Junction Conditions in General Relativity

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Table of Contents

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Notation

- We use the metric of signature (-1, 1, 1, 1)
- Greek indices $(\alpha, \beta, ...)$ run from 0 to 3
- Lower case Latin indices $(a, b, ...)$ run from 1 to 3
- Upper case Latin indices $(A, B, ...)$ run from 2 to 3

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Table of Contents

General relativity as a theory of spacetime

- General relativity, a theory of gravitation, was introduced by Albert Einstein in 1915.
- It is based on the principle of equivalence.
- This leads to the concept of a curved spacetime, where the presence of mass and energy distort the geometry of the universe.

The metric tensor defines a spacetime manifold

- The metric tensor $g_{\mu\nu}(x)$ allows for a generalization of the dot product of the ordinary Euclidean space.
- The metric alone defines the nature of spacetime it encodes.
- It can also be written in terms of the line element:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

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The second derivatives of the metric describe curvature

- The connection $\Gamma^{\alpha}_{\beta\gamma}$ is a structure in the manifold that helps define a covariant derivative.
- Connection of the manifold depends on the first derivative of the metric.
- The Riemann tensor $R_{\alpha\beta\gamma\delta}$ describes the curvature of the spacetime.
- The Riemann tensor depends on the derivative of the connection and as such on the second derivative of the metric.

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Einstein tensor and energy-momentum tensor are related

- The Einstein tensor $G_{\mu\nu}$ is a symmetric tensor containing information about the manifold's curvature.

- The energy-momentum tensor $T_{\mu\nu}$ describes the matter content of the universe.

- These are related via the Einstein equation.

Einstein equation - the equation of general relativity

The Einstein equation in geometrized units (where $G = c = 1$) is

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{1}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{2}$$

This equation relates the geometry of spacetime to the distribution of matter and energy within it.

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Solutions to the Einstein equation

There are many exact solutions to the Einstein equations:

- Minkowski space, which is familiar from special relativity.
- FLRW spacetime, which is used in cosmology.
- Schwarzschild spacetime, which predicts the existence of black holes.

Minkowski space - the simplest solution

- Minkowski space is the spacetime of special relativity, where there is no gravity.
- The metric of Minkowski space is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{3}$$

We say that the distance is

- timelike, when $ds^2 < 0$
- spacelike, when $ds^2 > 0$
- and lightlike (null), when $ds^2 = 0$

Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime

- The FLRW universe is familiar from cosmology.
- It describes a homogeneous and isotropic expanding (or contracting) universe.

The FLRW metric is

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right), \tag{4}$$

where K is the spatial curvature of the universe and $a(t)$ is a scale factor.

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Table of Contents

Hypersufaces - the basics

Definition of Hypersurface

A 3-dimensional subsurface of 4-dimensional spacetime.

Defining equations in the 4D spacetime

$$\Phi(x^\alpha) = 0 \qquad \text{or} \qquad x^\alpha = x^\alpha(y^a)$$

A Hypersurface

- can be thought of as some surface that exists in 4D-spacetime.
- has a metric that describes it.
- can be timelike, spacelike or null.

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Why hypersurfaces: to study the junction conditions geometrically

- We take the boundary between the spacetimes to be a hypersurface.
- Then the intrinsic properties of hypersurface, that are coordinate independent, become interesting.
- Using these intrinsic properties we can formulate the junction conditions in a purely geometric way.

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First fundamental form of the hypersurface

The intrinsic metric of hypersurface, also called the first fundamental form, is a projection of the metric in 4D spacetime:

$$h_{ab} = g_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b}, \tag{5}$$

for time- and spacelike case, and

$$\sigma_{AB} = g_{\alpha\beta} e^{\alpha}_{A} e^{\beta}_{B}, \tag{6}$$

for the null case. Here $e^{\alpha}_{a} = \frac{\partial x^{\alpha}}{\partial y^{a}}$ are the basis vectors in hypersurface.

Second fundamental form of the hypersurface

The second fundamental form, also known as extrinsic curvature, tells us how the hypersurface is embedded in the external 4D spacetime:

$$K_{ab} \equiv n_{\alpha;\beta} e^{\alpha}_{a} e^{\beta}_{b} \tag{7}$$

This carries information about the derivative of the metric in the normal direction. But this is not useful in null case, and we need to define transverse curvature:

$$C_{AB} = -N_{\alpha} e^{\alpha}_{A;\beta} e^{\beta}_{B} \tag{8}$$

where N^{α} is a transverse null vector of the hypersurface.

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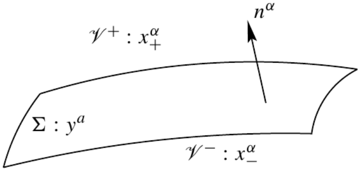
Table of Contents

What is the situation like?

We have the following situation:



We now want to see if the following distribution forms a valid distributional solution to the Einstein equation:

$$g_{\alpha\beta} = \Theta(l) g^+_{\alpha\beta} + \Theta(-l) g^-_{\alpha\beta} \tag{9}$$

Figure from *A Relativist's Toolkit* by Poisson

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The derivative of the metric gives the first junction condition

The derivative of the metric becomes

$$g_{\alpha\beta,\gamma} = \Theta(l)g^{+}_{\alpha\beta,\gamma} + \Theta(-l)g^{-}_{\alpha\beta,\gamma} + \varepsilon\delta(l)[g_{\alpha\beta}]n_\gamma \tag{10}$$

where $[g_{\alpha\beta}]$ is the jump of metric across the hypersurface. This means that the connection of the spacetime would have a term proportional $\Theta(l)\delta(l)$, which is not distributionally defined. This gives the first junction condition:

$$[g_{\alpha\beta}] = 0 \tag{11}$$

$$\Rightarrow [g_{\alpha\beta}]e^{\alpha}_{a}e^{\beta}_{b} = 0 \tag{12}$$

$$[h_{ab}] = 0 \tag{13}$$

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Are there any Singularities?

- The first junction condition itself won't be enough to rid us of singularities in curvature.
- This is because the Riemann tensor depends on the second derivative of the metric.
- The singularity appears in the form of the Dirac delta function.

Second junction condition gets rid of all singularities

- To get rid of this singular part, the term proportional to the delta function must be zero.
- Turns out this is the same as demanding that the jump in extrinsic curvature across the hypersurface be zero.
- This gives us the second junction condition:

$$[K_{ab}] = 0$$

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Is there a physical explanation for the singularity?

- In the case that the second junction condition is not satisfied, we can provide a physical explanation.
- The delta function in the Riemann tensor implies a delta function in the energy-momentum tensor.
- Thus the delta function singularity can be interpreted as there being a singular thin shell of matter at the hypersurface.

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Example: Oppenheimer-Snyder collapse

- Model of the collapse of a star into a black hole.
- The exterior spacetime (V^+) is the Schwarzschild and the interior spacetime (V^-) is FLRW.

$$ds_+^2 = -fdt^2 + f^{-1}dr^2 + r^2 d\Omega^2 \tag{14}$$

$$ds_-^2 = -d\tau^2 + a^2(\tau)(d\chi^2 + \sin^2\chi d\Omega^2) \tag{15}$$

Example: Oppenheimer-Snyder collapse

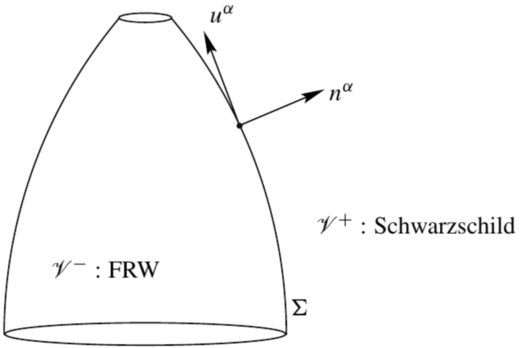

Figure: The Oppenheimer-Snyder spacetime.

Figure from *A Relativist's Toolkit* by Poisson

Example: Oppenheimer-Snyder collapse

From the junction conditions, we get the relationships between the coordinates of the two spacetimes and by combining them, we get

$$M = \frac{4\pi}{3}\rho R^3 \tag{16}$$

where R is the radius of the hypersurface in the region V^+. It can also be written in terms of coordinates of V^-:

$$R(\tau) = a(\tau)\sin\chi_0 \tag{17}$$

where χ_0 is the value of χ at hypersurface in the coordinates of V^-.

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Other applications

- Black hole universe
- Cosmological phase shift
- Brane worlds picture of the universe

Summary

- Hypersurfaces are convenient for describing the junction conditions.
- The junction conditions demand the jump in intrinsic metric and extrinsic curvature to be zero across the two sides of the hypersurface.
- When the jump in extrinsic curvature is not zero, we find that there is a singular thin layer of matter on the hypersurface.

Thank you for your attention!