Decidability for Priorean Linear Time using a Fixed-Point Labelled Calculus

Bianca Boretti[1] and Sara Negri[2]

[1] Dept. of Electronics and Information, Politecnico di Milano
[2] Dept. of Philosophy, University of Helsinki

Abstract. A labelled sequent calculus is proposed for Priorean linear time logic, the rules of which reflect a natural closure algorithm derived from the fixed-point properties of the temporal operators. All the rules of the system are finitary, but proofs may contain infinite branches. Soundness and completeness of the calculus are stated with respect to a notion of provability based on a condition on derivation trees: A sequent is provable if and only if no branch leads to a 'fulfilling sequent,' the syntactical counterpart of a countermodel for an invalid sequent. Decidability is proved through a terminating proof search procedure, with an exponential bound to the branches of derivation trees for valid sequents, calculated on the length of the characteristic temporal formula of the endsequent.

1 Introduction

What is commonly known as unary propositional linear time logic (LTL) is the future-oriented reflexive version of Priorean linear time logic: Only the future operators **G**, **F** and **T** are considered in unary LTL, and **G** and **F** have the intuitive meanings of 'it is and will always be the case' and 'it is or will be the case', respectively. LTL is known to be decidable [13]. Decidability has been established by several authors [15], [6], [7] through 2-phase tableau systems: In such systems, after the construction of the tableau graph, a second phase is required in order to check whether every eventuality formula has been satisfied.

In [11] a tableau system has been proposed, in which the termination of proof search can be determined locally, but the system covers only a limited fragment of LTL. In [12] a decision procedure for the whole logic has been achieved through a tableau calculus in which the second phase is incorporated into the rules by annotating sets of formulas with history information. However, this system contains a loop rule which hides a non-local closing condition: In fact, whereas the rules of the system act top-down, the "the result part [...] is synthesized bottom-up (from children to parents)" (p. 286), thus it is necessary to inspect previous nodes in order to verify if there is a loop. In [4] there are local rules with history annotation; Decidability of the calculus is not explicitly stated and would require a similar non-local closing condition in the form of a loop check.

In [2] a labelled sequent calculus G3LT for reflexive Priorean linear time was defined through the method of internalization of the possible world semantics within the syntax of sequent calculi, as developed by the second author in [8, 9]. The calculus has all the structural rules admissible, but it requires an infinitary rule to the effect that between any two points there are only finitely many other points. By replacing the infinitary rule with two weaker finitary rules a system for non-standard discrete frames was obtained and a conservativity result for an appropriate fragment of the original calculus proved.

In the present work, a labelled calculus $\mathrm{G3LT}_{cl}$ is defined, the rules of which are justified by a closure algorithm that exploits the fixed-point properties of temporal operators, as proposed for example in [5]. All the rules of the system $\mathrm{G3LT}_{cl}$ are finitary, however, proofs generally require infinite descent in the sense of [3]. Admissibility of cut for $\mathrm{G3LT}_{cl}$ is not established syntactically but as a consequence of completeness. This is unproblematic, because the calculus is conceived as an instrument for establishing decidability of Priorean linear time.

Decidability is proved through a terminating proof search procedure: If a sequent is not a theorem of Priorean linear time logic, then root-first application of the rules of $\mathrm{G3LT}_{cl}$ leads, by the use of labels, to another sequent that supplies an immediate and simple construction of a countermodel. If, on the other hand, we start with a derivable sequent, a finite bound allows to truncate any potentially infinite branch. This establishes at the same time a direct proof of completeness with respect to Kripke semantics. The definition of proofs in $\mathrm{G3LT}_{cl}$ is completely local, and termination is determined with no need of checking previous parts of the derivation because every sequent keeps all the information required.

The calculus $\mathrm{G3LT}_{cl}$ contains also past temporal operators and the decision procedure is given in the strong form of an explicit bound on proof search, although the absence of a global condition on derivations imposes an exponential size on it. We remark, however, that the main purpose of this paper is not to establish decidability, but to illustrate a very general method through its uniform application to linear temporal logic. Although a proof-theoretic approach is followed, the use of labels permits to formalize model-theoretic arguments and to obtain direct proofs of validity and completeness.

The paper is organized as follows: In Section 2 we give the definition of the fixed-point proof system $\mathrm{G3LT}_{cl}$; In Section 3 we identify the proofs in the system; We prove soundness in Section 4 and completeness in Section 5; Decidability through termination of proof search is established in Section 6. For background, and the treatment of Until and Since that have not been included here, we refer to the first author's Ph.D. thesis [1]. For a concise illustration of the general method employed in this work, including the system G3LT, see [2].

2 A fixed-point proof system

The presence of induction constitutes an intrinsic obstacle to the possibility of establishing decidability of Priorean linear time logic through a terminating proof-search procedure. In this paper, we present a labelled calculus $\mathrm{G3LT}_{cl}$ for Priorean linear time. All the rules are finitary, but proofs generally require arguments by infinite descent in the sense of [3]. In a temporal frame for Priorean linear time, between any two points there are only finitely many other points, therefore any model that appeals to an infinite increasing or decreasing sequence of points between two instants can be ignored. The proof-theoretic counterpart of that occurs, for example, when root-first applications of the rules do never realize a future formula $x : \mathbf{F}B$ in the antecedent with a labelled formula $y : B$ and a finite chain $x \prec y_0, \ldots, y_n \prec y$.

A particular class of sequents, which correspond to the syntactic counterparts of countermodels for unprovable purely logical sequents (defined below), is identified and used for giving a sound and complete definition of proofs in G3LT$_{cl}$. Termination of proof search is then obtained thanks to the analogy of the rules of the calculus to the algorithm that produces saturated subsets of formulas.

The basic idea is to formulate a labelled calculus G3LT$_{cl}$ from the fixed-point properties of temporal operators:

$$\mathbf{G}A \supset\subset \mathbf{T}A \,\&\, \mathbf{TG}A \qquad \mathbf{F}A \supset\subset \mathbf{T}A \vee \mathbf{TF}A$$
$$\mathbf{H}A \supset\subset \mathbf{Y}A \,\&\, \mathbf{YH}A \qquad \mathbf{P}A \supset\subset \mathbf{Y}A \vee \mathbf{YP}A$$

In a standard proof of decidability for LTL, as given for example in [13], [14], [5], a countermodel for an invalid sentence is constructed as a relational structure where a saturated set of closure formulas Δ is the immediate successor of a saturated set of closure formulas Γ if $A \in \Delta$ whenever $\mathbf{T}A \in \Gamma$, and a fairness condition is satisfied, namely that all the eventualities of the form $\mathbf{F}A$ are fulfilled at some point. Here the notion of ($\prec$-)saturated label (see Definitions 11, 12) will be defined in order to identify the class of sequents which correspond to countermodels for invalid sequents.

In initial sequents, ϕ is either an atomic formula or a formula prefixed by $\mathbf{T}$ or $\mathbf{Y}$. The propositional rules are identical to those of G3LT in [2]. Repetition of the principal formula in the premisses of $R\mathbf{G}_{cl}$, $L\mathbf{F}_{cl}$, $R\mathbf{H}_{cl}$ and $L\mathbf{P}_{cl}$ is required for the definition of fulfilling sequent (see Definition 19). If the flow of time is linear and unbounded, the next-time operator $\mathbf{T}$ and the previous-time operator $\mathbf{Y}$ satisfy the following, where $x \prec y$ means that x is the immediate predecessor of y (or y the immediate successor of x):

$x \Vdash \mathbf{T}A$ *iff for all* y, $x \prec y$ *implies* $y \Vdash A$, *iff for some* y, $x \prec y$ *and* $y \Vdash A$
$x \Vdash \mathbf{Y}A$ *iff for all* y, $y \prec x$ *implies* $y \Vdash A$, *iff for some* y, $y \prec x$ *and* $y \Vdash A$

In analogy with the rules for the quantifiers, the universal semantic explanation for $\mathbf{T}$ and $\mathbf{Y}$ would give a variable condition in the right rule, whereas the existential semantic explanation would give a variable condition in the left rule. Because of linearity, both explanations are available, and the rules for $\mathbf{T}$ and $\mathbf{Y}$ can conveniently be formulated in the form given in Table 1, which uses the universal semantic explanation for the left rule and the existential explanation for the right one. Thus, no variable condition is required.

The calculus G3LT$_{cl}$ does not permit syntactic cut elimination. This is because the rules for $\mathbf{T}$ and $\mathbf{Y}$ are given in a non-harmonious way, that is, the left and the right rules are justified by different semantical explanations. However, it is precisely because of this particular choice of rules that the essential properties of G3LT$_{cl}$ hold. Also, we will show that the system without cut is complete, and thus prove that G3LT$_{cl}$ is closed with respect to cut.

The notion of derivability in the calculus G3LT$_{cl}$ is defined in the standard way: A derivation is an initial sequent, or an instance of $L\bot$, or is obtained by an application of a logical or mathematical rule to the derivation(s) concluding its premiss(es). In Section 3 we shall introduce a generalized notion of provability in G3LT$_{cl}$, which admits derivation trees with infinite branches.

Table 1. The rules of the calculus G3LT$_{cl}$

Initial sequents and $L\bot$

$$x:\phi,\Gamma\Rightarrow\Delta,x:\phi \qquad\qquad \frac{}{x:\bot,\Gamma\Rightarrow\Delta}\,{\scriptstyle L\bot}$$

Fixed-point rules

$$\frac{x:\mathbf{T}A,x:\mathbf{TG}A,\Gamma\Rightarrow\Delta}{x:\mathbf{G}A,\Gamma\Rightarrow\Delta}\,{\scriptstyle L\mathbf{G}_{cl}} \qquad \frac{\Gamma\Rightarrow\Delta,x:\mathbf{G}A,x:\mathbf{T}A \quad \Gamma\Rightarrow\Delta,x:\mathbf{G}A,x:\mathbf{TG}A}{\Gamma\Rightarrow\Delta,x:\mathbf{G}A}\,{\scriptstyle R\mathbf{G}_{cl}}$$

$$\frac{x:\mathbf{T}A,x:\mathbf{F}A,\Gamma\Rightarrow\Delta \quad x:\mathbf{TF}A,x:\mathbf{F}A,\Gamma\Rightarrow\Delta}{x:\mathbf{F}A,\Gamma\Rightarrow\Delta}\,{\scriptstyle L\mathbf{F}_{cl}} \qquad \frac{\Gamma\Rightarrow\Delta,x:\mathbf{T}A,x:\mathbf{TF}A}{\Gamma\Rightarrow\Delta,x:\mathbf{F}A}\,{\scriptstyle R\mathbf{F}_{cl}}$$

$$\frac{x:\mathbf{Y}A,x:\mathbf{YH}A,\Gamma\Rightarrow\Delta}{x:\mathbf{H}A,\Gamma\Rightarrow\Delta}\,{\scriptstyle L\mathbf{H}_{cl}} \qquad \frac{\Gamma\Rightarrow\Delta,x:\mathbf{H}A,x:\mathbf{Y}A \quad \Gamma\Rightarrow\Delta,x:\mathbf{H}A,x:\mathbf{YH}A}{\Gamma\Rightarrow\Delta,x:\mathbf{H}A}\,{\scriptstyle R\mathbf{H}_{cl}}$$

$$\frac{x:\mathbf{Y}A,x:\mathbf{P}A,\Gamma\Rightarrow\Delta \quad x:\mathbf{YP}A,x:\mathbf{P}A,\Gamma\Rightarrow\Delta}{x:\mathbf{P}A,\Gamma\Rightarrow\Delta}\,{\scriptstyle L\mathbf{P}_{cl}} \qquad \frac{\Gamma\Rightarrow\Delta,x:\mathbf{Y}A,x:\mathbf{YP}A}{\Gamma\Rightarrow\Delta,x:\mathbf{P}A}\,{\scriptstyle R\mathbf{P}_{cl}}$$

Tomorrow and Yesterday rules

$$\frac{x\prec y,y:A,x:\mathbf{T}A,\Gamma\Rightarrow\Delta}{x\prec y,x:\mathbf{T}A,\Gamma\Rightarrow\Delta}\,{\scriptstyle L\mathbf{T}} \qquad \frac{x\prec y,\Gamma\Rightarrow\Delta,x:\mathbf{T}A,y:A}{x\prec y,\Gamma\Rightarrow\Delta,x:\mathbf{T}A}\,{\scriptstyle R\mathbf{T}_{cl}}$$

$$\frac{y\prec x,y:A,x:\mathbf{Y}A,\Gamma\Rightarrow\Delta}{y\prec x,x:\mathbf{Y}A,\Gamma\Rightarrow\Delta}\,{\scriptstyle L\mathbf{Y}} \qquad \frac{y\prec x,\Gamma\Rightarrow\Delta,x:\mathbf{Y}A,y:A}{y\prec x,\Gamma\Rightarrow\Delta,x:\mathbf{Y}A}\,{\scriptstyle R\mathbf{Y}_{cl}}$$

Mathematical rules:

$$\frac{y\prec x,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta}\,\textit{L-Ser} \qquad \frac{x\prec y,\Gamma\Rightarrow\Delta}{\Gamma\Rightarrow\Delta}\,\textit{R-Ser}$$

Rules *L-Ser* and *R-Ser* have the condition that y is not in the conclusion.

A rule is height-preserving admissible if, whenever its premiss(es) is (are) derivable, also its conclusion is derivable with the same bound on the derivation height; A rule is height-preserving invertible if, whenever its conclusion is derivable, also its premiss(es) is (are) derivable with the same bound on the derivation height. The proofs of the following structural results are detailed in [1].

Proposition 1. *Substitution of labels is height-preserving admissible in* G3LT$_{cl}$*. All the rules of* G3LT$_{cl}$ *are height-preserving invertible. Weakening and contraction are height-preserving admissible in* G3LT$_{cl}$*.*

Definition 2. *In an instance of rule* R-Ser *(resp.* L-Ser*) with active formula $x\prec y$ (resp. $y\prec x$), the label x is called* side label.

Lemma 3. *A derivation in* G3LT$_{cl}$ *can be transformed into a derivation with all instances of* R-Ser *and* L-Ser *applied on side labels that appear in the conclusion of the rule.*

Root-first proof search can, without loss of generality, be restricted to *minimal derivations*, that is, derivations which cannot be shortened through height-preserving admissibility of contraction or other local modifications: In particular, applications of rules that produce duplications of atoms when read from conclusion to premisses can be dispensed with by height-preserving admissibility of contraction. The same holds if a duplication occurs modulo fresh replacement of eigenvariables, so we have:

Lemma 4. *In a minimal derivation in* G3LT$_{cl}$*, rule* R-Ser *(resp.* L-Ser*) need not be applied on a relational atom $x \prec y$ (resp. $y \prec x$) if its conclusion contains an atom $x \prec z$ (resp. $z \prec x$) in the antecedent.*

Lemma 5. *The rules* L-Ser *and* R-Ser *permute up with respect to all the rules of* G3LT$_{cl}$ *in case their eigenvariable is not contained in the active formula(s) of the latter.*

Lemma 6. *On any branch of a minimal derivation in* G3LT$_{cl}$*, a given temporal rule with the repetition of the principal formula(s) in the premiss(es) need not be applied more than once on the same formulas.*

A *purely logical sequent* is a sequent that contains no relational atoms and in which every formula is labelled by the same variable. Every purely logical sequent $\Gamma \Rightarrow \Delta$ with all its formulas labelled by x corresponds to a characteristic formula $\wedge\Gamma^x \supset \vee\Delta^x$, where $\Gamma^x = \{A \mid x : A \in \Gamma\}$, and similarly Δ^x. With this identification, the rules of the system G3LT$_{cl}$, read root first, correspond to the algorithm for producing the saturated subsets of closure formulas from a given formula.

Definition 7. *The set $cl(A)$ of* closure formulas *of a formula A is defined inductively as follows:*

- *$B \in cl(A)$ for every subformula B of A;*
- *$\mathbf{T}B$ and $\mathbf{TG}B \in cl(A)$ if $\mathbf{G}B \in cl(A)$;*
- *$\mathbf{T}B$ and $\mathbf{TF}B \in cl(A)$ if $\mathbf{F}B \in cl(A)$;*
- *$\mathbf{Y}B$ and $\mathbf{YH}B \in cl(A)$ if $\mathbf{H}B \in cl(A)$;*
- *$\mathbf{Y}B$ and $\mathbf{YP}B \in cl(A)$ if $\mathbf{H}B \in cl(A)$.*

Lemma 8. *Let $|A|$ be the number of subformulas of A. The cardinality of $cl(A)$ is linearly bounded by $|A|$, namely $|cl(A)| \leq 3 \cdot |A|$.*

Proof. By induction on the length of A.

Corollary 9. *The number of subsets of $cl(A)$ is at most $2^{3|A|}$.*

The definition of a satured set of formulas is an extension of the classical definition, obtained for the temporal modalities from their fixed-point properties.

Definition 10. *A set S of formulas is* saturated *if the following conditions are satisfied:*

- $\bot$ *is not in* S*;*
- *For every formula* B*, it is not possible that both* B *and* $\neg B$ *are in* S*;*
- $\neg\neg B$ *in* S *implies that* B *is in* S*;*
- $B \& C$ *in* S *implies that both* B *and* C *are in* S*;*
- $\neg(B \& C)$ *in* S *implies that either* $\neg B$ *or* $\neg C$ *is in* S*;*
- $B \vee C$ *in* S *implies that* B *or* C *is in* S*;*
- $\neg(B \vee C)$ *in* S *implies both* $\neg B$ *and* $\neg C$ *are in* S*;*
- $B \supset C$ *in* S *implies that either* $\neg B$ *or* C *is in* S*;*
- $\neg(B \supset C)$ *in* S *implies that both* B *and* $\neg C$ *are in* S*;*
- $\mathbf{G}B$ *in* S *implies that both* $\mathbf{T}B$ *and* $\mathbf{TG}B$ *are in* S*;*
- $\neg\mathbf{G}B$ *in* S *implies that either* $\neg\mathbf{T}B$ *or* $\neg\mathbf{TG}B$ *is in* S*;*
- $\mathbf{F}B$ *in* S *implies that* $\mathbf{T}B$ *or* $\mathbf{TF}B$ *is in* S*;*
- $\neg\mathbf{F}B$ *in* S *implies that both* $\neg\mathbf{T}B$ *and* $\neg\mathbf{TF}B$ *are in* S*;*
- $\mathbf{H}B$ *in* S *implies that both* $\mathbf{Y}B$ *and* $\mathbf{YH}B$ *are in* S*;*
- $\neg\mathbf{H}B$ *in* S *implies that either* $\neg\mathbf{Y}B$ *or* $\neg\mathbf{YH}B$ *is in* S*;*
- $\mathbf{P}B$ *in* S *implies that* $\mathbf{Y}B$ *or* $\mathbf{YP}B$ *is in* S*;*
- $\neg\mathbf{P}B$ *in* S *implies that both* $\neg\mathbf{Y}B$ *and* $\neg\mathbf{YP}B$ *are in* S*.*

The notions of saturated and $\prec$-saturated label in a sequent are then given as follows:

Definition 11. *A label* x *in* $\Gamma \Rightarrow \Delta$ *is* saturated *if the set* $\Gamma^x \cup \overline{\Delta^x}$ *is saturated, where* $\Gamma^x\{B \mid x : B \in \Gamma\}$, $\overline{\Delta^x} = \{\overline{B} \mid x : B \in \Delta\}$*, and* $\overline{B} \equiv \neg B$ *if* $B \neq \neg C$*,* $\overline{B} \equiv C$ *otherwise.*

Definition 12. *A label* x *in* $\Gamma \Rightarrow \Delta$ *is* $\prec$-saturated *if it is saturated and:*

- $x : \mathbf{T}B$ *in* $\Gamma(\Delta)$ *implies that, if* $x \prec y$ *is in* Γ*, then* $y : B$ *is in* $\Gamma(\Delta)$*;*
- $x : \mathbf{Y}B$ *in* $\Gamma(\Delta)$ *implies that, if* $y \prec x$ *is in* Γ*, then* $y : B$ *is in* $\Gamma(\Delta)$*.*

3 Proofs in G3LT$_{cl}$

In this Section we shall define the proofs in G3LT$_{cl}$ through the identification of a particular class of sequents, which can be considered finite syntactical counterparts of countermodels for invalid sequents.

Given a purely logical sequent $\Gamma \Rightarrow \Delta$, we start a proof search by applying root-first the rules of G3LT$_{cl}$ for the propositional connectives and for $\mathbf{G}$, $\mathbf{F}$, $\mathbf{H}$, and $\mathbf{P}$, whenever possible. When x becomes saturated, we apply once the rules *R-Ser* and *L-Ser* with side label x, thus introducing new labels y and y' and the accessibility relations $x \prec y$ and $y' \prec x$. By Lemma 5 we are allowed to postpone the application of the rules for seriality until no more logical rule can be applied, and by Lemmas 3 and 4 we do not need to apply a seriality rule with side label z, if z is not a label in the sequent or the antecedent already contains an atom $z \prec z'$ (resp. $z' \prec z$). Next, we apply the rules $L\mathbf{T}$ and $R\mathbf{T}_{cl}$ (resp. $L\mathbf{Y}$

and $R\mathbf{Y}_{cl}$) on the formulas with $\mathbf{T}$ (resp. $\mathbf{Y}$) as their outermost operator until x becomes $\prec$-saturated. Note that by Lemma 6, we need not apply more than once a temporal rule on the same principal formula(s). We repeat the procedure with the formulas marked by y and y'. We continue as before with all the labels possibly introduced by *R-Ser* and *L-Ser*, and so on. This procedure motivates the following definition:

Definition 13. *A* pre-proof *of a purely logical sequent in* G3LT$_{cl}$ *is a (possibly infinite) tree obtained by applying root-first the logical and mathematical rules of the calculus, whenever possible.*

Before giving the definition of a proof in G3LT$_{cl}$, we need some preliminary notions. We shall construct the syntactic counterpart of a countermodel from a failed proof search and therefore define syntactic entities through their correspondence to a Kripke model for Priorean linear time.

Definition 14. *A* discrete linear temporal frame $\mathcal{F} = (\mathcal{K}, \prec^{\mathcal{K}}, <^{\mathcal{K}})$ *is a linearly ordered set, with the order relation* $<^{\mathcal{K}}$ *defined as the transitive closure of the immediate successor relation* $\prec^{\mathcal{K}}$*, functional and unbounded in both directions.*

Definition 15. *Let* $\mathcal{F} = (\mathcal{K}, \prec^{\mathcal{K}}, <^{\mathcal{K}})$ *be a discrete linear temporal frame. An* evaluation *of atomic formulas in a frame is a map* $\mathcal{V} : AtFrm \to \mathcal{P}(\mathcal{K})$*, assigning to any atom* P *the set of instants in which* P *holds. The standard notation for* $k \in \mathcal{V}(P)$ *is* $k \Vdash P$*. Evaluations are extended to arbitrary formulas by the following inductive clauses:*

For all $k \in \mathcal{K}$*, it is not the case that* $k \Vdash \bot$ *(abbr.* $k \nVdash \bot$*);*
$k \Vdash A\&B$ *if* $k \Vdash A$ *and* $k \Vdash B$*;*
$k \Vdash A \vee B$ *if* $k \Vdash A$ *or* $k \Vdash B$*;*
$k \Vdash A \supset B$ *if* $k \Vdash A$ *implies* $k \Vdash B$*;*
$k \Vdash \mathbf{G}A$ *(resp.* $k \Vdash \mathbf{H}A$*) if for all* k'*,* $k <^{\mathcal{K}} k'$ *(resp.* $k' <^{\mathcal{K}} k$*) implies* $k' \Vdash A$*;*
$k \Vdash \mathbf{F}A$ *(resp.* $k \Vdash \mathbf{P}A$*) if for some* k'*,* $k <^{\mathcal{K}} k'$ *(resp.* $k' <^{\mathcal{K}} k$*) and* $k' \Vdash A$
$k \Vdash \mathbf{T}A$ *(resp.* $k \Vdash \mathbf{Y}A$*) if for all* k'*,* $k \prec^{\mathcal{K}} k'$ *(resp.* $k' \prec^{\mathcal{K}} k$*) implies* $k' \Vdash A$

The definition of evaluation of formulas justifies the notion of interpretation of the labels of a sequent and of validity for labelled formulas and relational atoms in a discrete linear temporal frame:

Definition 16. *Let* $\mathcal{F} = (\mathcal{K}, \prec^{\mathcal{K}}, <^{\mathcal{K}})$ *be a linear discrete frame with accessibility relations* $<^{\mathcal{K}}$ *and* $\prec^{\mathcal{K}}$*. Let* W *be the set of labels used in the derivation of the sequent* $\Gamma \Rightarrow \Delta$ *in* G3LT$_{cl}$*. An* interpretation *of the labels from* W *in* $\mathcal{K}$ *is a function* $[\![\cdot]\!] : W \to \mathcal{K}$*. A* countermodel *to* $\Gamma \Rightarrow \Delta$ *is a discrete linear temporal frame* $(\mathcal{K}, \prec^{\mathcal{K}}, <^{\mathcal{K}})$ *together with an interpretation and an evaluation that validates all the formulas and relational atoms in* Γ *and no formula in* Δ*; Namely, for all labelled formulas* $z : A$ *and relational atoms* $x \prec y$ *in the antecedent,* $[\![z]\!] \Vdash A$ *and* $[\![x]\!] \prec^{\mathcal{K}} [\![y]\!]$ *but for no* $w : B$ *in the succedent* $[\![w]\!] \Vdash B$

The semantic explanations for the possibility-like temporal operators **F**, **P** and the definition of the order relation $<^{\mathcal{K}}$ as the transitive closure of the immediate successor relation $\prec^{\mathcal{K}}$ justify the following notion of future and past witness. We use the standard symbol for syntactic identity "$x \equiv y$" to denote that x and y are the same syntactic object.

Definition 17. *Given a labelled formula $z : \mathbf{F}B$ in the antecedent of a sequent $\Gamma \Rightarrow \Delta$ (resp. $z : \mathbf{G}B$ in the succedent), we say that a label z' is a* future witness *for $z : \mathbf{F}B$ (resp. $z : \mathbf{G}B$) if $z' : B$ is in Γ (resp. $z' : B$ is in Δ) and the relational atoms $z \prec z_0, \ldots, z_{n-1} \prec z_n \equiv z'$ are in Γ for some n.*
Given a labelled formula $z : \mathbf{P}B$ in the antecedent of a sequent $\Gamma \Rightarrow \Delta$ (resp. $z : \mathbf{H}B$ in the succedent), we say that a label z' is a past witness *for $z : \mathbf{P}B$ (resp. $z : \mathbf{H}B$) if $z' : B$ is in Γ (resp. $z' : B$ is in Δ) and the relational atoms $z' \prec z_0, \ldots, z_{n-1} \prec z_n \equiv z$ are in Γ for some n.*

In the syntactic object that corresponds to a Priorean linear time model, we have to ensure that every possibility-like formulas is realized by some label:

Definition 18. *A chain $z_i \prec z_{i+1}, \ldots, z_{j-1} \prec z_j$ (with $j \geq i+1$) in a sequent $\Gamma \Rightarrow \Delta$ is a* future loop *if z_j marks exactly the same formulas as the label z_i and, for every labelled formula $z_q : \mathbf{F}B$ in Γ (resp. $z_q : \mathbf{G}B$ in Δ) with $i \leq q \leq j$, there exists z_k such that $i \leq k \leq j$ and $z_k : B$ is in Γ (resp. in Δ). We call z_j the* future looping label *with respect to z_i.*
A chain $z_i \prec z_{i+1}, \ldots, z_{j-1} \prec z_j$ (with $j \geq i+1$) in a sequent $\Gamma \Rightarrow \Delta$ is a past loop *if z_i marks exactly the same formulas as the label z_j and, for every labelled formula $z_q : \mathbf{P}B$ in Γ (resp. $z_q : \mathbf{H}B$ in Δ) with $i \leq q \leq j$, there exists some variable z_k such that $i \leq k \leq j$ and $z_k : B$ is in Γ (resp. in Δ). We call z_i the* past looping label *with respect to z_j.*

A root-first proof search succeeds when a derivation is found, namely all the leaves of the derivation tree are initial sequents or instances of $L\bot$. However, a failed proof search does not in general ensure that an endsequent $\Gamma \Rightarrow \Delta$ is invalid unless a countermodel can be constructed from it. Here comes into play the notion of fulfilling sequent for a purely logical sequent $\Gamma \Rightarrow \Delta$:

Definition 19. *Let the sequent $\Gamma^* \Rightarrow \Delta^*$ be obtained by root-first proof search from the purely logical sequent $\Gamma \Rightarrow \Delta$ (with all its formulas labelled by x). Then, $\Gamma^* \Rightarrow \Delta^*$ is a* fulfilling sequent *if the following conditions are satisfied:*

- *(i) Every label in it is $\prec$-saturated;*
- *(ii) It contains a chain of relational atoms $z_{-m} \prec z_{-(m-1)}, \ldots, z_{-1} \prec z_0 \equiv x$, $z_0 \prec z_1, \ldots, z_{n-1} \prec z_n$, such that for some i with $-m < i \leq 0$ the subchain $z_{-m} \prec z_{-(m-1)}, \ldots, z_{i-1} \prec z_i$ is a past loop, and for some j with $0 \leq j < n$, the subchain $z_j \prec z_{j+1}, \ldots, z_{n-1} \prec z_n$ is a future loop;*
- *(iii) Every labelled formula $z : \mathbf{F}B$ in Γ^* (resp. $z : \mathbf{G}B$ in Δ^*) is either witnessed by a future witness label z', or has z inside a future loop;*
- *(iv) Every labelled formula $z : \mathbf{P}B$ in Γ^* (resp. $z : \mathbf{H}B$ in Δ^*) is either witnessed by a past witness label z', or has z inside a past loop.*

Intuitively, a fulfilling sequent corresponds to a structure constituted by a (possibly empty) linear chain with two simple loops at the ends, with the left and the right loop obtained by identifying the first and the last label of the past and of the future loop, respectively.

In Section 4 we shall prove that, given a model for Priorean linear time, it is possible to extract the corresponding fulfilling sequent, and in Section 5 we shall show how to linearize the future and the past loop in order to obtain an appropriate model.

Proposition 20. *Let $\Gamma' \Rightarrow \Delta'$ be obtained by applying root-first the rules of* G3LT$_{cl}$ *from the purely logical sequent $\Gamma \Rightarrow \Delta$ with x as the uniform label that marks all the formulas in the latter. Then $\Gamma' \Rightarrow \Delta'$ contains a unique chain $z_{-m} \prec z_{-(m-1)}, \ldots, z_{-1} \prec z_0 \equiv x$, $z_0 \prec z_1, \ldots, z_{n-1} \prec z_n$ with z_i different from z_j for $i \neq j$.*

Proof. Since the root sequent $\Gamma \Rightarrow \Delta$ is purely logical, the result follows by Lemmas 3, 4 and the fact that only seriality rules can introduce relational atoms.

While searching for a fulfilling sequent, we want to find one as small as possible. Therefore we should try to avoid useless circles, namely those exploring instants reachable as well through a more direct path. This motivates the following definition:

Definition 21. *Let $\Gamma' \Rightarrow \Delta'$ be obtained by applying root-first the rules of* G3LT$_{cl}$ *from the purely logical sequent $\Gamma \Rightarrow \Delta$ with x as the uniform label that marks all the formulas in the latter. A chain $y_0 \prec y_1, \ldots, y_{n-1} \prec y_n$ (resp. $y_{-n} \prec y_{-(n-1)}, \ldots, y_{-1} \prec y_0$) with $y_0 \equiv x$ in $\Gamma' \Rightarrow \Delta'$ is* roundabout *if it contains labels y_i, y_j with $0 \leq i < j \leq n$ such that y_i and y_j mark the same formulas, $y_i \prec y_{i+1}, \ldots, y_{j-1} \prec y_j$ is not the future loop (resp. the past loop) and either $j = i + 1$ or for every y_k with $i < k < j$ there exists some y_l such that $l > j$ (resp. $l < i$) and y_k and y_l mark the same formulas. We say that the subchain $y_i \prec y_{i+1}, \ldots, y_{j-1} \prec y_j$ is* dispensable. *A fulfilling sequent is* reduced *if it does not contain dispensable subchains.*

Note that by Definition 21 a chain can be roundabout also in the case that y_i and y_j mark no formulas.

Theorem 22. *If a proof search for a purely logical sequent $\Gamma \Rightarrow \Delta$ (with all its formulas labelled by x) leads to a fulfilling sequent $\Gamma^* \Rightarrow \Delta^*$, then it also leads to a reduced fulfilling sequent.*

Proof. (*Sketch*) Note that for every label z introduced by *R-Ser* (resp. *L-Ser*) a labelled formula $z : C$ in $\Gamma^* \Rightarrow \Delta^*$ either is introduced by applying root-first the rules $L\mathbf{T}$ and $R\mathbf{T}_{cl}$ (resp. $L\mathbf{Y}$ and $R\mathbf{Y}_{cl}$) or is the result of root-first application of the other rules on a formula introduced in the former way. If the chain $z_0 \prec z_1, \ldots, z_{n-1} \prec z_n$ with $x \equiv z_0$ contains a dispensable subchain $z_i \prec z_{i+1}, \ldots, z_{j-1} \prec z_j$, then the labels z_i and z_j mark the same formulas; Therefore $z_{j+1} : B$ is introduced by $L\mathbf{T}$ (resp. $R\mathbf{T}_{cl}$) with principal formulas

$z_j \prec z_{j+1}, z_j : \mathbf{T}B$ iff $z_{i+1} : B$ can be introduced by $L\mathbf{T}$ (resp. $R\mathbf{T}_{cl}$) with principal formulas $z_i \prec z_{i+1}, z_i : \mathbf{T}B$. Given a set of formulas marked by a label z, the rules of G3LT$_{cl}$ explore different subsets of closure formulas that possibly $\prec$-saturate z: While applying root-first the rules of G3LT$_{cl}$ we have to continue along the branch in which the label z_{i+1} is $\prec$-saturated by the same subset of closure formulas that $\prec$-saturates z_{j+1} in the original fulfilling sequent. By choosing the appropriate premiss of a branching rule whenever a roundabout chain is met, we finally reach the desired reduced fulfilling sequent.

Definition 23. *A pre-proof of a purely logical sequent is a* proof *if no branch in it leads to a fulfilling sequent. A sequent is* provable *if there is a proof for it.*

Every G3LT$_{cl}$ derivation is a G3LT$_{cl}$ proof, but the converse does not hold. Observe that, contrary to the definition of proof in cyclic calculi for induction and infinite descent of [3], our definition in G3LT$_{cl}$ is completely local, i.e. there is no need of checking previous parts of the tree: At any step of the proof search we simply have to consider the sequents introduced by root-first application of the rules and check if they are initial sequents, fulfilling sequents, or neither.

4 Soundness

Soundness for G3LT$_{cl}$ cannot be proved simply by showing that the initial sequents and the rules of the system are sound because, by Definition 23, proofs in G3LT$_{cl}$ can contain infinitely long branches. Therefore, we prove soundness by contraposition: If there exists a countermodel for $\Gamma \Rightarrow \Delta$, then the corresponding proof search should contain a fulfilling sequent and so $\Gamma \Rightarrow \Delta$ is unprovable in G3LT$_{cl}$. Thus, the absence of a fulfilling sequent in a derivation tree is a global soundness condition for a proof.

Some preliminary results concerning standard models are needed: We have to prove that, given a countermodel $\mathcal{M}$ for A, it is possible to extract a fulfilling sequent all the labels of which mark $\prec$-saturated sets of closure formulas of A. The lemmas below show how to construct a future and a past loop from $\mathcal{M}$. In the following, we write $s \leqslant^{\mathcal{K}} s'$ if $s = s'$ or $s <^{\mathcal{K}} s'$ in a model $\mathcal{M} = (\mathcal{K}, \prec^{\mathcal{K}}, <^{\mathcal{K}}, \Vdash)$.

Lemma 24. *Let $\mathcal{M} = (\mathcal{K}, \prec^{\mathcal{K}}, <^{\mathcal{K}}, \Vdash)$ be a model for Priorean linear time and suppose that, for some instant w, $w \nVdash A$. Then for some s such that $w \leqslant^{\mathcal{K}} s$, there exists s' such that $s <^{\mathcal{K}} s'$, s and s' satisfy the same subset $H \subseteq cl(A)$, and for every t if $s \leqslant^{\mathcal{K}} t \leqslant^{\mathcal{K}} s'$ and $t \Vdash \mathbf{F}B$ and $\mathbf{F}B \in cl(A)$ (resp. $t \nVdash \mathbf{G}B$ and $\mathbf{G}B \in cl(A)$) there exists u such that $s \leqslant^{\mathcal{K}} u \leqslant^{\mathcal{K}} s'$ and $u \Vdash B$ (resp. $u \nVdash B$).*

Proof. Since every model for Priorean linear time is isomorphic to the integers, there are infinitely many instants greater than w. However, by Corollary 9, there are only $2^{3|A|}$ subsets of $cl(A)$. By an application of Ramsey's Theorem, for some instant(s) greater than w there exist infinitely many instants satisfying the same subset H of closure formulas of A. Let s be the first instant of the infinite set of instants $s_0 <^{\mathcal{K}} s_1 <^{\mathcal{K}} s_2 <^{\mathcal{K}} s_3 <^{\mathcal{K}} \dots$ all satisfying the same subset $H \subseteq cl(A)$

and such that $w \leqslant^{\mathcal{K}} s$. Let $s \leqslant^{\mathcal{K}} t$ and $t \Vdash \mathbf{F}B$ and $\mathbf{F}B \in cl(A)$ (resp. $t \nVdash \mathbf{G}B$ and $\mathbf{G}B \in cl(A)$). If there exists a u such that $u \Vdash B$ and $s \leqslant^{\mathcal{K}} u \leqslant^{\mathcal{K}} t$, we are done. Otherwise, since $t \Vdash \mathbf{F}B$ (resp. $t \nVdash \mathbf{G}B$), there exists some u such that $t <^{\mathcal{K}} u$ and $u \Vdash B$ (resp. $u \nVdash B$). Since, by hypothesis, there are infinitely many instants greater than s satisfying H, but u can be reached from t by finitely many iterations of the relation $\prec^{\mathcal{K}}$, for some $i = 1, 2, \ldots$, we have $s <^{\mathcal{K}} u \leqslant^{\mathcal{K}} s_i$. For every i there are only finitely many closure formulas of A of the form $\mathbf{F}B$ (resp. $\mathbf{G}B$) validated (resp. invalidated) by an instant t such that $s \leqslant^{\mathcal{K}} t \leqslant^{\mathcal{K}} s_i$, and for every such t we can find a k and a u such that $s \leqslant^{\mathcal{K}} u \leqslant^{\mathcal{K}} s_{i+k}$ and $u \Vdash B$ (resp. $u \nVdash B$). Since the set of closure formulas of A is finite, the process eventually ends with the determination of a s' such that $s <^{\mathcal{K}} s'$ and for every t if $s \leqslant^{\mathcal{K}} t \leqslant^{\mathcal{K}} s'$ and $t \Vdash \mathbf{F}B$ and $\mathbf{F}B \in cl(A)$ (resp. $t \nVdash \mathbf{G}B$ and $\mathbf{G}B \in cl(A)$) there exists u such that $s \leqslant^{\mathcal{K}} u \leqslant^{\mathcal{K}} s'$ and $u \Vdash B$ (resp. $u \nVdash B$).

Lemma 25. *Let $\mathcal{M} = (\mathcal{K}, \prec^{\mathcal{K}}, <^{\mathcal{K}}, \Vdash)$ be a model for Priorean linear time such that for some instant w, $w \nVdash A$. Then for some instant s such that $s \leqslant^{\mathcal{K}} w$, there exists s' such that $s' <^{\mathcal{K}} s$, s and s' satisfy the same subset $H \subset cl(A)$ and for every t if $s' \leqslant^{\mathcal{K}} t \leqslant^{\mathcal{K}} s$ and $t \Vdash \mathbf{P}B$ and $\mathbf{P}B \in cl(A)$ (resp. $t \nVdash \mathbf{H}B$ and $\mathbf{H}B \in cl(A)$) there exists u such that $s' \leqslant^{\mathcal{K}} u \leqslant^{\mathcal{K}} s$ and $u \Vdash B$ (resp. $u \nVdash B$).*

Proof. Analogous to the proof of Lemma 24.

Lemma 26. *All the rules of* $\mathrm{G3LT}_{cl}$ *are sound.*

Proof. The case of the initial sequents and the propositional rules is straightforward. The rules for $\mathbf{G}$, $\mathbf{F}$, $\mathbf{H}$ and $\mathbf{P}$ are sound by definition, since they are justified by their fixed-point interpretations. Similarly, the rules for $\mathbf{T}$ and $\mathbf{Y}$ are justified by their semantic explanations, and the mathematical rules correspond to the frame properties of left and right seriality for $\prec$.

Theorem 27. *If a purely logical sequent $\Gamma \Rightarrow \Delta$ (with all its formulas labelled by x) has a countermodel, then it is not provable in* $\mathrm{G3LT}_{cl}$.

Proof. Let us suppose that there exists a countermodel $\mathcal{M} = (\mathcal{K}, \prec^{\mathcal{K}}, <^{\mathcal{K}}, \Vdash)$ for the purely logical sequent $\Gamma \Rightarrow \Delta$, namely there exists $w \in \mathcal{K}$ such that $[\![x]\!] = w$ and $w \nVdash \wedge\Gamma^x \supset \vee\Delta^x$. By Lemma 26, every countermodel for the conclusion of any of the rules of $\mathrm{G3LT}_{cl}$ is a countermodel for (at least one of) the premiss(es). By choosing the appropriate branch we eventually find a sequent with a chain

$$z_{-m} \prec z_{-(m-1)}, \ldots, z_{-1} \prec z_0 \equiv x,\ z_0 \prec z_1, \ldots, z_{n-1} \prec z_n$$

every label of which matches an instant in the corresponding position in $\mathcal{M}$. To show that this sequent is a fulfilling sequent for $\Gamma \Rightarrow \Delta$, we have to check that the conditions of Definition 19 are satisfied:

(i) By induction on the length of formulas, it is easy to prove that evey label z appearing in the tree can be $\prec$-saturated by applying the rules root-first;

(ii) The presence of a future and a past loop follows from Lemmas 24 and 25, and the fact that we can go on applying right and left seriality rules and introduce new labels until the conditions of the lemmas are satisfied;

(iii) If the formula $z : \mathbf{F}B$ (resp. $z : \mathbf{G}B$) is in the antecedent (resp. succedent), then $[\![z]\!] \Vdash \mathbf{F}B$ (resp. $[\![z]\!] \nVdash \mathbf{G}B$). Therefore, either there exists an instant s such that $[\![z]\!] <^{\mathcal{K}} s$ and $s \Vdash B$ (resp. $s \nVdash B$), and for some z', $[\![z']\!] = s$ and z' is the future witness of $z : \mathbf{F}B$ (resp. $z : \mathbf{G}B$), or $[\![z]\!]$ falls under the conditions of Lemma 24, and thus z is inside a future loop;

(iv) If the formula formula $z : \mathbf{P}B$ (resp. $z : \mathbf{H}B$) is in the antecedent (resp. succedent), then $[\![z]\!] \Vdash \mathbf{P}B$ (resp. $[\![z]\!] \nVdash \mathbf{H}B$). Therefore, either there exists an instant s such that $s <^{\mathcal{K}} [\![z]\!]$ and $s \Vdash B$ (resp. $s \nVdash B$), and for some z', $[\![z']\!] = s$ and z' is the past witness of $z : \mathbf{P}B$ (resp. $z : \mathbf{H}B$), or $[\![z]\!]$ falls under the conditions of Lemma 25, and so z is inside a past loop.

5 Completeness

Completeness is also proved by contraposition: If $\Gamma \Rightarrow \Delta$ is not provable in G3LT$_{cl}$, i.e. if the root-first proof search leads to a fulfilling sequent, then a countermodel for $\Gamma \Rightarrow \Delta$ can be constructed. Our completeness result follows the method in [10]. However, the definition of fulfilling sequents allows to consider only finite objects, and not (possibly) infinite reduction tree; Furthermore, the presence of the fixed-point rules for the temporal operators requires additional work in proving the inductive steps for temporal formulas, since we cannot appeal directly to the semantic explanations for the corresponding operators.

Let us consider the standard frame $\mathcal{F} = (\mathcal{K}, \prec^{\mathcal{K}}, <^{\mathcal{K}})$ for Priorean linear time, with $\mathcal{K} = \{s_i \mid i \in \mathbb{Z}\}$, $s_i \prec^{\mathcal{K}} s_{i+1}$ and $s_i <^{\mathcal{K}} s_j$ for $i < j$. Given a fulfilling sequent $\Gamma^* \Rightarrow \Delta^*$ for the purely logical sequent $\Gamma \Rightarrow \Delta$, we construct a countermodel $\mathcal{M}$ by defining an appropriate interpretation for the set of labels in $\Gamma^* \Rightarrow \Delta^*$ into the domain $\mathcal{K}$ as follows: We put $[\![x]\!] = s_0$ if x is the label that marks all the formulas in $\Gamma \Rightarrow \Delta$, and for every label z if the relational atoms $x \equiv z_0 \prec z_1, \dots, z_{n-1} \prec z_n \equiv z$ are in Γ, we put $[\![z]\!] = s_n$. Analogously, if $z \equiv z_{-n} \prec z_{-(n-1)}, \dots, z_{-1} \prec z_0 \equiv x$ are in Γ, we put $[\![z]\!] = s_{-n}$. We evaluate the atomic formulas by putting $[\![z]\!] \Vdash P$ if $z : P$ is in Γ^* and $[\![z]\!] \nVdash P$ if $z : P$ is in Δ^*. Furthermore, if z_{n+l} is the future looping label with respect to z_n, $[\![z_{n+l}]\!] = s_{n+l}$ and $[\![z_n]\!] = s_n$, then for every instant $s_{n+m\cdot l+q}$ (with $m \geq 0$ and $0 \leq q \leq l-1$) we put $s_{n+m\cdot l+q} \Vdash P$ if $z_{n+q} : P$ is in Γ^* and $s_{n+m\cdot l+q} \nVdash P$ if $z_{n+q} : P$ is in Δ^*. Analogously, if $z_{-(n+l)}$ is the past looping label with respect to z_{-n}, $[\![z_{-(n+l)}]\!] = s_{-(n+l)}$ and $[\![z_{-n}]\!] = s_{-n}$, then for every instant $s_{-(n+m\cdot l+q)}$ (with $m \geq 0$ and $0 \leq q \leq l-1$) we put $s_{-(n+m\cdot l+q)} \Vdash P$ if $z_{-(n+q)} : P$ is in Γ^* and $s_{-(n+m\cdot l+q)} \nVdash P$ if $z_{-(n+q)} : P$ is in Δ^*.

Lemma 28. *$\mathcal{M}$ is a countermodel for $\Gamma^* \Rightarrow \Delta^*$.*

Proof. By definition, if $z \prec z'$ is in Γ^*, then $[\![z]\!] \prec^{\mathcal{K}} [\![z']\!]$. We have to show that, for arbitrary formulas B, if $z : B$ is in Γ^*, then $[\![z]\!] \Vdash B$, and if $z : B$ is in Δ^*, then $[\![z]\!] \nVdash B$. We proceed by induction on the length of the formula B. If B is an atomic formula P and $z : P$ is in Γ^*, then $[\![z]\!] \Vdash P$ by construction. If $z : P$ is in Δ^*, then $[\![z]\!] \nVdash P$ by construction. Since z is $\prec$-saturated, $z : P$ cannot be both in Γ^* and in Δ^*. If $B \equiv \bot$, then it cannot be in Γ^* by definition of fulfilling sequent. If $z : \bot$ is in Δ^*, then $[\![z]\!] \nVdash \bot$ by Definition 15. The case of propositional connectives is straightforward. We consider in detail only the cases of $B \equiv \mathbf{T}C$ and $B \equiv \mathbf{G}C$, all the other cases being analogous.

If $B \equiv \mathbf{T}C$ and $z : \mathbf{T}C$ is in Γ^* (resp. Δ^*), then we have two cases: (i) If the label z is not the future looping label z_f, then it is connected to it by a chain $z \equiv z_{n+l-i} \prec z_{n+l-(i-1)}, \ldots, z_{n+l-1} \prec z_{n+l} \equiv z_f$ and, since the label z_{n+l-i} is $\prec$-saturated, we have $z_{n+l-(i-1)} : C$ in Γ^* (resp. Δ^*). Therefore, by construction, we have $[\![z_{n+l-i}]\!] \prec^{\mathcal{K}} [\![z_{n+l-(i-1)}]\!]$ and by inductive hypothesis $[\![z_{n+l-(i-1)}]\!] \Vdash C$ (resp. $[\![z_{n+l-(i-1)}]\!] \nVdash C$). So $[\![z]\!] \Vdash \mathbf{T}C$ (resp. $[\![z]\!] \nVdash \mathbf{T}C$). (ii) If z is the future looping label, then by definition for no label z' the atom $z \prec z'$ is in Γ^*. However, we have some label z_n such that $x \equiv z_0 \prec z_1, \ldots, z_{n-1} \prec z_n, z_n \prec z_{n+1}, \ldots, z_{n+l-1} \prec z_{n+l} \equiv z$ are in Γ^* for $l > 0$ and z_n marks the same formulas as z; In particular $z_n : \mathbf{T}C$ is in Γ^* (resp. Δ^*). Since z_n is $\prec$-saturated, $z_{n+1} : C$ is in Γ^* (resp. Δ^*). By construction $[\![z]\!] = s_{n+l}$, so $[\![z]\!] \prec^{\mathcal{K}} s_{n+l+1}$ and, by construction and inductive hypothesis, $s_{n+l+1} \Vdash C$ (resp. $s_{n+l+1} \nVdash C$). Therefore $[\![z_{n+l}]\!] \Vdash \mathbf{T}C$ (resp. $[\![z_{n+l}]\!] \nVdash \mathbf{T}C$).

If $B \equiv \mathbf{G}C$ and $z : \mathbf{G}C$ is in Γ^*, then, since z is $\prec$-saturated, both $z : \mathbf{T}C$ and $z : \mathbf{TG}C$ are in Γ^*, and, if the label $z \prec z'$ is in Γ^*, both $z' : C$ and $z' : \mathbf{G}C$ are in Γ^*. Therefore, by repeating this argument, we have that for every z'', if $z \prec z_i, \ldots, z_{i+j-1} \prec z_{i+j} \equiv z''$ are in Γ^* for some $i, j \geq 0$, then $z'' : C$ and $z'' : \mathbf{G}C$ are in Γ^*. Note that, if z is the future looping label or z'' is inside a future loop $z_m \prec z_{m+1}, \ldots, z_{n-1} \prec z_n$ (with $n > m$) both $z_k : C$ and $z_k : \mathbf{G}C$ are in Γ^* for every $m \leq k \leq n$. By inductive hypothesis, for every s, if $[\![z]\!] <^{\mathcal{K}} s$ then $s \Vdash C$, therefore $[\![z]\!] \Vdash \mathbf{G}C$.

If $z : \mathbf{G}C$ is in Δ^* then, by Definitions 18 and 19, we have two cases: (i) There exists some future witness label z' such that $z' : C$ is in Δ^* and the atoms $z \prec z_i, \ldots, z_{i+j-1} \prec z_{i+j} \equiv z'$ are in Γ^* for some $i, j \geq 0$. So, by construction and inductive hypothesis there is some $s = [\![z']\!]$ such that $[\![z]\!] <^{\mathcal{K}} s$ and $s \nVdash C$, so $[\![z]\!] \nVdash \mathbf{G}C$. (ii) z is inside a future loop $z_n \prec z_{n+1}, \ldots, z_{n+i-1} \prec z_{n+i} \equiv z$, $z_{n+i} \prec z_{n+i+1}, \ldots, z_{n+l-1} \prec z_{n+l}$ (with $l \geq i$). Then there exists some label z' such that either $z_n \equiv z'$ or the atoms $z_n \prec z_{n+1}, \ldots, z_{n+q-1} \prec z_{n+q} \equiv z'$ are in Γ^* for $0 \leq q \leq i$ and the formula $z' : C$ is in Δ^*. By construction $[\![z']\!] = s_{n+q}$, so $[\![z]\!] <^{\mathcal{K}} s_{n+l+q}$ and, by inductive hypothesis, $s_{n+l+q} \nVdash C$. Therefore $[\![z]\!] \nVdash \mathbf{G}C$.

By the following result, every countermodel for the fulfilling sequent $\Gamma^* \Rightarrow \Delta^*$ is a countermodel for the corresponding endsequent $\Gamma \Rightarrow \Delta$:

Lemma 29. *All the rules of* $\mathrm{G3LT}_{cl}$ *preserve countermodels, that is, a countermodel for (at least one of) the premisses is a countermodel for the conclusion.*

Proof. Immediate for the rules for **T** and **Y** and for the rules of seriality. For the propositional rules, by definition of validity for the propositional connectives. For the rules for **G**, **F**, **H** and **P**, by their fixed-point interpretation.

Theorem 30. *If the purely logical sequent* $\Gamma \Rightarrow \Delta$ *has no countermodels, then it is provable in* $\mathrm{G3LT}_{cl}$.

Corollary 31. *Provability of purely logical sequents in* $\mathrm{G3LT}_{cl}$ *is closed with respect to cut.*

Proof. By soundness of the cut rule and completeness of $\mathrm{G3LT}_{cl}$.

6 Termination of proof search

In root-first application of the rules of $\mathrm{G3LT}_{cl}$, two possibilities arise: (i) The proof search terminates because we find a fulfilling sequent or because every branch leads to an initial sequent or an instance of $L\bot$; (ii) The proof search does not terminate and, by König's Lemma, there is at least one infinite branch.

However, we can truncate a potentially infinite proof search as shown below. By Theorem 22, if $\Gamma \Rightarrow \Delta$ is not provable, then the proof search leads to a reduced fulfilling sequent. Whenever a branch leads to a sequent with a roundabout chain, we can drop that branch and start a new one: If every branch in the proof search for $\Gamma \Rightarrow \Delta$ leads to either an initial sequent or a sequent with a roundabout chain, then $\Gamma \Rightarrow \Delta$ is provable in $\mathrm{G3LT}_{cl}$.

Lemma 32. *Suppose that the proof search for a purely logical sequent* $\Gamma \Rightarrow \Delta$*, with all the formulas labelled by* x*, leads to a sequent* $\Gamma' \Rightarrow \Delta'$*: If the chain* $y_{-m} \prec y_{-(m-1)}, \ldots, y_{-1} \prec y_0 \equiv x$ *and the chain* $x \equiv y_0 \prec y_1, \ldots, y_{n-1} \prec y_n$ *are not roundabout then the number of labels has an exponential bound on the order of the length of* $A \equiv \wedge\Gamma^x \supset \vee\Delta^x$*, namely* $m, n \leq \sum_{i=1}^{2^{3|A|}} i$*.*

Proof. (*Sketch*) We recall here that the rules of $\mathrm{G3LT}_{cl}$ reflect the closure algorithm that from a formula A gives the set of its closure formulas and, by Corollary 9, the number of subsets of closure formulas of A is at most $2^{3|A|}$. Let us consider the longest case of a non-roundabout chain $y_0 \prec y_1, \ldots, y_{n-1} \prec y_n$ such that for every k with $0 \leq k \leq n$, y_k labels a subset of closure formulas of A. It contains a first subchain $y_0 \prec y_1, \ldots, y_{i-2} \prec y_{i-1}$ such that $i = 2^{3|A|}$ and every subset of closure formulas of A is labelled by some y_k, for $0 \leq k \leq i-1$. Then we have a second subchain $y_i \prec y_{i+1}, \ldots, y_{i+j-2} \prec y_{i+j-1}$, such that $j = 2^{3|A|}-1$ and every subset of closure formulas of A except one is marked by y_k for $i \leq k \leq i+j-1$. Thus, the subchain in the $l+1$st position contains $j = 2^{3|A|} - l$ labels, that mark the same subsets of *cl(A)* marked by the members of the chain in the lth position, except one. Summing up the numbers of the members of each subchain, we finally obtain that $n = \sum_{i=1}^{2^{3|A|}} i$. The same argument applies to the chain $y_{-m} \prec y_{-(m-1)}, \ldots, y_{-1} \prec y_0$, therefore $m = \sum_{i=1}^{2^{3|A|}} i$.

Theorem 33. *Proof search for* $\mathrm{G3LT}_{cl}$ *terminates.*

Proof. Let us suppose that the proof search for the purely logical sequent $\Gamma \Rightarrow \Delta$ (with all its formulas labelled by x) does not terminate. Since every rule of $\mathrm{G3LT}_{cl}$ has a finite number of premisses, any derivation tree is finitely branching, so by König's Lemma there is at least one infinite branch. Obviously it cannot lead to an initial sequent, nor to a conclusion of L$\bot$, nor to a fulfilling sequent, because otherwise it would be finite. We have to show that it contains a sequent with a roundabout chain. Note that the endsequent contains a finite number of formulas: The logical rules for connectives and for temporal operators can introduce only a finite number of new formulas, and by Lemma 6 temporal rules cannot be applied more than once with the same principal formula(s). Furthermore, by Lemmas 3 and 4 we need not apply a seriality rule with side label z, if z is not a label in the sequent or the antecedent already contains an atom $z \prec z'$ (resp. $z' \prec z$). Consequently, an infinite branch should contain a sequent with an infinite $\prec$-chain. However, by Lemma 32 if a chain is not roundabout, then it is finite and exponentially bounded on the order of the length of the formula corresponding to the endsequent $\Gamma \Rightarrow \Delta$. Therefore, any potentially infinite branch can be truncated as soon as a sequent is met that contains a chain $z_{-m} \prec z_{-(m-1)}, \ldots, z_{-1} \prec z_0 \equiv x, z_0 \prec z_1, \ldots, z_{n-1} \prec z_n$ with $m > \sum_{i=1}^{2^{3|\wedge \Gamma^x \supset \vee \Delta^x|}} i$ or $n > \sum_{i=1}^{2^{3|\wedge \Gamma^x \supset \vee \Delta^x|}} i$.

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