Proof analysis beyond geometric theories: from rule systems to systems of rules

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Abstract

A class of axiomatic theories with arbitrary quantifier alternations is identified and a conversion to normal form is provided in terms of *generalized geometric implications*. The class is also characterized in terms of Glivenko classes as those first-order formulas that do not contain implications or universal quantifiers in the negative part. It is shown how the methods of proof analysis can be extended to cover such axioms by means of conversion to *systems of rules*. The structural properties for the resulting extensions of sequent calculus are established and a generalization of the first-order Barr theorem is shown to follow as an immediate application. The method is also applied to obtain complete labelled proof systems for logics defined through their relational semantics. In particular, the method provides analytic proof systems for all the modal logics in the Sahlqvist fragment.

1 Introduction

The central goal of the *method of proof analysis* is the design of appropriate proof systems that go beyond pure logic. What the adjective 'appropriate' exactly denotes can be often identified with the property of being *analytic*, or, more generally, with the property of preserving the structural features of basic logical calculi and the consequences of analyticity. When the goal is achieved, one can extract crucial information from the analysis of proofs in a formal inference system for a given theory, in the same way as in pure logic. In particular, the proof systems can be used not only as calculi for finding formal derivations, but also for showing the underivability of certain statements and thus obtaining proofs of independence; further on, they can be used for establishing conservativity of extensions, faithfulness of embeddings between logical systems, and in general for results that involve the simultaneous analysis of derivability in different theories.

The starting point of the investigation is given by an analytic proof system for pure logic, taken as the basis of the extension procedure. Natural deduction is often the privileged proof system because it is the one that most closely resembles the human way of making logical inferences. On the other hand, such naturality works to perfection only with intuitionistic logic, whereas sequent calculus has a much greater flexibility. Not only can sequent calculi be defined in an elegant way also for classical—and sometimes non-classical—logics, but usually they come with properties that allow their direct use as systems of automated deduction; the latter is the case for the remarkable class of *G3 sequent calculi.*

When sequent calculi are used for the analysis of mathematical theories, a first limitation is encountered: if the theories are formulated as axioms, or equivalently as axiomatic sequents, full elimination of cuts is lost. What one can obtain is a *generalized Hauptsatz* that reduces all cuts in derivations to cuts on axioms, so analyticity is not preserved. If instead axioms appear as assumptions in the antecedent of sequents, full cut elimination is maintained, but analyticity in root-first proof search fails because there is usually an infinity of instances of universal axioms to choose from.

In earlier work, we have shown how the problem can be overcome by converting axioms into rules of inference. The method allows full cut elimination for all classical theories with universal axioms and for intutionistic theories with universal axioms without implications in the negative part, that is, axioms equivalent to conjunctions with conjuncts of the form $\forall x(P_1 \& \dots \& P_n \supset Q_1 \vee \dots \vee Q_m)$, where the P_i and the Q_j are atomic formulas, and with a degenerate form where n or m can be zero and thus replaced by $\top$ or $\bot$. The idea is to convert axioms into rules of sequent calculus in such a way that *the logical content of the axiom is replaced by the meta-linguistic meaning of sequent rules.* Such rules can be formulated so that the active and principal formulas occur only on the left or only on the right part of sequents. If the left rule scheme paradigm is chosen, conjunction on the left is replaced by the commas in the antecedents of sequents in the conclusion and disjunction on the right by a plurality of premisses in the rule, each with one of the disjuncts as in the rule:

$$\frac{Q_1, \Gamma \to \Delta \quad \dots \quad Q_m, \Gamma \to \Delta}{P_1, \dots, P_n, \Gamma, \to \Delta} RL$$

The sequent $\to \forall x(P_1 \& \dots \& P_n \supset Q_1 \vee \dots \vee Q_m)$ is clearly derivable by the rule.

If the right rule scheme is instead chosen, we have a plurality of premisses each with one of the conjuncts in the succedent, and a conclusion with commas in the succedent to replace the disjunction:

$$\frac{\Gamma \to \Delta, P_1 \quad \dots \quad \Gamma \to \Delta, P_n}{\Gamma, \to \Delta, Q_1, \dots, Q_m} RR$$

Again, the sequent $\to \forall x(P_1 \& \dots \& P_n \supset Q_1 \vee \dots \vee Q_m)$ is derivable by the rule.

The universal quantifier does not occur explicitly in the conversion to either the left or the right rule schemes because the rules have an inherently universal interpretation. In this way, the added rules do not interfere with the conversion step of the procedure of cut elimination. Also other structural properties such as the height-preserving admissibility of contraction are preserved with an appropriate formulation of the rules. All the details of the procedure and several examples and applications are found in Negri and von Plato (1998 and 2001, chapter 6).

Later, we have shown that the axiom-as-rules method can be extended to *geometric theories* (Negri 2003). Geometric theories are theories with $\forall\exists$-axioms of the form $\forall x_1 \dots \forall x_n(A \supset B)$ where A and B are geometric formulas, that is, formulas that contain no $\supset$ nor $\forall$. Such axioms, known as geometric implications, can be equivalently written as conjunctions of universal closures of formulas of the form

$$P_1 \& \dots \& P_m \supset \exists y_{1_1} \dots \exists y_{1_k} M_1 \vee \dots \vee \exists y_{n_1} \dots \exists y_{n_l} M_n$$

Here the P_i are atoms, the M_j conjunctions of atoms, and the variables y do not occur in the P_i. Geometric implications are transformed into rules by the same guidelines as for universal axioms. In addition, the existential quantifier disappears in the conversion through a condition on the existential variables: in left rules, we have a variable condition stating that these variables should not appear anywhere else in the rule (cf. Section 3 below). It is then seen that all the structural properties of the basic sequent calculi are maintained by the addition of rules that arise from geometric implications.

A first striking application of the method of conversion of geometric axioms into rules was a trivialization of the Barr theorem, a central result of constructive mathematics by which a classical proof of a geometric implication in a geometric theory can be transformed into a constructive proof. It turned out that in a cut-free system of sequent calculus with any collection of geometric rules, a classical proof of a geometric implication is already a constructive proof (cf. Negri 2003 and Negri and von Plato 2011). Although the term 'geometric' for these axiomatizations does not originate from geometry but from category theory, geometric theories and their proof-theoretic treatment through the geometric rule scheme have been employed for a formalization of Euclidean geometry in Avigad et al. (2009) and for projective and affine geometry in Negri and von Plato (ch. 10, 2011).

A second striking application of geometric axioms stems from the fact that they cover the defining properties, formulated in terms of relational semantics, of the most common systems of modal logics. As such, they have been particularly useful for the proof-theoretical investigation of modal logic for *labelled* systems of deduction. They have thus been used to obtain analytic deductive systems for modal logic, through systems of natural deduction in Simpson (1994) and systems of sequent calculus in Negri (2005).

A dual of the class of geometric theories is given by the class of *co-geometric theories*. These are theories axiomatized by formulas of the form (called co-geometric implications) $\forall x_1 \dots \forall x_n (A \supset B)$ where A and B are co-geometric formulas, that is, formulas that contain no $\supset$ nor $\exists$. Co-geometric implications can be written in the following equivalent way: They are conjunctions of universal closures of formulas of the form

$$\forall z_1 \dots \forall z_n (\forall x_1 M_1 \& \dots \& \forall x_n M_n \supset P_1 \vee \dots \vee P_m)$$

Here the M_j are conjunctions of atoms. Variable conditions are dual to those for the geometric implications and consequently the corresponding rule is a right rule. The duality between rules of geometric and of co-geometric theories is useful to transfer proof-theoretic results from a geometric to a co-geometric theory. Such transfer occurs when the basic notions of a theory are replaced by their duals, as in the passage from axiomatizations based on equality to axiomatizations based on apartness (cf. Negri and von Plato 2005, 2011 ch. 9).

The way the logical constants get eliminated in the axiom-to-rules conversion can be summarized as the table that follows: In the first column we have the logical constant with its polarity in the axiom, in the second the corresponding meta-linguistic connector in left rules and in the third the one in the right rule. The first three lines give the correspondence for universal axioms, the fourth and the fifth for geometric and co-geometric axioms, respectively, the sixth has been open so far, and the last, the property of Noetherianity stating that every chain of accessible elements is eventually stationary, has been obtained through the use of labels (Negri 2005, Dyckhoff and Negri 2013):

Logical constant/property	**Left rule**	**Right rule**
negative &	,	branching
positive $\vee$	branching	,
positive $\supset$	split in conclusion/premiss	split in premiss/conclusion
positive $\exists$ (geometric axiom)	variable condition	—
negative $\forall$ (cogeometric axiom)	—	variable condition
quantifier alternations beyond $\forall\exists$	?	?
Noetherianity	labels	labels

The applicability of the method of proof analysis to logics characterized by a relational semantics has brought a wealth of applications to the proof theory of non-classican logics, including provability logic (Negri 2005), substructural logic (Negri 2008), intermediate logics (Dyckhoff and Negri 2012), conditional logics (Olivetti et al. 2007), description logics (de Paiva et al. 2011), systems for collective intentionality (Hakli and Negri 2011), and dynamic logics such as the logic of public announcement (Negri and Maffezioli 2010) and the epistemic logic of programs (Maffezioli and Naibo 2013). In all these applications, the geometric rule scheme suffices for the extra mathematical rules; however, for provability logics, the characterizing condition is not first order and the property of Noetherianity is absorbed into the calculus by means of a suitable modification of the rules for the modality.

Recently, we investigated the principles of the verificationist theory of truth by the methods of proof analysis and studied how the ground logic, either classical or intuitionistic, affects their liability to paradoxes. The study, focused on the well known Church-Fitch paradox, brought forward a new challenge to the method of conversion of axioms into rules. The *knowability principle*, which states that whatever is true can be known, is rendered in a standard multi-modal alethic/epistemic language by the axiom $A \supset \Diamond\mathcal{K}A$. This axiom corresponds, in turn, to the frame property $\forall x \exists y(xRy \,\&\, \forall z(yR_{\mathcal{K}}z \supset x \leqslant z))$ where R, $R_{\mathcal{K}}$, and $\leqslant$ are the alethic, epistemic, and intuitionistic accessibility relations respectively. This frame property goes beyond the scheme of geometric implication and therefore the conversion into rules cannot be carried through with the usual rule scheme for geometric implications. In this specific case, we succeeded with a combination of two rules that make up a *system of rules* linked together by the requirement of appearing in a certain order in the derivation and by a side condition on the eigenvariable. The resulting calculus enjoyed all the structural properties of the ground logical system and led to definite answers to the questions raised by the Church-Fitch paradox by means of a complete control over the structure of derivations for knowability logic (cf. Maffezioli, Naibo, and Negri 2012).

The generalization and systematization of the method of system of rules is the main task in this article. After some preliminaries, we shall provide in Section 3 an extension of the axiomatic class of geometric theories, the class of *generalized geometric implications*. We shall prove through an operative conversion to normal form that they can also be characterized in terms of Glivenko classes as those first-order formulas that do not contain implications or universal quantifiers in the negative part. As an intermediate step to giving the system of rules that correspond to generalized geometric implications, we shall demonstrate the conversion procedure on a simple example from a first-order relational axiomatization of lattices. The system of rules for generalized geometric implications will then be defined and the equivalence with the axioms proved. In Section 4 we shall establish the structural properties for the extensions with systems of rules, admissibility of cut, weakening, and contraction. As an immediate application, we shall provide in Section 5 an extension of Barr's theorem. In Section 6 we shall connect the extension of the axiom-as-rules method to the labelled proof theory of non-classical logic and provide a proof of completeness of the proof systems obtained by enriching the ground labelled calculus **G3K** with systems of rules for the frame properties. We shall then use the characterization in terms of Glivenko classes to prove that generalized geometric implications contain the class of Kracht's formulas, i.e. the class of frame correspondents of Sahlqvist formulas. It follows that the corresponding systems of rules provide analytic proof systems for all the modal logics in the Sahlqvist fragment.

2 Preliminaries

We refer to Negri and von Plato (1998, 2001) for the necessary background on sequent calculus and its extension with nonlogical (alias mathematical) rules. The sequent calculus we shall be using here is the contraction- and cut-free sequent calculus **G3**. We list below the rules for its classical version **G3c** and the modifications for obtaining its intuitionistic version **G3im**. Here the letter "m" stands for multi-succedent.

G3c

Initial sequents:

$P, \Gamma \to \Delta, P$

Logical rules:

$$\frac{A, B, \Gamma \to \Delta}{A \& B, \Gamma \to \Delta}\, L\& \qquad \frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \& B}\, R\&$$

$$\frac{A, \Gamma \to \Delta \quad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}\, L\vee \qquad \frac{\Gamma \to \Delta, A, B}{\Gamma \to \Delta, A \vee B}\, R\vee$$

$$\frac{\Gamma \to \Delta, A \quad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}\, L\supset \qquad \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}\, R\supset$$

$$\frac{}{\bot, \Gamma \to \Delta}\, L\bot$$

$$\frac{A(t/x), \forall x A, \Gamma \to \Delta}{\forall x A, \Gamma \to \Delta}\, L\forall \qquad \frac{\Gamma \to \Delta, A(y/x)}{\Gamma \to \Delta, \forall x A}\, R\forall$$

$$\frac{A(y/x), \Gamma \to \Delta}{\exists x A, \Gamma \to \Delta}\, L\exists \qquad \frac{\Gamma \to \Delta, \exists x A, A(t/x)}{\Gamma \to \Delta, \exists x A}\, R\exists$$

G3im

$$\frac{A \supset B, \Gamma \to A \quad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}\, L\supset \qquad \frac{A, \Gamma \to B}{\Gamma \to \Delta, A \supset B}\, R\supset$$

$$\frac{\Gamma \to A(y/x)}{\Gamma \to \Delta, \forall x A}\, R\forall$$

In the initial sequents, P is any atomic formula. Greek upper case Γ, Δ stand for multisets of formulas. The restriction in $R\forall$ is that y must not occur free in Γ, Δ, $\forall x A$ (Γ, $\forall x A$ for **G3im**). The restriction in $L\exists$ is that y must not occur free in $\exists x A, \Gamma, \Delta$. We may summarize these conditions by the requirement that y must not occur free in the conclusion of the two rules.

All the structural rules (weakening, contraction and cut) are admissible in **G3c** and in **G3im**. The calculi are thus complete for classical and intuitionistic first order logic, respectively.

3 Generalized geometric theories as systems of rules

We recall that a formula in the language of first-order logic is called *geometric* if it does not contain $\supset$ or $\forall$. A *geometric implication* is a formula of the form

$$\forall \overline{x}(A \supset B)$$

where A and B are geometric formulas. A *geometric theory* is a theory axiomatized by geometric implications.

It is known (cf., e.g., Simpson 1994 or Palmgren 2002) that any geometric implication can be reduced to a conjunction of formulas of the form:

$$\forall \overline{x}(\& P_i \supset \exists \overline{y}_1 M_1 \vee \ldots \vee \exists \overline{y}_n M_n) \qquad GA$$

Here the P_i range over a finite set of atomic formulas and all the M_j are conjunctions of atomic formulas and the variables in the vector $\overline{y}_j$ are not free in the P_i. We shall call a formula of type *GA* a *geometric axiom.* Let M_j be $Q_{j_1} \& \ldots \& Q_{j_{k_j}}$ where Q_{j_l} are atomic formulas. With a slight abuse of notation, we shall use the vector notation for multisets of formulas instead of lists, and write $\overline{P}$ for the multiset $P_1, \ldots, P_k$ and $\overline{Q}_j$ for $Q_{j_1}, \ldots, Q_{j_{k_j}}$. We shall also sometime for better readability omit the vector notation for lists of variables and denote by $\overline{Q}_j(z_j/y_j)$ the result of the substitution of the variables $\overline{y}_j$ by $\overline{z}_j$ in each of the Q_{j_l}, that is, $Q_{j_1}(z_j/y_j), \ldots, Q_{j_{k_j}}(z_j/y_j)$.

The rule scheme that corresponds to the geometric axiom *GA* is

$$\frac{\overline{Q}_1(z_1/y_1), \overline{P}, \Gamma \to \Delta \quad \ldots \quad \overline{Q}_k(z_k/y_k), \overline{P}, \Gamma \to \Delta}{\overline{P}, \Gamma \to \Delta}\ {\scriptstyle GRS}$$

Here the variables y_i are called the *replaced variables* of the scheme, and the (lists of) variables z_i the *proper variables* or *eigenvariables* of the scheme. The scheme has the condition that the proper variables are not free in $\overline{P}, \Gamma, \Delta$. We shall call a rule scheme of the above form a *geometric rule scheme*, *GRS* for short.

As explained in Negri and von Plato (1998), the *principal formulas* $P_1, \ldots, P_m$ of the scheme must be repeated in the antecedent of each premiss for proving the rule of contraction admissible. We recall also another condition, the *closure condition*, which has to be satisfied to obtain admissibility of contraction for extensions: It can happen that a substitution in the atoms of a rule produces duplications among the formulas P_i. Then in order to ensure that contraction is admissible in the system, we need to add the contracted rule, that is, we must make sure that the following condition is satisfied:

Closure condition: *Given a system with geometric rules, if it has a rule with an instance of form*

$$\frac{\overline{Q}_1(y_1/x_1), P_1, \ldots, P_{m-2}, P, P, \Gamma \Rightarrow \Delta \quad \ldots \quad \overline{Q}_n(y_n/x_n), P_1, \ldots, P_{m-2}, P, P, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_{m-2}, P, P, \Gamma \Rightarrow \Delta}$$

then also the rule

$$\frac{\overline{Q}_1(y_1/x_1), P_1, \ldots, P_{m-2}, P, \Gamma \Rightarrow \Delta \quad \ldots \quad \overline{Q}_n(y_n/x_n), P_1, \ldots, P_{m-2}, P, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_{m-2}, P, \Gamma \Rightarrow \Delta}$$

has to be included in the system.

The condition is unproblematic because the number of rules to be added to a given system of nonlogical rules is finite. The closure condition will always be assumed in the extension by rules.

The notion of geometric axiom will be taken as the base case in the inductive definition of a *generalized geometric axiom.* So we take GA_0 to be *GA* and GRS_0 to be *GRS.* We then define

$$GA_1 \equiv \forall \overline{x}(\, \& \, P_i \supset \exists y_1 \, \& \, GA_0 \vee \ldots \vee \exists y_m \, \& \, GA_0)$$

Next we define by induction

$$GA_{n+1} \equiv \forall \overline{x}(\,\&\, P_i \supset \exists y_1 \,\&\, GA_{k_1} \vee \ldots \vee \exists y_m \,\&\, GA_{k_m})$$

Here $\&\, GA_i$ denotes a conjunction of GA_i-axioms and $k_1, \ldots, k_m \leqslant n$.

The following holds:

Proposition 3.1. *Generalized geometric implications do not contain negative occurrences of implications nor of universal quantifiers.*

Proof: By induction on the generation of generalized geometric implications. Formulas of type GA_0 clearly do not contain negative occurrences of implications nor of universal quantifiers. For the inductive step, the conclusion follows from the observation that the components for which the inductive hypothesis holds are in the positive part of GA_{n+1} and the construction does not add any negative implication nor any negative universal quantifier. QED

The above lemma indeed gives a characterization of generalized geometric implications in terms of "Glivenko classes" (cf. Orevkov 1968), because we can prove the following

Theorem 3.2. *Any first-order formula A that does not contain negative occurrences of implications nor of universal quantifiers is intuitionistically (and even minimally) equivalent to a conjunction of generalized geometric implications and can be converted to equivalent generalized geometric rules.*

Proof: We shall give an operative procedure for proving the result, by a root-first decomposition of the given first-order formula in the sequent calculus **G3im**. The procedure will at the same time provide the canonical form of A as a conjunction of generalized geometric implications and a system of rules that corresponds to A. The latter part is postponed after the formal definition of the generalized geometric rule scheme in Subsection 3.2.

Start with the sequent $\rightarrow A$ as a root in a proof-search tree. Observe that because of the assumptions on A, the possible rules that are applicable are $L\&$, $R\&$, $L\vee$, $R\vee$, $R\supset$, $R\forall$, $L\exists$, $R\exists$. Among these, we consider the invertible ones. Rules $R\supset$ and $R\forall$ are classically invertible and they are invertible in **G3im** only for single-succedent conclusions, but the root-first proof search can, in the presence of disjunction and positive existentials, produce a multisuccedent sequent, so they are applied root-first only if there is an empty right context. Rule $R\exists$ is not even classically invertible (invertibility in the G3-calculi is obtained artificially through the repetition of the principal formula in its premiss). All the other rules, namely $L\&$, $R\&$, $L\vee$, $R\vee$, $L\exists$, are both classically and intuitionistically invertible.

After applying all the invertible steps we obtain the following decomposition:

$$\begin{array}{c} \{\overline{P_i} \rightarrow \exists y_{i,1} A_{i,1}, \ldots, \exists y_{i,n_i} A_{i,n_i}\}_{i=1,\ldots,k} \\ \vdots \\ \rightarrow A \end{array}$$

In the upper sequents, all the $\overline{P_i}$ are multisets of atomic formulas. By the invertibility of the rules, the search tree provides a derivation of the equivalence of A with the conjunction of the formulas $\&\overline{P_i} \supset \exists y_{i,1} A_{i,1} \vee \ldots \vee \exists y_{i,n_i} A_{i,n_i}$. The detailed proof is done along the lines of the proof of Theorem 3.1.4 (conjunctive normal form for propositional classical logic) in Negri and von Plato (2001). Formula A is a generalized geometric implication if each of the $A_{i,j}$ is.

The procedure is then repeated for each of the formulas $A_{i,j}$ by a root-first proof search with all the available invertible rules. Each of these has the same original restrictions on implications and universal quantifiers and is a proper subformula of A, thus the procedure terminates and the claim is proved by well-founded induction. QED

In Negri (2003), we have established an equivalence between axiomatic systems based on geometric axioms and contraction- and cut-free sequent systems with geometric rules. The equivalence can be extended by a suitable definition of *systems of rules* for generalized geometric axioms. Here the word "system" is used in the same sense as in linear algebra where there are systems of equations with variables in common, and each equation is meaningful and can be solved only if considered together with the other equations of the system. In the same way, the systems of rules considered here will consist of rules connected to each other by some variables and will in addition be subject to the condition of appearing in a certain order in a derivation.

3.1 From generalized geometric axioms to rules and back: An example.

Before proceeding to the definition of GRS_n, we consider a simple and well known example of an axiom that belongs to the first step of our extension beyond the geometric axioms, the class GA_1. Consider the axiom of *join semi-lattices*:

$$\forall xy \exists z((x \leqslant z \,\&\, y \leqslant z) \,\&\, \forall w(x \leqslant w \,\&\, y \leqslant w \supset z \leqslant w)) \qquad \textit{lub-A}$$

Observe that the axiom can be equivalently written in the form

$$\forall xy \exists z \forall w((x \leqslant z \,\&\, y \leqslant z) \,\&\, (x \leqslant w \,\&\, y \leqslant w \supset z \leqslant w))$$

One thus recognizes this as an axiom of the form GA_1, with the first antecedent of atomic formulas empty.

It will be clear, however, that it is convenient to push the quantifiers as deep as possible into the formula, so we consider the first form of the axiom for the conversion into a system of rules.

In the case of the axiom for join semilattices the system of rules consists of the following two rules of existence and uniqueness of the least upper bound:

$$\frac{x \leqslant z, y \leqslant z, \Gamma \to \Delta}{\Gamma \to \Delta}\,{\scriptstyle lub\text{-}E} \qquad \frac{z \leqslant w, x \leqslant w, y \leqslant w, \Gamma \to \Delta}{x \leqslant w, y \leqslant w, \Gamma \to \Delta}\,{\scriptstyle lub\text{-}U}$$

Rule *lub-E* has the condition that z is fresh (i.e., not in the conclusion of the rule), whereas rule *lub-U* has the condition that in a derivation it should always be applied above (but not necessarily immediately above) rule *lub-E* . This means that any derivation that uses rule *lub-U* must have a branch of the form:

$$\begin{array}{c} \dfrac{z \leqslant w, x \leqslant w, y \leqslant w, \Gamma' \to \Delta'}{x \leqslant w, y \leqslant w, \Gamma' \to \Delta'}\,{\scriptstyle lub\text{-}U} \\ \vdots \\ \mathcal{D} \\ \vdots \\ \dfrac{x \leqslant z, y \leqslant z, \Gamma \to \Delta}{\Gamma \to \Delta}\,{\scriptstyle lub\text{-}E} \end{array} \tag{1}$$

The condition is that z is not free in Γ, Δ.

Observe that a system of rules departs from the usual handling of variables used in the (left) regular or geometric rule scheme, where the variables are implicitly universal unless subject to a variable condition, which makes them existential. Here we have the standard convention of the geometric rule scheme for the lower rule (and specifically, x and y are universal, z existential), but in the upper one only the variable w that did not appear in the lower rule is implicitly universal.

We may use the notation familiar from linear algebra to indicate systems of rules. In the case of system of rules for join semilattices we have the system:

$$\begin{cases} lub\text{-}U \\ \\ lub\text{-}E \end{cases}$$

Here, unlike in linear algebra, the order of the lines in the system of rules matters, and the convention is that the order is the same as the order of occurrence in a derivation imposed by the side condition. Alternatively, we shall use the more compact notation $\langle lub\text{-}E, lub\text{-}U \rangle$ to indicate a system of rules, with the convention that the latter rule in the derivation is written first, with the order suggested by the order of quantifiers in the generalized geometric axiom.

As an illustration of the transition from a GA_1 axiom to a GRS_1 system of rules, we show that the system **G3c** (**G3im**) extended by axiom *lub-A* and the structural rules is equivalent to **G3c** (**G3im**) extended by the system of rules $\langle lub\text{-}E, lub\text{-}U \rangle$.

First, given the system of rules $\langle lub\text{-}E, lub\text{-}U \rangle$, axiom *lub-A* is derivable as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x \leqslant z, y \leqslant z \to x \leqslant z \quad x \leqslant z, y \leqslant z \to y \leqslant z}{x \leqslant z, y \leqslant z \to x \leqslant z \,\&\, y \leqslant z}\,{\scriptstyle R\&} \quad \dfrac{\dfrac{\dfrac{\dfrac{z \leqslant w, x \leqslant z, y \leqslant z, x \leqslant w, y \leqslant w \to z \leqslant w}{x \leqslant z, y \leqslant z, x \leqslant w, y \leqslant w \to z \leqslant w}\,{\scriptstyle lub\text{-}U}}{x \leqslant z, y \leqslant z, x \leqslant w \,\&\, y \leqslant w \to z \leqslant w}\,{\scriptstyle L\&}}{x \leqslant z, y \leqslant z \to x \leqslant w \,\&\, y \leqslant w \supset z \leqslant w}\,{\scriptstyle R\supset}}{x \leqslant z, y \leqslant z \to \forall w(x \leqslant w \,\&\, y \leqslant w \supset z \leqslant w)}\,{\scriptstyle R\forall}}{x \leqslant z, y \leqslant z \to x \leqslant z \,\&\, y \leqslant z \,\&\, \forall w(x \leqslant w \,\&\, y \leqslant w \supset z \leqslant w)}\,{\scriptstyle R\&}}{x \leqslant z, y \leqslant z \to \exists z(x \leqslant z \,\&\, y \leqslant z \,\&\, \forall w(x \leqslant w \,\&\, y \leqslant w \supset z \leqslant w))}\,{\scriptstyle R\exists}}{\to \exists z(x \leqslant z \,\&\, y \leqslant z \,\&\, \forall w(x \leqslant w \,\&\, y \leqslant w \supset z \leqslant w))}\,{\scriptstyle lub\text{-}E}}{\to \forall x \forall y \exists z(x \leqslant z \,\&\, y \leqslant z \,\&\, \forall w(x \leqslant w \,\&\, y \leqslant w \supset z \leqslant w))}\,{\scriptstyle R\forall}$$

Observe that the conditions of the system of rules are respected.

Conversely, we show that any derivation that uses the system of rules $\langle lub\text{-}E, lub\text{-}U \rangle$ can be transformed into one with axiom *lub-A* and cuts.

A derivation in the theory for join semi-lattices can use rule *lub-E* in isolation, whereas any use of *lub-U* subsumes a use of *lub-E* further down. The transformation has to be performed for both rules, *lub-E* subject to the eigenvariable condition, and *lub-U* subject to the condition of the system of rules. For the former, we assume a derivation that contains *lub-E* in isolation, i.e. contains the step

$$\dfrac{\overset{\vdots}{x \leqslant z, y \leqslant z, \Gamma \to \Delta}}{\Gamma \to \Delta}\,{\scriptstyle lub\text{-}E}$$

This is transformed as follows:

$$\dfrac{\dfrac{\dfrac{\to \forall xy \exists z(x \leqslant z \,\&\, y \leqslant z \,\&\, \forall w(x \leqslant w \,\&\, y \leqslant w \supset z \leqslant w))}{\to \exists z \forall w(x \leqslant z \,\&\, y \leqslant z \,\&\, (x \leqslant w \,\&\, y \leqslant w \supset z \leqslant w))}\,{\scriptstyle Der}}{\to \exists z(x \leqslant z \,\&\, y \leqslant z)}\,{\scriptstyle Der} \qquad \dfrac{\dfrac{\overset{\vdots}{x \leqslant z, y \leqslant z, \Gamma \to \Delta}}{x \leqslant z \,\&\, y \leqslant z, \Gamma \to \Delta}\,{\scriptstyle L\&}}{\exists z(x \leqslant z \,\&\, y \leqslant z), \Gamma \to \Delta}\,{\scriptstyle L\exists}}{\Gamma \to \Delta}\,{\scriptstyle Cut}$$

Here and below we have denoted with *Der* a sequence of omitted steps in the purely logical calculus. The transformation for *lub-U* used in conjunction with *lub-E* as in (1) above is as follows:

$$\dfrac{\dfrac{\to \forall xy \exists z(x \leqslant z \,\&\, y \leqslant z \,\&\, \forall w(x \leqslant w \,\&\, y \leqslant w \supset z \leqslant w))}{\to \exists z(x \leqslant z \,\&\, y \leqslant z \,\&\, \forall w(x \leqslant w \,\&\, y \leqslant w \supset z \leqslant w))}\,{\scriptstyle Der} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\dfrac{\dfrac{z \leqslant w, x \leqslant w, y \leqslant w, \Gamma' \to \Delta'}{x \leqslant w, y \leqslant w, x \leqslant w \,\&\, y \leqslant w \supset z \leqslant w, \Gamma' \to \Delta'}\,{\scriptstyle Der}}{x \leqslant w, y \leqslant w, \forall w(x \leqslant w \,\&\, y \leqslant w \supset z \leqslant w), \Gamma' \to \Delta'}\,{\scriptstyle L\forall} \\ \vdots \\ \mathcal{D}' \\ \vdots \\ x \leqslant z, y \leqslant z, \forall w(x \leqslant w \,\&\, y \leqslant w \supset z \leqslant w), \Gamma \to \Delta\end{matrix}}{x \leqslant z \,\&\, y \leqslant z, \forall w(x \leqslant w \,\&\, y \leqslant w \supset z \leqslant w), \Gamma \to \Delta}\,{\scriptstyle L\&}}{x \leqslant z \,\&\, y \leqslant z \,\&\, \forall w(x \leqslant w \,\&\, y \leqslant w \supset z \leqslant w), \Gamma \to \Delta}\,{\scriptstyle L\&}}{\exists z(x \leqslant z \,\&\, y \leqslant z \,\&\, \forall w(x \leqslant w \,\&\, y \leqslant w \supset z \leqslant w)), \Gamma \to \Delta}\,{\scriptstyle L\exists}}{\Gamma \to \Delta}\,{\scriptstyle Cut}$$

Here $\mathcal{D}'$ is obtained from $\mathcal{D}$ by weakening with $\forall w(x \leqslant w \,\&\, y \leqslant w \supset z \leqslant w)$.

3.2 The generalized geometric rule scheme.

We shall now proceed with the definition of the generalized geometric rule scheme for GA_n axioms for any n. The definition is by induction on n. The base case, namely the geometric rule scheme, has been given in Negri (2003) (see also Negri and von Plato 2011, ch. 8). The example above shows how the generalized geometric rule scheme is obtained in the special case of $n = 1$. We observe that in the example considered above there is an occurrence of the existentially quantified variables which is not in the scope of the inner universal quantifier. The occurrence of this variable allows to isolate the *E*-part of the system of rules from the *U*-part. This is not however generally the case, and we may have to deal with a situation in which the existential variable is in the scope of an inner universal quantifier, as in the axiom of terminal elements in a preorder:

$$\exists x \forall y(x \leqslant y \supset x = y) \qquad \textit{Term}$$

In order to isolate the introduction of the existential variable from the universal clause, we put the axiom into the equivalent form:

$$\exists x(x = x \,\&\, \forall y(x \leqslant y \supset x = y)) \qquad \textit{Term}'$$

The equivalence is granted by the fact that we assume to be working within a system of first-order logic with equality; the system can be presented as a cut-free system of sequent calculus as in Negri and von Plato (2001).

Next we proceed to the translation of the axiom into a system of rules as follows:

$$\cfrac{\cfrac{x = y, x \leqslant y, \Gamma' \to \Delta'}{x \leqslant y, \Gamma' \to \Delta'} \\ \vdots \\ x = x, \Gamma \to \Delta}{\Gamma \to \Delta}$$

The condition is that x does not appear free in Γ, Δ.

With this proviso, we are now ready to give the general rule scheme for axioms of the form GA_n for an arbitrary n.

We consider, to start with, the case of GA_1. The scheme GRS_1 is as follows:

$$\cfrac{\begin{array}{ccc} \Gamma'_1 \to \Delta'_1 & & \Gamma'_m \to \Delta'_m \\ \vdots & & \vdots \\ \mathcal{D}_0^1 & & \mathcal{D}_0^m \\ \vdots & & \vdots \\ \Gamma''_1 \to \Delta''_1 & & \Gamma''_m \to \Delta''_m \\ \vdots & & \vdots \\ \mathcal{D}^1 & & \mathcal{D}^m \\ \vdots & & \vdots \\ z_1 = z_1, \overline{P}, \Gamma \to \Delta & \ldots & z_m = z_m, \overline{P}, \Gamma \to \Delta \end{array}}{\overline{P}, \Gamma \to \Delta} \qquad (2)$$

Here z_i are eigenvariables in the last inference step, the derivations indicated by $\mathcal{D}_0^i$ use rules of the form $GRS_0(z_i)$ that correspond to the geometric axioms $GA_0(z_i/x_i)$ in addition to logical rules, and the $\mathcal{D}^i$ use only logical rules.

Observe that the scheme accounts for both of the cases we have considered in the above examples, the first in which the existential variable can be isolated as in *lub-A*, where $\overline{P}$ is $x \leqslant z, y \leqslant z$ and the $z_i = z_i$ are absent, and the second, such as in axiom *Term*, where there are no atoms $\overline{P}$ and the existential variable x is declared by the equality $x = x$ in the equivalent axiom *Term′* before conversion rules. Since a generalised geometric implication has several layers and in each one of the two situations may occur, the rule scheme has to be given in a form that accounts for both of them.

The scheme GRS_{n+1} is defined inductively, once the schemes GRS_{k_i} have been defined for $k_i \leqslant n$ and with the same conditions as above, as follows

$$
\begin{array}{c}
\begin{array}{ccc}
\Gamma_1' \to \Delta_1' & & \Gamma_m' \to \Delta_m' \\
\vdots & & \vdots \\
\mathcal{D}^1_{k_1} & & \mathcal{D}^m_{k_m} \\
\vdots & & \vdots \\
\Gamma_1'' \to \Delta_1'' & & \Gamma_m'' \to \Delta_m'' \\
\vdots & & \vdots \\
\mathcal{D}^1 & & \mathcal{D}^m \\
\vdots & & \vdots \\
z_1 = z_1, \overline{P}, \Gamma \to \Delta & \dots & z_m = z_m, \overline{P}, \Gamma \to \Delta
\end{array} \\
\hline
\overline{P}, \Gamma \to \Delta
\end{array}
\qquad (3)
$$

We are now in a position to complete the proof of Theorem 3.2 by giving the system of rules that corresponds to a formula A without implications or universal quantifiers in the negative part.

Proof of Theorem 3.2, second part: Observe that each block of invertible steps of the decomposition of A defines an additional layer in the inductive definition of a system of rules. If $GRS(A_{i,j})$ are the systems of rules that correspond to $A_{i,j}$, then the following scheme gives the system of rules that corresponds to A:

$$
\begin{array}{c}
\begin{array}{ccc}
\vdots & & \vdots \\
GRS(A_{i,1}) & & GRS(A_{i,n_i}) \\
\vdots & & \vdots \\
y_{i,1} = y_{i,1}, \overline{P_i}, \Gamma \to \Delta & \dots & y_{i,n_i} = y_{i,n_i}, \overline{P_i}, \Gamma \to \Delta
\end{array} \\
\hline
\overline{P_i}, \Gamma \to \Delta
\end{array}
$$

Here the variables y_{j,n_j} are eigenvariables, i.e. not in $\overline{P_i}, \Gamma, \Delta$. QED

We introduce a little bit of compact notation for denoting system of rules. This is done inductively. For $n = 0$, GRS_n is GRS_0, the usual geometric rule scheme. For $n = 1$ we have the system

$$
GRS_1 = \begin{cases} GRS_0(z_1), \dots, GRS_0(z_m) \\ \\ z_1, \dots, z_m \end{cases}
$$

Here the z_i are the eigenvariables of the last step of the system; these are included in the parametric variables (those other than the eigenvariables) of the rule schemes GRS_0. For $n+1$ we have the system

$$
GRS_{n+1} = \begin{cases} GRS_{k_1}(z_1), \dots, GRS_{k_m}(z_m) \\ \\ z_1, \dots, z_m \end{cases}
$$

Here for all $i = 1, \dots, m$ we have $k_i \leqslant n$. The set of eigenvariables of the system is given by the z_i together with the eigenvariables of the systems $GRS_{k_i}(z_i)$. We observe that each z_i may indeed indicate a vector of variables, but the vector symbol is omitted.

Because of the equivalence of the geometric rule scheme and geometric axioms (cf. Negri 2003), we know that by the use of rules that belong to the scheme $GRS_0(z_i)$, we can derive $\& \, GA_0(z_i/x_i)$, and conversely that a derivation that uses the scheme $GRS_0(z_i)$ can be converted into a purely logical derivation from the same assumptions, with $\& \, GA_0(z_i/x_i)$ added in the antecedent of the conclusion.

We are now in a position to prove:

Proposition 3.3. *Axiom schemes of the form GA_1 are derivable in* **G3c** *(***G3im***) extended by a rule scheme of the form GRS_1. Conversely, any (cut-free) derivation that uses rule schemes of the form GRS_1 can be converted into (cut-free) derivations with the same assumptions and with conclusion weakened with GA_1 in the antecedent of conclusion, and therefore in a derivation with cuts in the system extended by axioms in the class GA_1.*

Proof: For the first part, consider the derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\to \&\, GA_0(z_1/y_1)}}{z_1 = z_1, \overline{P} \to \&\, GA_0(z_1/y_1)}\,L\text{-}Wk}{z_1 = z_1, \overline{P} \to \exists y_1 \&\, GA_0}\,R\exists}{z_1 = z_1, \overline{P} \to \exists y_1 \&\, GA_0, \dots, \exists y_m \&\, GA_0}\,R\text{-}Wk \quad \dots \quad \dfrac{\dfrac{\dfrac{\vdots}{\to \&\, GA_0(z_m/y_m)}}{z_m = z_m, \overline{P} \to \&\, GA_0(z_m/y_m)}\,L\text{-}Wk}{\dfrac{z_m = z_m, \overline{P} \to \exists y_m \&\, GA_0}{z_m = z_m, \overline{P} \to \exists y_1 \&\, GA_0, \dots, \exists y_m \&\, GA_0}\,R\text{-}Wk}\,R\exists}{\overline{P} \to \exists y_1 \&\, GA_0, \dots, \exists y_m \&\, GA_0}\,Ex}{\overline{P} \to \exists y_1 \&\, GA_0 \vee \dots \vee \exists y_m \&\, GA_0}\,R\vee}{\to \overline{P} \supset \exists y_1 \&\, GA_0 \vee \dots \vee \exists y_m \&\, GA_0}\,R\supset}{\to \forall \overline{x}(\&\, \overline{P} \supset \exists y_1 \&\, GA_0 \vee \dots \vee \exists y_m \&\, GA_0)}\,R\forall
$$

The dotted parts are derivations in the system with the rule scheme GRS_0 that correspond to the geometric axioms $GA_0(z_i/y_i)$. These are determined as in section 3 of Negri (2003). *Ex* indicates the existential part of GRS_1.

For the converse, we have to show that given the premisses of GRS_1 as in (2) above, we can derive its conclusion modulo left weakening with axioms in GA_1. We have:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\Gamma_1' \to \Delta_1' \\ \vdots \\ \mathcal{D}_0^{1\prime} \\ \vdots \\ \&\, GA_0, \Gamma_1'' \to \Delta_1'' \\ \vdots \\ \mathcal{D}^{1\prime} \\ \vdots \\ y_1 = y_1, \&\, GA_0, \overline{P}, \Gamma \to \Delta\end{matrix}}{y_1 = y_1 \,\&\, \&\, GA_0, \overline{P}, \Gamma \to \Delta}\,L\&}{\exists y_1(y_1 = y_1 \,\&\, \&\, GA_0), \overline{P}, \Gamma \to \Delta}\,L\exists \quad \dots \quad \dfrac{\dfrac{\begin{matrix}\Gamma_m' \to \Delta_m' \\ \vdots \\ \mathcal{D}_0^{m\prime} \\ \vdots \\ \&\, GA_0, \Gamma_m'' \to \Delta_m'' \\ \vdots \\ \mathcal{D}^{m\prime} \\ \vdots \\ y_m = y_m, \&\, GA_0, \overline{P}, \Gamma \to \Delta\end{matrix}}{y_m = y_m \,\&\, \&\, GA_0, \overline{P}, \Gamma \to \Delta}\,L\&}{\exists y_m(y_m = y_m \,\&\, \&\, GA_0), \overline{P}, \Gamma \to \Delta}\,L\exists}{\exists y_1(y_1 = y_1 \,\&\, \&\, GA_0) \vee \dots \vee \exists y_m(y_m = y_m \,\&\, \&\, GA_0), \overline{P}, \Gamma \to \Delta}\,L\vee}{\&\, \overline{P} \supset \exists y_1(y_1 = y_1 \,\&\, \&\, GA_0) \vee \dots \vee \exists y_m(y_m = y_m \,\&\, \&\, GA_0), \overline{P}, \Gamma \to \Delta}\,Der}{\forall \overline{x}(\&\, \overline{P} \supset \exists y_1(y_1 = y_1 \,\&\, \&\, GA_0) \vee \dots \vee \exists y_m(y_m = y_m \,\&\, \&\, GA_0)), \overline{P}, \Gamma \to \Delta}\,L\forall
$$

Here the derivations $\mathcal{D}_0^{i\prime}$ are obtained from the $\mathcal{D}_0^i$ by the known equivalence of geometric axioms and geometric rules and the derivations $\mathcal{D}^{i\prime}$ are obtained from the $\mathcal{D}^i$ by left weakening with $\&\, GA_0(z_i/y_i)$. The conclusion follows by a cut with the axiom scheme GA_1. QED

Clearly, the proof of the above theorem can be restated as a proof of the inductive step for the proof of correspondence between axiom schemes GA_n and rule schemes GRS_n. We therefore have:

Theorem 3.4. Equivalence of axiomatic systems and systems of rules. *For all n, axiom schemes of the form GA_n are derivable in* **G3c** *(***G3im***) extended by rule schemes of the form GRS_n. Conversely, any cut-free derivation that uses rule schemes of the form GRS_n can be converted into cut-free derivations with the same assumptions and with conclusion weakened with GA_n in the antecedent of conclusion, and therefore in a derivation with cuts in the system extended by axioms in the class GA_n.*

Remark. We conclude this section with the analysis of an example that shows that the class GA_1 need not contain quantifier alternations; in fact there are purely propositional axioms that are in GA_1 but not in GA_0. An example is the linearity axiom $(P \supset Q) \vee (Q \supset P)$, with the corresponding system of rules of the form

$$\dfrac{\begin{array}{c} \dfrac{Q, P, \Gamma' \to \Delta'}{P, \Gamma' \to \Delta'} \\ \vdots \\ \mathcal{D}^1 \\ \vdots \\ \Gamma \to \Delta \end{array} \qquad \begin{array}{c} \dfrac{P, Q, \Gamma'' \to \Delta''}{Q, \Gamma'' \to \Delta''} \\ \vdots \\ \mathcal{D}^2 \\ \vdots \\ \Gamma \to \Delta \end{array}}{\Gamma \to \Delta}\, {\scriptstyle Lin}$$

The correspondence of the above system of rules with the linearity axiom follows as a special case from the correspondence between rule schemes of the form GRS_n and axiom schemes GA_n, but we show it for completeness in this special case. First the axiom is derivable from the system of rules added to **G3im** as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{Q, P \to Q}{P \to Q}}{\to P \supset Q, Q \supset P}\, {\scriptstyle R\supset}}{\to (P \supset Q) \vee (Q \supset P)}\, {\scriptstyle R\vee} \qquad \dfrac{\dfrac{\dfrac{P, Q \to P}{Q \to P}}{\to P \supset Q, Q \supset P}\, {\scriptstyle R\supset}}{\to (P \supset Q) \vee (Q \supset P)}\, {\scriptstyle R\vee}}{\to (P \supset Q) \vee (Q \supset P)}\, {\scriptstyle Lin}$$

Second, we show that given the premisses of *Lin* we can derive its conclusion through cuts with the axiomatic sequent $\to (P \supset Q) \vee (Q \supset P)$ and with sequents derived in pure logic:

$$\dfrac{\to (P \supset Q) \vee (Q \supset P) \qquad \dfrac{\begin{array}{c} \dfrac{\begin{array}{c}\vdots \\ P, P \supset Q \to Q\end{array} \quad Q, P, \Gamma' \to \Delta'}{P, P \supset Q, \Gamma' \to \Delta'}\, {\scriptstyle Cut} \\ \vdots \\ \mathcal{D}^{1\prime} \\ \vdots \\ P \supset Q, \Gamma \to \Delta \end{array} \qquad \begin{array}{c} \dfrac{\begin{array}{c}\vdots \\ Q, Q \supset P \to P\end{array} \quad P, Q, \Gamma'' \to \Delta''}{Q, Q \supset P, \Gamma'' \to \Delta''}\, {\scriptstyle Cut} \\ \vdots \\ \mathcal{D}^{2\prime} \\ \vdots \\ Q \supset P, \Gamma \to \Delta \end{array}}{(P \supset Q) \vee (Q \supset P), \Gamma \to \Delta}\, {\scriptstyle L\vee}}{\Gamma \to \Delta}\, {\scriptstyle Cut}$$

Here $\mathcal{D}^{1'}$ and $\mathcal{D}^{2'}$ are obtained from $\mathcal{D}^1$ and $\mathcal{D}^2$ by left weakening with $P \supset Q$ and $Q \supset P$, respectively[1].

A simple argument shows that rule *Lin* is analytical, i.e. P and Q are found among the subformulas of the conclusion. Consider Q and the derivation $\mathcal{D}^2$ of the right premiss of the rule. Either there are no other applications of linearity or, if there are, either Q is always a side formula or is a principal formula in some of them. In the first two cases, it is found as a subformula of the conclusion, else if it is principal somewhere we choose among the two branches of linearity the one in which it does not disappear in the upper inference line. Since the derivation is a finite objects, and in particular, there is only a finite number of applications of linearity, Q cannot disappear altogether. In an identical way, one concludes that P cannot disappear[2].

A typical, non-degenerate example of an axiom in GA_1 is the continuity axiom of analysis, $\forall\epsilon\exists\delta(\forall x(x \in B(\delta) \supset f(x) \in B(\epsilon)))$.

4 Admissibility of structural rules

In analogy with the extension by geometric rules, we shall denote by **G3cGT** (resp. **G3imGT**) the extension of **G3c** (resp. **G3im**) by systems of rules that follow the generalized geometric rule scheme. Likewise, in defining the extension, care has to be paid to the closure condition to maintain the admissibility of the rule of contraction. In order to show that the systems **G3cGT** and **G3imGT** are complete with respect to the classical and intuitionistic generalized geometric theory GT, respectively, we need to extend the results on Negri (2003) from systems with the geometric rule scheme to systems with systems of rules. The leading idea for the extension is that it is not enough to prove results of height-preserving admissibility, but there should be stronger "system-preserving" admissibilities. This is needed to guarantee that the transformations used in proving height-preserving admissibility of substitution, weakening and contraction, as well as admissibility of cut, transform a correct derivation into another correct derivation, that is, into one in which the added rules respect the conditions on the eigenvariables. We include for completeness the otherwise routine proofs.

Lemma 4.1. System-preserving substitution. *Given a derivation of* $\Gamma \to \Delta$ *in* **G3cGT** *(***G3imGT***, resp.), with* x *a free variable in* Γ, Δ *and a term* t *free for* x *in* Γ, Δ *not containing*

[1] This example suggests a possible connection with the method of hypersequents, where linearity is translated by the rule of *communication*

$$\frac{G \mid \Delta_1, \Gamma_1 \to \Pi_1 \quad G \mid \Delta_2, \Gamma_1 \to \Pi_2}{G \mid \Delta_2, \Gamma_1 \to \Pi_1 \mid \Delta_1, \Gamma_2 \to \Pi_2}$$

and the identification of two branches that lead to the same conclusion in the sequent calculus derivation by the rule of *external contraction* $\frac{G \mid \Gamma \to \Delta \mid \Gamma \to \Delta \mid H}{G \mid \Gamma \to \Delta \mid H}$ between identical components of an hypersequent (cf. Avron 1996 for the presentation of the method of hypersequents for non-classical logics with the specific analysis of linearity on p. 9, and Ciabattoni, Galatos and Terui 2008 for other extensions of hypersequents by rules.

[2] This proof of analyticity may bring to mind the proof of the subterm property for the theory of linear order, which also originates from the idea that the only way to have a term disappear from a derivation would be to have an infinite derivation (Theorem 7.9 in Negri and von Plato 2011 or Theorem 6.3 in Negri, von Plato, Coquand 2004).

any of the eigenvariables in the derivation, we can find a derivation of $\Gamma(t/x) \to \Delta(t/x)$ *in* **G3cGT** *(***G3imGT***, resp.) with the same height and system structure.*

Proof: By induction on the height of the given derivation, considering the last rule applied. If the last rule is a scheme of the form GRS_0, the proof goes as in Negri (2003), with the additional observation that the structure is voidly preserved. If the last rule is of the form GRS_{n+1} (with the notation as in the definition of the scheme), we apply the inductive hypothesis to the conclusions of the rules $GRS_{k_1}(z_i)$, which gives derivations of $\Gamma''_i(t/x) \to \Delta''_i(t/x)$ and conclusions $z_i = z_i(t/x), \overline{P}, \Gamma(t/x) \to \Delta(t/x)$, which is the same as $z_i = z_i, \overline{P}, \Gamma(t/x) \to \Delta(t/x)$ by the hypothesis on t and x. QED

Theorem 4.2. *The rules of weakening*

$$\frac{\Gamma \to \Delta}{A, \Gamma \to \Delta}\, LW \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, A}\, RW$$

are height- and system-preserving admissible in **G3cGT** *and in* **G3imGT**.

Proof: By induction on the height of the derivation of the premiss, as in Negri and von Plato (1998). In case the last step is of the form GRS_n and A contains some of its eigenvariables, the substitution lemma is applied to the eigenvariables of the system of rules to have new free variables not clashing with those in A. The conclusion is then obtained by applying the inductive hypothesis and the geometric rule scheme. QED

The proof of admissibility of the contraction rules for **G3cGT** and **G3imGT** requires the use of inversion lemmas for all those rules that do not copy the principal formula into their premisses. Again, here as in the other admissibilities above, we need to establish the stronger height and system preserving inversion. All the inversion lemmas for the propositional rules that hold for **G3c** and **G3im** hold for the extension by systems of rules, for the same reason as for the geometric rule scheme, that is, the fact that systems of rules have only atomic formulas as principal and active formulas. For the inversions of $L\exists$ and $R\forall$, we need to add a condition on the variable to avoid clashes with the eigenvariables of the systems of rules in the derivation.

Let $\vdash_n \Gamma \to \Delta$ denote a derivation of the sequent $\Gamma \to \Delta$ in **G3cGT**, with derivation height bounded by n. We have:

Lemma 4.3. Inversion for quantifier rules.
(i) If $\vdash_n \exists x A, \Gamma \to \Delta$ *and* y *is not among the eigenvariables of the systems of rules in the derivation, then* $\vdash_n A(y/x), \Gamma \to \Delta$*, with the same system structure.*
(ii) If $\vdash_n \Gamma \to \Delta, \forall x A$ *and* y *is not among the eigenvariables of the systems of rules in the derivation, then* $\vdash_n \Gamma \to \Delta, A(y/x)$*, with the same system structure.*

Proof: (i) Similar to the proof of Lemma 3 in Negri (2003), with the additional observation of system preserving admissibility for all cases. The new case, in which the last rule in the derivation of $\exists x A, \Gamma \to \Delta$ is the last step in a system of rules GRS_n, is treated, variable-wise, in the same way as the case of geometric rules as last step.

(ii) Similar to (i). QED

We remark that a similar statement holds for **G3imGT**, with (ii) modified to an empty Δ.

By Lemma 4.1, we are allowed to assume the following:

Purity condition. *In a derivation in* **G3cGT** (**G3imGT**) *the sets of proper variables (eigenvariables) of the generalized geometric rules are pairwise disjoint and appear only in the subtrees of the derivation above such rules.*

Theorem 4.4. *The rules of contraction*

$$\frac{A, A, \Gamma \to \Delta}{A, \Gamma \to \Delta}\, LC \qquad \frac{\Gamma \to \Delta, A, A}{\Gamma \to \Delta, A}\, RC$$

are admissible and height- and system-preserving in **G3cGT** *and in* **G3imGT**.

Proof: For left contraction, the proof is by induction on the height of the derivation of the premiss. If it is an initial sequent or a zero-premiss rule, the conclusion is also an an initial sequent or a zero-premiss rule. If the last rule is a propositional rule, then $A, \Gamma \to \Delta$ follows as in theorem 3.2 of Negri and von Plato (1998). If it is $L\forall$, we apply the induction hypothesis to the premiss of the rule, and then the rule, and similarly if it is $L\exists$ with A not principal in it. If it is $L\exists$ with $A \equiv \exists x B$ and premiss $B(y/x), \exists x B, \Gamma \to \Delta$, by the variable condition on the generalized geometric rule scheme, $y \notin \Gamma, \Delta$, so y is not among the eigenvariables in any geometric rule in the derivation of $B(y/x), \exists x B, \Gamma \to \Delta$ (or even of the whole derivation if we assume the pureness condition), so we can apply the inversion lemma for $L\exists$ instantiated to y and obtain a derivation of $B(y/x), B(y/x), \Gamma \to \Delta$. By the induction hypothesis, we get $B(y/x), \Gamma \to \Delta$ and by $L\exists$, $\exists x B, \Gamma \to \Delta$. The rest of the proof is similar to the proof for geometric extensions as detailed in Negri (2003). In particular, nothing more than the same care with variables that was already needed for that proof is needed to establish system-preserving admissibility in the extensions with generalized geometric rules. QED

Theorem 4.5. *The cut rule*

$$\frac{\Gamma \to \Delta, A \quad A, \Gamma' \to \Delta'}{\Gamma, \Gamma' \to \Delta, \Delta'}\, Cut$$

is admissible and system-preserving in **G3cGT** *and in* **G3imGT**.

Proof: Also this proof has the same structure as the corresponding proof in Negri (2003), with an induction on the length of A with subinduction on the sum of the heights of the derivations of $\Gamma \to \Delta, A$ and $A, \Gamma' \to \Delta'$. Here we have to check that the additional requirement of preservation of system structure is met by that procedure. We observe that, in general, the permutation of cuts preserves the system structure, because if a step (a mathematical rule) is followed in the original derivation by another step (another mathematical rule such as one that removes eigenvariables), this order is maintained after the permutation. In addition, the pureness condition guarantees that the permutation does not introduce spurious variables that could interfere with the variable condition of the generalized geometric rule scheme.

To illustrate the argument for one specific conversion type, we shall consider in detail the case in which the left premiss of cut is the conclusion of a generalized geometric rule and the cut formula A principal in it. In this case A is necessarily an atomic formula and therefore it cannot be principal in the left premiss of cut:

$$
\begin{array}{ccc}
 & \Gamma_1 \to \Delta_1 & \Gamma_m \to \Delta_m \\
 & \vdots\ \mathcal{D}^1_{k_1} & \vdots\ \mathcal{D}^m_{k_m} \\
 & \Gamma'_1 \to \Delta'_1 & \Gamma'_m \to \Delta'_m \\
 & \vdots\ \mathcal{D}^1 & \vdots\ \mathcal{D}^m
\end{array}
$$

$$
\dfrac{\dfrac{\Gamma'' \to \Delta'', A}{\Gamma \to \Delta, A}\,R \qquad \dfrac{z_1 = z_1, A, \overline{P}, \Gamma' \to \Delta' \quad \dots \quad z_m = z_m, A, \overline{P}, \Gamma' \to \Delta'}{A, \overline{P}, \Gamma' \to \Delta'}\,GGRS}{\overline{P}, \Gamma, \Gamma' \to \Delta, \Delta'}\,cut
$$

The derivation is converted as follows:

$$
\begin{array}{ccc}
 & \Gamma_1 \to \Delta_1 & \Gamma_m \to \Delta_m \\
 & \vdots\ \mathcal{D}^1_{k_1} & \vdots\ \mathcal{D}^m_{k_m} \\
 & \Gamma'_1 \to \Delta'_1 & \Gamma'_m \to \Delta'_m \\
 & \vdots\ \mathcal{D}^1 & \vdots\ \mathcal{D}^m
\end{array}
$$

$$
\dfrac{\dfrac{\Gamma'' \to \Delta'', A \qquad \dfrac{z_1 = z_1, A, \overline{P}, \Gamma' \to \Delta' \quad \dots \quad z_m = z_m, A, \overline{P}, \Gamma' \to \Delta'}{A, \overline{P}, \Gamma' \to \Delta'}\,GGRS}{\overline{P}, \Gamma'', \Gamma' \to \Delta'', \Delta'}\,cut}{\overline{P}, \Gamma, \Gamma' \to \Delta, \Delta'}\,R
$$

Observe that in case rule R is a rule with variable conditions, the purity condition is used to guarantee that the new context does not contain any of the eigenvariables of the rules. QED

5 Generalized first-order Barr's theorem

The first-order Barr theorem is a conservativity result stating that if a geometric implication is provable classically in a geometric theory, then it is provable intuitionistically. We can extend this result, with generalized geometric implications in place of geometric ones. The generalization can be attempted both in the theory part and in the conservative class; however, it is seen immediately that generalized geometric implications do not fall under the possible extensions of the conservative class. A counterexample is $(A \supset B) \vee (B \supset A)$. This is in GA_1 and has a level 1 disjunction and is provable classically but not intuitionistically. We can, on the other hand, liberalize the theory part to the whole class of generalized geometric theories and obtain the following result:

Theorem 5.1 (Generalized first-order Barr's theorem)**.** *For all n, if a geometric implication is derivable in* **G3c** $+GRS_n$*, it is derivable in* **G3im** $+GRS_n$.

Proof: *Nothing to prove.* Any derivation uses only rules from GRS_n and logical rules. Because of the shape of the conclusion, none of the rules that violates the intuitionistic single-succedent restrictions (i.e., the classical multisuccedent $R \supset, R\forall$) can have been used, so the derivation is already an intuitionistic one. QED

The conclusion of the above theorem can be strengthened to a conservativity over *minimal* logic if an additional assumption is added, namely that the geometric implication does not contain $\bot$ in the negative part. The reason is that the derivation will not contain $\bot$ in any antecedent of sequents, and a fortiori the rule $L\bot$ of intuitionistic logic cannot have been used. We recall (cf. e.g. Troelstra and Schwichtenberg 2000 or Negri 2003) that a multi-succedent sequent calculus **G3mm** for minimal logic is obtained from **G3im** by replacing $L\bot$ with the initial sequent $\bot, \Gamma \to \Delta, \bot$.

By the characterization of generalized geometric implications in terms of Glivenko classes and the conversion of generalized geometric implications into generalized geometric rules in Theorem 3.2, the above results can be restated as follows:

Theorem 5.2. *Suppose that Γ is a multiset of formulas that do not contain $\supset$ or $\forall$ in the negative part, and that A is a geometric implication. If* **G3c**$\vdash \Gamma \to A$*, then* **G3im**$\vdash \Gamma \to A$*. If in addition A does not contain $\bot$ in the negative part, then* **G3mm**$\vdash \Gamma \to A$

6 Labelled systems with systems of rules

So far, the most general classes of logics defined through their Kripke semantics, and thus amenable to a formulation in term of labelled systems, have been those characterized by geometric frame conditions and by the Noetherian condition. These include all the intermediate interpolable logics (Dyckhoff and Negri 2012), all the common systems of normal modal logics and the provability logics of Gödel-Löb and of Grzegorczyk (cf. Negri 2005, Dyckhoff and Negri 2013). They do not include certain intermediate logics such as the *Kreisel-Putnam logic* (cf. Dyckhoff and Negri 2012) and *knowability logic* (cf. Maffezioli, Naibo and Negri 2012), both characterized by frame conditions in GA_1.

We are now in a position to extend labelled deduction to all modal (and multimodal) logics with frame conditions that are generalized geometric implications.

The formulation of these systems is obtained as follow:

1. Start with a basic labelled modal system, such as the system for classic modal logic **G3K** (as in Negri 2005), or the system for intuitionistic modal logic **G3I** (as in Negri and von Plato 2011 or Dyckhoff and Negri 2012), or a multimodal system (as in Hakli and Negri 2011, Maffezioli, Naibo and Negri 2012).

2. Add the frame properties in the form of generalized geometric rules for accessibility relations.

The methodological difference with respect to the method exposed in Negri (2005) is that the extra rules for labels can be systems of rules that correspond to generalized geometric implications.

The proofs of the structural properties of the system thus obtained is extended almost routinely to all the systems obtained. A bit of care is needed in the soundness part of the proof of completeness. We sketch the argument for the case of an extension with a rule in GGR_1 to illustrate the subtle point.

We first recall some definitions, starting with the table of rules of **G3K**:

Initial sequents:

$x : P, \Gamma \to \Delta, x : P$ $\qquad\qquad$ $xRy, \Gamma \to \Delta, xRy$

Propositional rules:

$$\frac{x : A, x : B, \Gamma \to \Delta}{x : A\&B, \Gamma \to \Delta}\, L\& \qquad \frac{\Gamma \to \Delta, x : A \quad \Gamma \to \Delta, x : B}{\Gamma \to \Delta, x : A\&B}\, R\&$$

$$\frac{x : A, \Gamma \to \Delta \quad x : B, \Gamma \to \Delta}{x : A \vee B, \Gamma \to \Delta}\, L\vee \qquad \frac{\Gamma \to \Delta, x : A, x : B}{\Gamma \to \Delta, x : A \vee B}\, R\vee$$

$$\frac{\Gamma \to \Delta, x : A \quad x : B, \Gamma \to \Delta}{x : A \supset B, \Gamma \to \Delta}\, L\supset \qquad \frac{x : A, \Gamma \to \Delta, x : B}{\Gamma \to \Delta, x : A \supset B}\, R\supset$$

$$\frac{}{x : \bot, \Gamma \to \Delta}\, L\bot$$

Modal rules:

$$\frac{y : A, x : \Box A, xRy, \Gamma \to \Delta}{x : \Box A, xRy, \Gamma \to \Delta}\, L\Box \qquad \frac{xRy, \Gamma \to \Delta, y : A}{\Gamma \to \Delta, x : \Box A}\, R\Box$$

$$\frac{xRy, y : A, \Gamma \to \Delta}{x : \Diamond A, \Gamma \to \Delta}\, L\Diamond \qquad \frac{xRy, \Gamma \to \Delta, x : \Diamond A, y : A}{xRy, \Gamma \to \Delta, x : \Diamond A}\, R\Diamond$$

Rules $R\Box$ and $L\Diamond$ have the condition that y is not in the conclusion.

Table. The system G3K

In the first initial sequent, P is an arbitrary atomic formula. As remarked in Negri (2005), no rule removes an atom of the form xRy from the right-hand side of sequents, and such atoms are never active in the logical rules, therefore initial sequents of the form $xRy, \Gamma \to \Delta, xRy$ can as well be left out from the calculus without impairing completeness of the system. Such initial sequents are needed only for deriving properties of the accessibility relation, namely, the extra axioms corresponding to the rules for R given below. Then extra axioms are included in the system in the form of rules for the accessibility relation. They follow the regular or the geometric rule scheme recalled in the Introduction and in Section 3 above. For example, the rules that correspond to axiom **T** ($\Box A \supset A$) and **4** ($\Box A \supset \Box\Box A$) are obtained by the conversion of the frame properties of *reflexivity* and *transitivity*, respectively, and have the form

$$\frac{xRx, \Gamma \to \Delta}{\Gamma \to \Delta}\,{\scriptstyle Ref} \qquad \frac{xRz, xRy, yRz, \Gamma \to \Delta}{xRy, yRz, \Gamma \to \Delta}\,{\scriptstyle Trans}$$

The full strength of the geometric rule scheme is instead used for converting axioms, such as **2** ($\Diamond\Box A \supset \Box\Diamond A$) that corresponds to *directness*, a geometric frame condition with existential variables, $\forall xyz(xRy \& xRz \supset \exists w(yRw \& zRw)$; the rule to be added to **G3K** is

$$\frac{yRw, zRw, xRy, xRz, \Gamma \to \Delta}{xRy, xRz, \Gamma \to \Delta}\,{\scriptstyle Dir}$$

where w is a fresh variable.

We can go further and consider not just frame properties that correspond to geometric axioms, but also frame properties that correspond to generalized geometric implications. For example, the modal axiom $A \supset \Diamond\Box\Diamond A$ corresponds to the frame property

$$\forall x \exists y(xRy \& \forall z(yRz \supset zRx))$$

The property is in GA_1 and the corresponding system of rules has the form

$$\left\{ \begin{array}{l} \dfrac{zRx, yRz, \Gamma \to \Delta}{yRz, \Gamma \to \Delta} \\[2ex] \dfrac{xRy, \Gamma \to \Delta}{\Gamma \to \Delta} \end{array} \right.$$

with the condition is that y is not free in Γ, Δ and the side condition discussed in Section 3.1 on the order of application of the rules in a derivation, namely that whenever the upper rule is applied on any branch it is followed by the lower one, as in

$$\begin{array}{c} \dfrac{zRx, yRz, \Gamma' \to \Delta'}{yRz, \Gamma' \to \Delta'} \\ \vdots \\ \dfrac{xRy, \Gamma \to \Delta}{\Gamma \to \Delta} \end{array}$$

The reader can check that $A \supset \Diamond\Box\Diamond A$ is derivable in **G3K** extended by the above system of rules.

We shall extend the use of the terminology in Negri (2005) and denote by **G3K*** any extension of **G3K** with rules that follow the generalized geometric rule scheme.

Definition 6.1. *Let K be a frame with an accessibility relation $\mathcal{R}$ that satisfies the properties $*$. Let W be the set of variables (labels) used in derivations in* **G3K***. *An* **interpretation** *of the labels W in frame K is a function $[\![\cdot]\!] : W \to K$. A* **valuation** *of atomic formulas in frame K is a map $\mathcal{V} : AtFrm \to \mathcal{P}(K)$ that assigns to each atom P the set of nodes of K in which P holds; the standard notation for $k \in \mathcal{V}(P)$ is $k \Vdash P$.*

Definition 6.2. *A sequent* $\Gamma \to \Delta$ *is* **true for an interpretation and a valuation** *in* K *if for all labelled formulas* $x : A$ *and relational atoms* yRz *in* Γ*, whenever* $[\![x]\!] \Vdash A$ *and* $[\![y]\!]\mathcal{R}[\![z]\!]$ *in* K*, then for some* $w : B$ *in* Δ*,* $[\![w]\!] \Vdash B$*. A sequent is* **valid** *if it is true for every interpretation and every valuation in a frame.*

Theorem 6.3. *If the sequent* $\Gamma \to \Delta$ *is derivable in* **G3K****, then it is valid in every frame with properties* $*$ *ranging over generalized geometric axioms.*

Proof. Given a valuation $\mathcal{V}$ in a frame K, we show that each rule preserves the truth for the evaluation, from which the result follows by induction. For a generalized geometric axiom GA_n, $n > 0$, the proof is supplemented by an induction on n. The base case, with $n = 0$ has already been proved (cf. Theorem 11.27 in Negri and von Plato 2011 or Theorem 5.3 in Negri 2009). We show the case $n = 1$. For simplicity we assume that GA_1 is of the form $\forall x(\,\&\, P_i \supset \exists y GA_0)$ and we consider the last rule in the derivation. The cases of initial sequents, propositional rules, modal rules, and mathematical rules without eigenvariables, are dealt with as in the above mentioned proofs.

If the sequent is a conclusion of the rule system that corresponds to GA_1, we have

$$\dfrac{\begin{array}{c}\Gamma' \to \Delta' \\ \vdots \\ \mathcal{D}_0 \\ \vdots \\ \Gamma'' \to \Delta'' \\ \vdots \\ \mathcal{D} \\ \vdots \\ y = y, \overline{P}, \Gamma \to \Delta\end{array}}{\overline{P}, \Gamma \to \Delta}$$

Suppose that the premiss is true for $\mathcal{V}$ and consider an interpretation $[\![\cdot]\!]$ such that all the frame relations xRy hold under the interpretation and for all formulas $z : A$ in Γ, $[\![z]\!] \Vdash A$. In particular, all the relations P_i hold in the frame under the interpretion, so by GA_1, there is k in the frame such that $GA_0(k)$ holds. Then we have

1. By the induction on the first parameter, the conclusions of $GRS_0(k)$ is true.

2. The sequent $y = y, \overline{P}, \Gamma \to \Delta$ is true for all interpretations that verify the antecedent and with $[\![y]\!] \equiv k$. Observe that such an interpretation can be obtained as an extension of the one we started with, because by the variable condition $y \notin \Gamma, \Delta$. So there is $w : B \in \Delta$ such that $[\![w]\!] \Vdash B$ as requested.

We observe that the proof is generalized to an arbitrary axiom in the class GA_1 by considering as many branches that start with the eigenvariables y_i as there are disjunctions of conjunctions of GA_0 in the axiom and for each branch as many applications of rules in $GRS_0(y_i)$ as there are conjuncts in each of $GRS_0(y_i)$. The inductive step for $n + 1$ is done exactly in the same way, with GA_n and GRS_n in place of GA_0 and GRS_0. QED

The completeness part of the completeness proof does not present any difference with respect to the proof for geometric extensions: the reduction tree and the countermodel construction are defined as in the proof of theorem 11.28 of Negri and von Plato (2011), by considering closure with respect to all systems of rules rather than just all rules. We thus obtain the following theorem, where **G3K*** denotes any extension of **G3K** by generalized geometric rules and the properties $*$ are the corresponding frame rules:

Theorem 6.4. *Let* $\Gamma \to \Delta$ *be a sequent in the language of* **G3K***. *Then either the sequent is derivable in* **G3K*** *or it has a Kripke countermodel with properties* $*$.

The completeness theorem then follows as a corollary:

Corollary 6.5. *If a sequent* $\Gamma \to \Delta$ *is valid in every Kripke model with the frame properties* $*$, *then it is derivable in the system* **G3K***.

The question then naturally arises on the expressive power of this extension: Which classes of non-classical logics does the method capture? A well known class is that of modal logics axiomatized by Sahlqvist formulas. We can indeed easily show that the method provides a complete proof system of each modal logic in this fragment. To prove this claim, we first recall from Blackburn et al. (2001) a characterization of frame properties that correspond to Sahlqvist formulas, given in terms of *Kracht formulas*. Before giving the definition, we recall that a *restricted universal quantifier* $\forall^r$ is of the form

$$\forall^r y A(x, y) \equiv \forall y(xRy \supset A(x, y))$$

and a *restricted existential quantifier* $\exists^r$ is of the form

$$\exists^r y A(x, y) \equiv \exists y(xRy \& A(x, y))$$

Restricted quantifiers, either universal or existential, are denoted by Q^r. Formulas are, as usual, assumed to be *clean*, i.e. to satisfy the property that no variable is both free and bound and no two distinct occurrences of quantifiers bind the same variable (formulas with this property are sometimes called *pure*). A variable in such a formula is *inherently universal* if it is either free or bound by a restricted universal quantifier which is not in the scope of an existential quantifier. Kracht formulas are first order (clean) formulas with the following properties:

1. *Restrictedly positive*, i.e., built from atomic formulas and negation of atomic formulas using only &, $\vee$, and restricted quantifiers.

2. Every atomic subformula is either of the form $z = z$ or $z \neq z$, or else it contains at least one inherently universal variable.

In particular, we observe that the restrictors xRy of restricted quantifiers are atomic formulas. We can then proceed in two ways to show that Kracht formulas are indeed a special case of generalized geometric implications. The first way is to use a conversion the normal form for

Kracht formulas, as detailed in Blackburn et al. (2001, p. 171): it consists of conjunctions of formulas of the form

$$\forall^r x_1 \dots \forall^r x_n Q_1^r y_1 \dots Q_m^r y_m A(x_1, \dots, x_n, y_1, \dots, y_m)$$

where A is quantifier free and its atomic subformulas are either of the form $z = z$, or $z \neq z$ or contain one of the variables x_i. Then we can prove by induction on n and m that such formulas are indeed generalized geometric implications.

The second way is to observe that the very definition of Kracht formulas implies that they do not contain negative occurrences of $\supset$ or $\forall$ because they are built from atoms and negations of atoms using &, $\vee$, and Q^r. By Theorem 3.2 they can thus be converted to conjunctions of generalized geometric implications and to systems of generalized geometric rules.

We have thus proved:

Proposition 6.6. *Every Kracht formula is equivalent to a conjunction of generalized geometric implications.*

By Theorem 6.4, systems of rules for generalized geometric implications provide complete proof systems for the frame classes they correspond to, and thus, by Theorem 3.59 in Blackburn et al. (2001), they provide complete analytic proof systems for any modal logic axiomatized by Sahlqvist formulas.

7 Concluding remarks and related work

We have extended the method of proof analysis beyond the boundaries of geometric theories and highlighted a class of axioms for which the method works with equally strong structural results. The class is of particular relevance for the proof theory of modal logic because it includes the first-order formulas that correspond to modal axioms in the Sahlqvist fragment. The general *locality* of the rules has to be abandoned but the conditions that connect inference steps in a system of rules seem manageable enough because they involve only eigenvariables and the order of certain mathematical rules in a derivation. The manageability is confirmed by an application in epistemic logic that preceded and motivated this general investigation and involved a proper extension of the class of geometric axioms. We plan to investigate further applications of the method to the proof theory of non-classical logic, especially multimodal logics for which the frame conditions are found in the classes GA_n for $n \geq 1$. Frame conditions for a certain modal axiom are found by a heuristic method of root-first proof search in a basic labelled calculus. We plan to formalize these insights to develop a general method to establish correspondence results.

A conservativity results of classical over intuitionistic theories of the kind stated by our generalization of the first-order Barr theorem has been obtained in Ishihara (2000). The result is obtained by inductively defining four classes Q, R, J, and K of formulas, and proving that if a formula in K is classically provable from a collection of formulas in Q, then it is intuitionistically provable from the collection. It can be easily shown that the class Q contains the class of generalized geometric implications and the class K contains the class of geometric implications, and thus the result includes our Theorem 5.1. We point out, however, that whereas Ishihara's proof is obtained by a proof transformation through several intermediate lemmas that use the Gödel-Gentzen translation, our proof is reduced to a triviality and shows that a classical derivation in the theory obtained through the extension by rules *is* already an intuitionistic derivation. We have presented generalized geometric implications as normal forms for a certain Glivenko class of formulas. We point out that our conservativity result is not included in those of Orevkov (1968). On the other hand, a complete and simple reconstruction by means of G3 sequent calculi of the seven conservativity classes of Orevkov's work will appear in a separate paper (Negri 2014).

Another conservativity result related to our generalization of Barr's theorem has been presented by Schwichtenberg and Senjak (2013): they proved that if A is a formula without implications and Γ consists of formulas that contain disjunctions and falsity only negatively and implications only positively, then classical derivability implies minimal derivability. Their proof uses natural deduction and a method of elimination of stability axioms. In comparison, our conservativity result (Corollary 5.2) uses a cut-free sequent calculus, allows disjunctions in the positive part of formulas in Γ (but no universal quantifiers in the negative part), and is obtained with no effort as an immediate consequence of cut elimination.

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