (in brackets real consumption including ohmic losses)

synchrotron radiation spectrum:

(E.Wilson)

$F(\xi)$ = emission probability

$S(\xi)$ = (power) spectral density

$\xi = u/u_c$

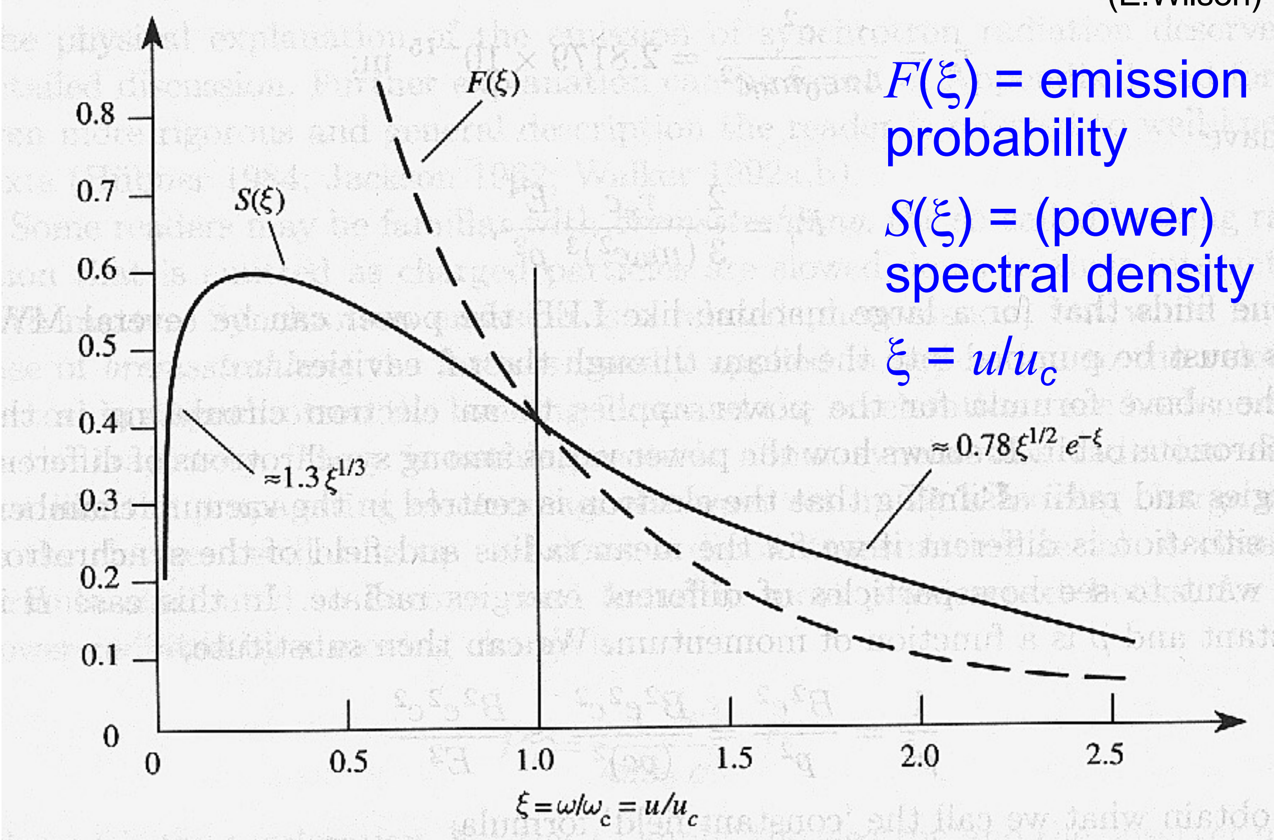


Fig. 8.1 Spectrum of synchrotron emission.

- spectrum broad; shape same when normalized to

$$u_c = \hbar\omega_c = 3\hbar c\gamma^3 / 2\rho$$

- however spectral density & broadness of spectrum depends on the energy of the critical quantum u_c.
if every quanta had the characteristic value u_c then rate of emitted synchrotron photons N would be

$$N = P_\gamma / u_c$$

- more careful averaging gives $N = \dfrac{15\sqrt{3}}{8}\dfrac{P_\gamma}{u_c}$

First figure shows a normal undamped synchrotron motion (protons).

Second figure shows damped motion in case of significant synchrotron radiation (electrons). Particles on upper upper half of their trajectory lose more energy than synchronous particle due to synchrotron radiation & migrates inwards in phase diagram – particles on lower half of their trajectory also migrates

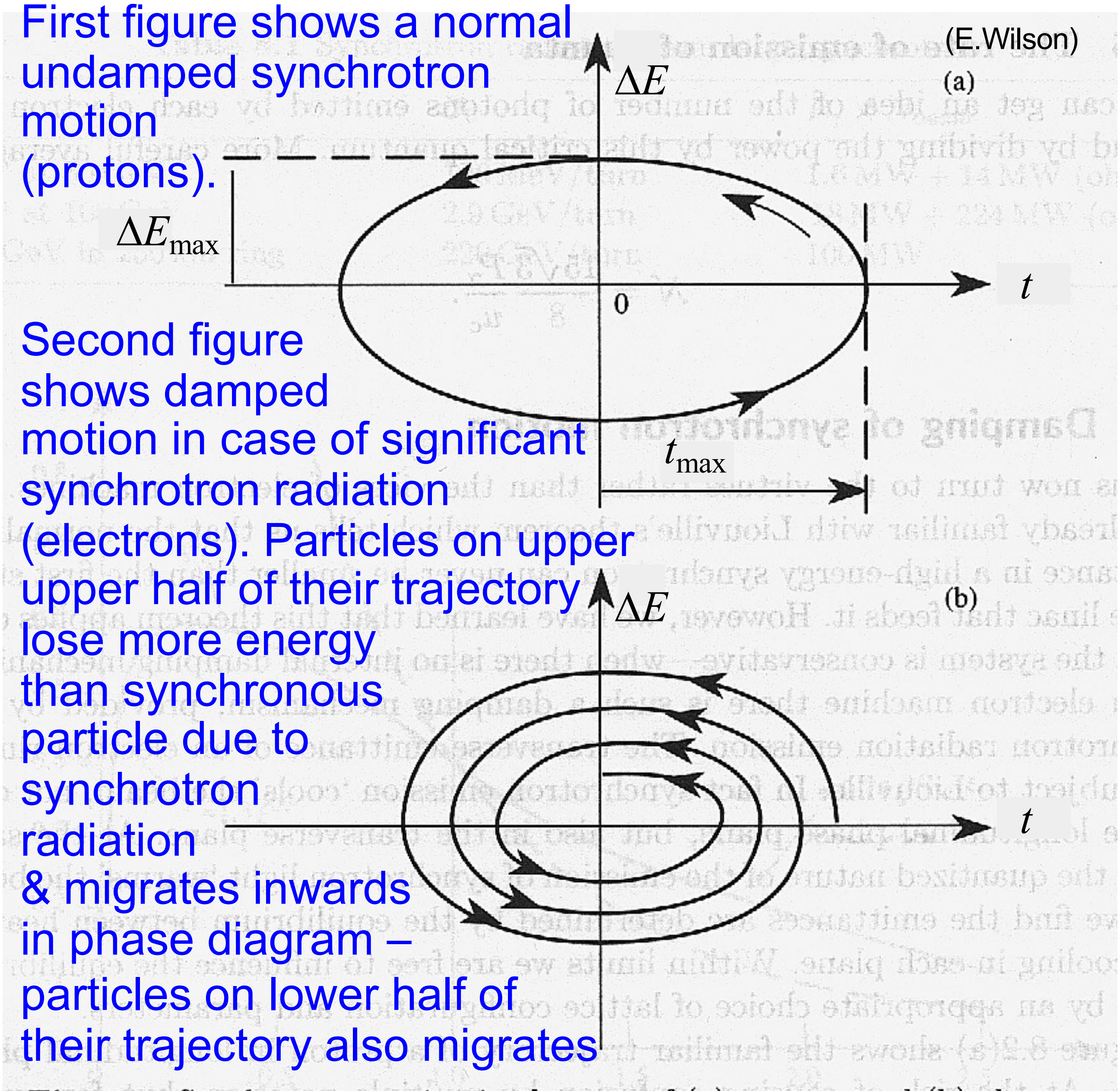


Fig. 8.2 Synchrotron motion in the case of (a) protons and (b) electrons.

inwards because now they lose less energy than synchronous particle – everything consequence of energy loss being proportional to cube of energy (E^4).

Argument assumes that synchronous particle gets an energy increment equal to what the particle lost due to synchrotron radiation each turn (to keep beam stable).

• start from emitted power expression for "constant field" ($B = p/\rho \approx E/\rho$) to avoid any assumption on the bending radius

$$P_\gamma = \frac{2}{3}\frac{r_e e^2}{(m_e c)^3}E^2 B^2$$

• particle with an energy difference ΔE w.r.t. synchronous particle

$$(E_S + \Delta E)^2 \approx E_S{}^2 + 2E_S\Delta E \Rightarrow$$

$$P_\gamma = \frac{2r_e e^2 B^2}{3(m_e c)^3}\left(E_S{}^2 + 2E_S\Delta E\right)$$

• rate of the energy loss

$$\frac{d\Delta E}{dt} = -\Delta E\frac{dP_\gamma}{d\Delta E} = -\frac{2P_\gamma}{E_S}\Delta E$$

• "damping time" i.e. time constant of the exponential damping = $1/\alpha$

$$\alpha = \frac{1}{\Delta E}\left|\frac{d\Delta E}{dt}\right| = \left\langle\frac{P_\gamma}{E_S}\right\rangle$$

• equation for the spiral path in ΔE vs. t plane

$$\Delta E(t) = \Delta E(t=0)e^{\alpha t}\cos\Omega t$$

damping rate for area quantities (like emittance) 2α.

damping time, inverse of α, i.e. time it takes particle to lose all its energy (linearly), can be as short as a few ms $\Rightarrow$ will dominate dynamics of an electron synchrotron, making it less prone to instabilities.

A competing mechanism exists, otherwise beam would become monochromatic. The mechanism causing growth of energy spread is due to the fact that photon emission occur in discrete quantized γ's.

- emission rate for photons & average loss

$$N = \frac{15\sqrt{3}}{8}\frac{P_\gamma}{u_c} \quad \& \quad u_c = \frac{3}{2}\frac{\hbar c \gamma^3}{\rho}$$

- axes chosen so that phase space a circle with radius *A*. Emission a random walk such that the change of radius is $\Delta A = \sqrt{N} u_c$

- area growth ∝ amplitude square $d\langle A^2 \rangle / dt = N \langle u^2 \rangle$

- careful averaging gives exponential growth rate

$$d\langle A^2 \rangle / A^2 dt = 11 N u_c^2 / 27 A^2 = 2\alpha$$

- equilibrium: put emittance growth due to emission of discrete quanta equal to emittance shrinkage rate

$$11 N u_c^2 / 27 A^2 = 2\alpha = 4 P_\gamma / E$$

remember: $\gamma = E/m_e$

- solve for an equilibrium amplitude

$$A^2 = \frac{11}{27}\frac{15\sqrt{3}}{8}\frac{1}{4}\frac{3}{2}\frac{\hbar c E^2 \gamma^2}{m_e \rho}$$

Equilibrium of synchrotron motion

• Fluctuations statistical & result in a Gaussian energy spread from electron/positron beams:

$$\left(\frac{\sigma_E}{E}\right)^2 = \frac{55}{64\sqrt{3}}\,\frac{\hbar c}{m_e c^2}\left(\frac{\gamma^2}{\rho}\right) = 1.92\cdot\,10^{-13}\left(\frac{\gamma^2}{\rho[m]}\right)$$

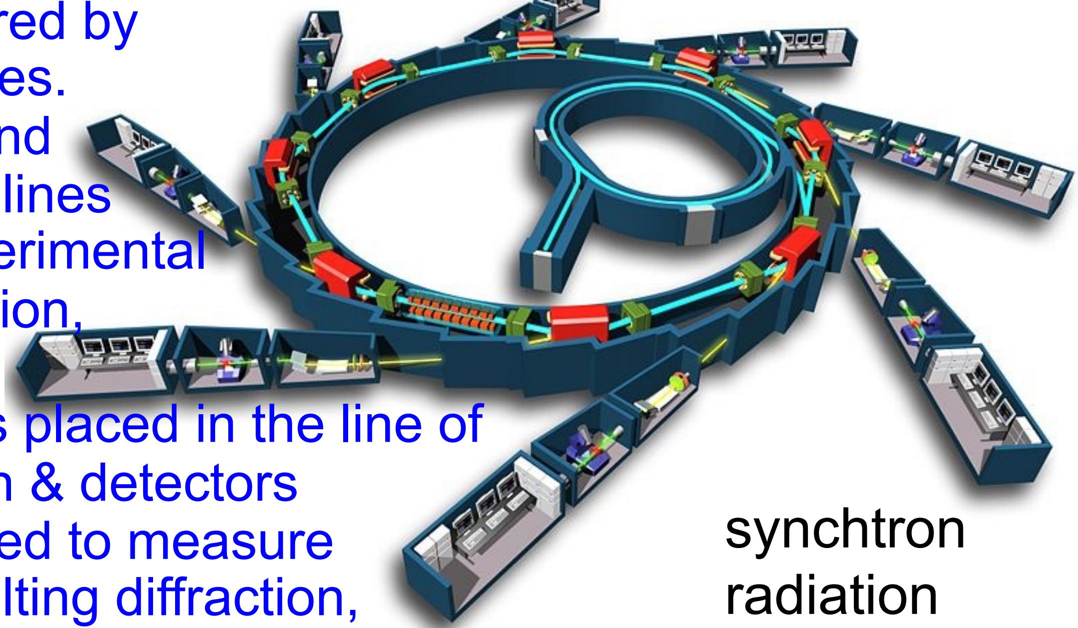
NB! Photon emission continously stirs up electron bunches & individual electrons will from time to time be in tails of the energy spread.Time it takes half of beam particles to have at some time as much as 6σ energy difference w.r.t. the synchronous particle not more than a few hours in an electron synchrotron.

applications of synchrotron light

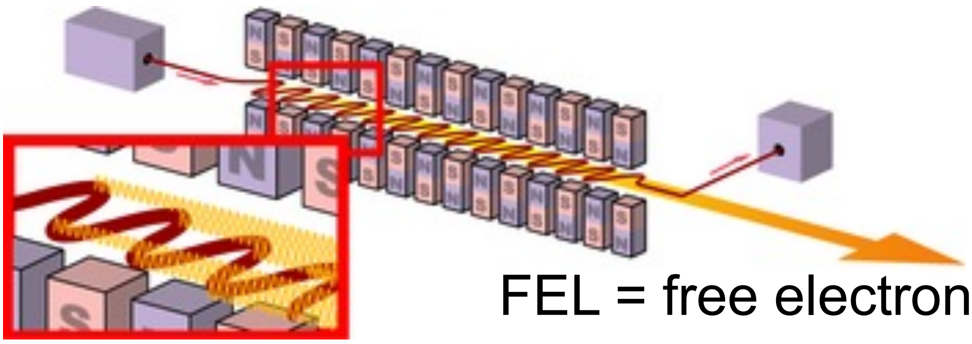
Synchrotron light = photons produced by bending magnets & insertion devices (“undulators”/“wigglers”) in storage rings. Major applications in condensed matter physics, materials science, biology & medicine. Experiments using synchrotron light involve probing structure of matter from nm level of electronic structure to μm & mm level important for medical imaging.

Some examples of synchrotron beamline applications:
- structural analysis of crystalline/amorphous material
- powder diffraction for structural characterization
- crystallography of proteins & other macromolecules
- photolithography for electronic structure production
- tomography for medical imaging
- residual stress analysis of materials…etc…

At a synchrotron facility, electrons usually accelerated by a synchrotron & then injected into a storage ring, in which they circulate, producing synchrotron radiation. Radiation projected at tangent to electron storage ring & captured by beamlines. At the end of beamlines are experimental end station, where samples placed in the line of radiation & detectors positioned to measure the resulting diffraction, scattering or secondary radiation.



synchtron radiation facility



FEL = free electron laser

electron beam led into “undulator” (set of magnets with alternating poles) forcing beam to take a sinusoidal path. Beam acceleration causes synchrotron radiation. Phase of electron motion automatically in phase with emitted photons. Already emitted photons interact with remaining electron beam & either accelerates or deaccelerates it → electrons become synchronized in phase → emitted photons make coherent beam.