One-point functions in defect Conformal Field Theories and Integrability

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Overview

1. Spectrum in a Conformal Field Theory
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   - Dilatation operator
2. Integrability in super-conformal filed theory
   - $\mathcal{N} = 4$ supersymmetric Yang-Mills theory
   - Underlying integrable spin chain
   - Defect filed theory
3. Generalize to the ABJM theory
   - Theory setup
   - Integrable $SU(4)$ alternating spin chain
4. Conclusion

Spectrum in a Conformal Field Theory

Scaling dimension

The scaling dimension of a local operator encodes how the operator transforms under a scaling of the coordinates:

$$\mathcal{O}(\lambda x) = \lambda^{-\Delta}\mathcal{O}(x) \tag{1}$$

For interacting conformal field theories, the scaling dimension depends on the coupling:

$$\Delta(g) = \Delta_0 + \gamma(g), \quad \gamma(0) \equiv 0 \tag{2}$$

$\gamma(g)$ is the anomalous dimension.

Define the dilatation operator that measures the scaling dimension:

$$\hat{D}\mathcal{O}_\Delta = \Delta\mathcal{O}_\Delta \tag{3}$$

Scaling dimension $\Leftrightarrow$ Spectrum of dilatation operator

Construct dilatation operator

Two-point function of a specific operator is fixed by conformal symmetry:

$$\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\rangle \sim \frac{1}{|x-y|^{2\Delta(g)}} \tag{4}$$

Expand Δ in g:

$$\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\rangle = \frac{1}{|x-y|^{2\Delta_0}}\left[1-\gamma(g)\log(\Lambda^2|x-y|^2)\right] \tag{5}$$

Now, we're able to derive dilatation operator by perturbation theory:

$$\hat{D}(g) = \sum_{n=0}^{\infty} g^{2n}\hat{D}^{(n)} \equiv \hat{D}^{(0)} + \Gamma(g), \quad \Delta(g) = \sum_{n=0}^{\infty} \Delta_n g^{2n} \tag{6}$$

Integrability in super-conformal filed theory

The standard $\mathcal{N} = 4$ SYM action in 4-dimension with gauge group $SU(N_c)$:

$$S_{\mathcal{N}=4} = 2\int d^4x\,\mathrm{Tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu\phi_i D^\mu\phi_i + \frac{i}{2}\bar{\psi}\Gamma^\mu D_\mu\psi + \frac{g_{\mathrm{YM}}}{2}\bar{\psi}\Gamma^i[\phi_i,\psi] + \frac{g_{\mathrm{YM}}^2}{4}[\phi_i,\phi_j][\phi_i,\phi_j]\right] \tag{7}$$

1. The theory is simpler than it looks and much simpler than QCD.
2. Simplicity comes from the huge symmetry: Conformal symmetry and maximal Supersymmetry in 4 dimension.
3. Integrable in the planar (large-N_c) limit.

The dilatation operator at one-loop

The correlation function of local operator $\mathcal{O}_{I_1,I_2\ldots I_L}(x) \sim Tr(\phi_{I_1}\phi_{I_2}\ldots\phi_{I_L})(x)$

$$\left\langle \mathcal{O}_{I_1,I_2\ldots I_L}(x)\overline{\mathcal{O}}_{J_1,J_2\ldots J_L}(y)\right\rangle_{one-loop} = \frac{1}{|x-y|^{2L}}\left(1-\log\left(\Lambda^2|x-y|^2\right)D^{(1)}_{IJ}\right) \quad (8)$$

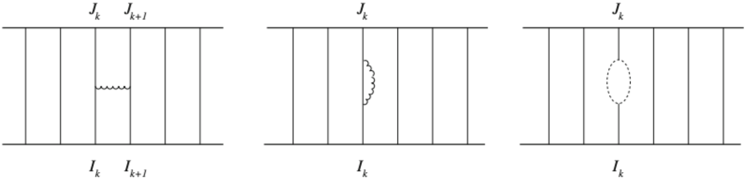


Figure: One-loop planar diagrams contributing to the correlation function

The dilatation operator at one-loop is:

$$D^{(1)}_{IJ} = \frac{\lambda}{16\pi^2}\sum_{n=1}^{L}(2-2\mathbb{P}_{n,n+1}+\mathbb{K}_{n,n+1})(\delta_{i_1,j_1}\delta_{i_2,j_2}\ldots\delta_{i_L,j_L}+\text{cyclic perm.}) \quad (9)$$

Map to spin-chain system

Gauge theory: Restrict to $SU(2)$ sub-sector, local operator consists of $Z \equiv \phi_1 + \mathrm{i}\phi_4$ and $X \equiv \phi_3 + \mathrm{i}\phi_5$. Thus, $\mathbb{K}_{n,n+1} = 0$

$$\Gamma(g) = \frac{\lambda}{16\pi^2} \sum_{n=1}^{L} (2 - 2\mathbb{P}_{n,n+1})$$

Diagonalize the dilatation operator

$$\Gamma \mathcal{O}_\Delta = \gamma(g) \mathcal{O}_\Delta$$
$$\mathcal{O}_\Delta = \mathrm{Tr}(ZZXZZX \cdots Z) + \cdots$$

Heisenberg spin-chain: Diagonalize the Hamiltonian

$$H|\psi\rangle = E|\psi\rangle$$
$$|\psi\rangle = |\uparrow\uparrow\downarrow\uparrow\uparrow\downarrow \cdots \uparrow\rangle + \cdots$$

Proposed by Minahan and Zarembo in 2002:

$$\begin{aligned} \Gamma(g) &\longleftrightarrow H \\ Z = \phi_1 + i\phi_4 &\longleftrightarrow |\uparrow\rangle \\ X = \phi_3 + i\phi_5 &\longleftrightarrow |\downarrow\rangle \end{aligned} \quad \Rightarrow \mathcal{N} = 4 \text{ SYM is integrable}$$

Heisenberg spin chain

The Hamiltonian of the Heisenberg spin-chain with L lattice sites:

$$H_{SU(2)} = \frac{\lambda}{8\pi^2} \sum_{\ell=1}^{L} \left(\frac{1}{2} - 2\vec{S}_\ell \cdot \vec{S_{\ell+1}} \right) = \Gamma(g)$$

The Hamiltonian is of size $2^L \times 2^L$

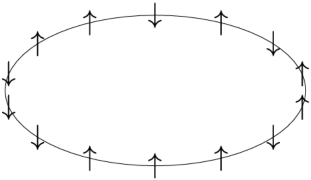


Figure: Heisenberg spin chain

Bethe ansatz equations (BAEs):

$$e^{ip_k L} = \prod_{j \neq k}^{N} S(p_j, p_k)$$

Map back to filed theory and find spectrum

Change variables from momenta to rapidity (just for simplicity):

$$e^{ip_k} = \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} \qquad u_k = \frac{1}{2}\cot\frac{p_k}{2}$$

Rewrite Bethe ansatz equations in a simpler form:

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{j\neq k}^{N} \frac{u_k - u_j + i}{u_k - u_j - i} \qquad k = 1, 2, ..., N \tag{10}$$

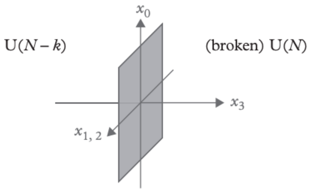
Map the energy spectrum to anomalous dimension:

$$E_N(\boldsymbol{p}) = E_0 + \sum_{k=1}^{N} \varepsilon(p_k) \quad \Rightarrow \quad \gamma(g) = g^2 \sum_{k=1}^{N} \frac{2}{4u_k^2 + 1}$$

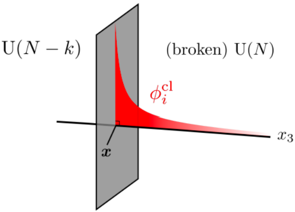
The correspondence between Bethe state and composite operator:

$$|\boldsymbol{u}\rangle = \Psi^{i_1 \dots i_L} |i_1, \dots, i_L\rangle \quad \Rightarrow \quad \mathcal{O}(x) = \Psi^{i_1 \dots i_L} \mathrm{tr}(\phi_{i_1} \dots \phi_{i_L})$$

Introduce co-dimension 1 defect:



defect filed theory



classical solutions of scalars

$$\nabla^2 \phi_i - [\phi_j, [\phi_j, \phi_i]] = 0 \quad \Rightarrow \quad \frac{d^2\phi_i}{dx_3^2} = [\phi_j, [\phi_j, \phi_i]]$$

$$\phi_i^{\text{cl}} = -\frac{1}{x_3}\begin{pmatrix} (t_i)_{k\times k} & 0_{k\times(N-k)} \\ 0_{(N-k)\times k} & 0_{(N-k)\times(N-k)} \end{pmatrix}, \; i = 1, 2, 3. \quad \phi_i^{\text{cl}} = 0, \; i = 4, 5, 6.$$

here t_i is k-dimensional representations of $\mathfrak{su}(2)$, recall $[t_i, t_j] = i\varepsilon_{ijk} t_k$

One-point functions in the defect field theory

For composite operator $\mathcal{O}(x) = \Psi^{i_1 \cdots i_L} \mathrm{tr}(\phi_{i_1} \ldots \phi_{i_L})$, inserting classical solutions gives us

$$\langle \mathcal{O}(x) \rangle^{cl} = (-1)^L \Psi^{i_1 \ldots i_L} \frac{\mathrm{tr}(t_{i_1} \ldots t_{i_L})}{x_3^L} \sim \frac{C}{|x_3|^{\Delta_0}} \tag{11}$$

Spin-chain picture: The defect $\Rightarrow$ Matrix Product State, for Heisenberg spin-chain, it takes the form:

$$|\,\mathrm{MPS}\rangle = \sum_{i_1, \ldots, i_L = 1}^{2} \mathrm{tr}[Z^{(i_1)} \ldots Z^{(i_L)}] |i_1, \ldots, i_L\rangle$$

Take $Z^{(1)} = t_1, Z^{(2)} = t_3$ by classical solutions, then

$$\langle \mathrm{MPS}\,|\boldsymbol{u}\rangle = \Psi^{i_1 \ldots i_L}\, \mathrm{tr}(t_{i_1} \ldots t_{i_L})$$

One-point functions in terms of overlap formula:

$$\langle \mathcal{O}(x) \rangle^{cl} \sim \frac{1}{x_3^L} \frac{\langle \mathrm{MPS}\,|\mathbf{u}\rangle}{\sqrt{\langle \mathbf{u}|\mathbf{u}\rangle}} \tag{12}$$

Integrable boundary states simplify overlap formula

Heisenberg spin-chain is a integrable system, from the point of view of conserved charges:

$$[Q_i, Q_j] = 0, \quad i, j = 1, \ldots, L$$

$Q_1 = \hat{P}, Q_2 = \hat{H}$ and

$$Q_3 = \sum_{l=1}^{L} Q_l, \quad Q_l = [H_{l-1,l}, H_{l,l+1}].$$

Definition of integrable boundary states (one proposal by Piroli, Pozsgay, Vernier 2017)

$$Q_{2n+1}|B\rangle = 0$$

In our case, we do have $Q_3|\text{MPS}\rangle = 0$ that implies the selection rule $\{u_k\} = \{-u_k\}$

Selection rule for Bethe roots ⇒ non-vanishing overlap $\langle \mathrm{MPS} \,|\, \boldsymbol{u} \rangle$

$$1 = \left(\frac{u_k - \frac{i}{2}}{u_k + \frac{i}{2}} \right)^L \prod_{\substack{j=1 \\ j \neq k}}^{N} \frac{u_k - u_j + i}{u_k - u_j - i} \equiv \exp[i \phi_k] \tag{13}$$

Gaudin matrix with size $N \times N$: $\quad G_{jk} = \frac{\partial \phi_j}{\partial u_k}$
Parity-symmetric Bethe roots $\{u_1, \ldots, u_{\frac{N}{2}}, -u_1, \ldots, -u_{\frac{N}{2}}\}$ lead to

$$\det G = \det G_+ \det G_-$$

here $G_\pm = \partial_{u_i} \phi_j \pm \partial_{u_{i+\frac{N}{2}}} \phi_j$ with $i, j = 1, 2, \ldots, \frac{N}{2}$, are $\frac{N}{2} \times \frac{N}{2}$ matrices.
Gaudin hypothesis: $\langle \boldsymbol{u} | \boldsymbol{u} \rangle \propto \det(G_{jk})$ for any integrable spin-chain
For Heisenberg spin-chain, we find:

$$\langle \boldsymbol{u} | \boldsymbol{u} \rangle = \left[\prod_{i=1}^{N} u_i^2 + \frac{1}{4} \right] \det G \qquad \frac{\langle \mathrm{MPS} \,|\, \mathbf{u} \rangle}{\sqrt{\langle \mathbf{u} | \mathbf{u} \rangle}} = 2^{1-L} \prod_{k=1}^{N} \sqrt{\frac{u_k - \frac{i}{2}}{u_k}} \sqrt{\frac{\det G_+}{\det G_-}}$$

Generalize to the ABJM theory

ABJM theory: setup

Super-Chern-Simons Theory

$$S = \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\lambda} \mathrm{tr} \left(A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda \right) + \cdots$$

Gauge group: $U(N)_k \times U(N)_{-k}$

Planar limit: $N, k \to \infty, \quad \lambda = \frac{k}{N}$ fixed

Field content:

Gauge field	$A_\mu, \quad \hat{A}_\mu$		[adjoint rep.]
Complex scalars	$Y_A^\dagger, \quad Y^A,$	$A = 1, 2, 3, 4$	[bi-fundamental rep.]
Weyl spinors	$\psi_A^\dagger, \quad \psi^A,$	$A = 1, 2, 3, 4$	[bi-fundamental rep.]

Scalar $SU(4)$ sector:

Gauge invariant operators

$$\mathcal{O} = \mathrm{tr} \left(Y^{I_1} Y_{J_1}^\dagger Y^{I_2} Y_{J_2}^\dagger \cdots \right) \quad \Rightarrow \quad \text{alternating spin chain}$$

vacuum $Y^1, Y_4^\dagger$ Excitations: Other scalar fields

The $SU(4)$ alternating spin chain

Hamiltonian

$$H = \frac{\lambda^2}{2} \sum_{n=1}^{2L} \left(2 - 2\mathbb{P}_{n,n+2} + \mathbb{P}_{n,n+2}\mathbb{K}_{n,n+1} + \mathbb{K}_{n,n+1}\mathbb{P}_{n,n+2}\right)$$

Basis states:

$$Y^A \mapsto |A\rangle \qquad \mathrm{tr}\left(Y^{A_1} Y^\dagger_{B_1} Y^{A_2} Y^\dagger_{B_2} \cdots\right) \mapsto \left|A_1 \bar{B}_1 A_2 \bar{B}_2\right\rangle$$

$$Y^\dagger_B \mapsto |\bar{B}\rangle$$

$$\mathbb{P}|A_1\rangle \otimes |A_2\rangle = |A_2\rangle \otimes |A_1\rangle \quad \mathbb{K}|A\rangle \otimes |\bar{B}\rangle = \delta_{AB} \sum_{C=1}^{4} |C\rangle \otimes |\bar{C}\rangle$$

The Hamiltonian can be diagonalized by Bethe ansatz.

Nested Bethe ansatz equations:

$$1 = e^{i\phi_{u_j}} = \left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^L \prod_{k\neq j}^{K_u} S(u_j, u_k) \prod_{k=1}^{K_w} \tilde{S}(u_j, w_k),$$

$$1 = e^{i\phi_{w_j}} = \prod_{k\neq j}^{K_w} S(w_j, w_k) \prod_{k=1}^{K_u} \tilde{S}(w_j, u_k) \prod_{k=1}^{K_v} \tilde{S}(w_j, v_k),$$

$$1 = e^{i\phi_{v_j}} = \left(\frac{v_j + \frac{i}{2}}{v_j - \frac{i}{2}}\right)^L \prod_{k\neq j}^{K_v} S(v_j, v_k) \prod_{k=1}^{K_w} \tilde{S}(v_j, w_k)$$

$$S(u,v) = \frac{u - v - i}{u - v + i} \qquad \tilde{S}(u,v) = \frac{u - v + \frac{i}{2}}{u - v - \frac{i}{2}}$$

Eigenstate: $|\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}\rangle = \sum_{\substack{\vec{\boldsymbol{s}} \in \text{all possible} \\ \text{distributions}}} \psi_{\vec{\boldsymbol{s}}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) |\vec{\boldsymbol{s}}\rangle$

$$E = \lambda^2 \left(\sum_{k=1}^{K_{\mathbf{u}}} \frac{1}{u_k^2 + \frac{1}{4}} + \sum_{k=1}^{K_{\mathbf{v}}} \frac{1}{v_k^2 + \frac{1}{4}} \right)$$

The $SU(4)$ alternating spin chain

The Gaudin matrix is of size $(K_u + K_w + K_v) \times (K_u + K_w + K_v)$

$$G = \begin{pmatrix} \partial_{u_i}\phi_{u_j} & \partial_{u_i}\phi_{w_j} & \partial_{u_i}\phi_{v_j} \\ \partial_{w_i}\phi_{u_j} & \partial_{w_i}\phi_{w_j} & \partial_{w_i}\phi_{v_j} \\ \partial_{v_i}\phi_{u_j} & \partial_{v_i}\phi_{w_j} & \partial_{v_i}\phi_{v_j} \end{pmatrix}$$

Gaudin conjecture: $\langle \boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v} | \boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v} \rangle \sim \det G$
Can be checked in specific case

Selection rules:
$K_u = K_w = K_v = L \quad \Rightarrow \{v_k\} = \{u_k\} \quad \{u_k\} = \{-u_k\} \quad \{w_k\} = \{-w_k\}$
leads to $\det G = \det G_+ \det G_-$ (same as $\mathcal{N} = 4$ SYM)

$$\frac{\langle \mathrm{MPS} | \boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v} \rangle}{\sqrt{\langle \boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v} | \boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v} \rangle}} \sim \sqrt{\frac{\det G_+}{\det G_-}}$$

Conclusion

Conclusion

1. For $\mathcal{N} = 4SYM$ and ABJM theory, there are underlying integrable spin chains that imply the integrability.
2. One-point functions in dCFT can be expressed as a overlap between Bethe states and boundary state.
3. Introduced domain wall corresponds to the integrable boundary state that yields selection rules for the overlap.
4. The overlap formula includes a universal determinant part and a model-dependent function, if the boundary state is integrable.

$$\frac{\langle B|\boldsymbol{u}\rangle}{\sqrt{\langle \boldsymbol{u}|\boldsymbol{u}\rangle}} = \prod_{j=1}^{N/2} f(u_j)\sqrt{\frac{\det G_+}{\det G_-}}$$

Thank you