Stone bases, alias the constructive content of Stone representation

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Abstract

This paper provides a constructive proof of Stone representation theorem for distributive lattices, in the framework of G. Sambin's formal topologies. In order to formalize the result wholly inside Martin-Löf's Intuitionistic Type Theory, the notion of *Stone base* is introduced, and it is proved to be equivalent to that of formal topology in which any cover of a basic open admits a finite subcover. The main theorem states that the category of distributive lattices with apartness is equivalent to the category of Stone bases. Finally the results are related to those of Johnstone and to the classical point-set representation.

1 Introduction

The purpose of this paper is to analyze the constructive content of Stone representation theorem for distributive lattices (cf. [ST]), asserting that any distributive lattice is isomorphic to the compact opens of a suitable topological space, called *coherent space*.

This is done in the setting of formal topologies, which have been introduced by G. Sambin in [S] with the purpose of developing pointfree topology inside the constructive foundational framework of Martin-Löf's Intuitionistic Type Theory, abbreviated ITT. The demand of expressibility inside ITT often brings to very "elementary" definitions and proofs. This is particularly evident in dealing with Stone representation theory for distributive lattices.

It is well known (cf. [J]) that distributive lattices form a category which is equivalent to the category of coherent frames. It will be shown here that the category of coherent frames, enriched with an intuitionistic predicate of apartness from 0 corresponding to the positivity predicate of formal topologies, is equivalent to the category of *Stone formal topologies*. Stone formal topologies are formal topologies in which every basic open is compact. Indeed we will prove that any Stone formal topology is uniquely determined by its behaviour on the finite subsets of the base. We are thus led to the notion of *Stone bases*, which represent the finitary content of Stone formal topologies, with the advantage of being wholly expressible inside ITT. In fact a cover, which is an infinitary relation, i.e. a relation between elements and subsets, is replaced by a *finitary cover*, which is a relation between elements of types, with no reference to arbitrary subsets. So the representation theorem for distributive lattices here takes the form of an equivalence with the category of Stone bases.

This ground result allows us to obtain the usual pointfree representation via coherent frames (cf. [J]) simply by "set-theoretic sugaring".

Finally, in order to obtain the usual point-set Stone representation by means of coherent spaces, we need to use the Prime Filter Theorem, which is seen to be equivalent to extensionality of Stone formal topologies.

Coherent spaces arise as spaces of (formal) points on a Stone formal topology, or equivalently on a Stone base; so we might say that (cf. [SVV]) Stone bases are to coherent spaces what Information bases are to Scott domains.

Even if we do not use the ITT notation explicitly throughtout the paper, we have been careful to distinguish which notions are expressible inside ITT and which are not. The various equivalences, proved outside ITT, tell us that our notion of Stone base is a good replacement, inside ITT, of the notion of coherent frame. We can thus forget about coherent frames, whose treatment in the constructive context of ITT would require an awkward amount of specifications, and retain the simpler notion of Stone base.

We conclude with observing that this work represents a continuation of some ideas already in [S] and [S2].

2 Stone formal topologies and the category of Stone bases

We start with some preliminary definitions concerning our approach to pointfree topology. For references, further details and notation the reader is referred to [S] (where sometimes a different notation is used), [S2], [SVV].

Definition 2.1 *A* formal topology *is a structure* $\mathcal{A} = (S, \cdot, 1, \lhd_{\mathcal{A}}, Pos_{\mathcal{A}})$ *where*[1]

- $(S, \cdot, 1)$ *is a* (formal) base*, i.e. a commutative monoid with unit;*
- $\lhd_{\mathcal{A}}$ *is a relation between elements and subsets of* S*, called* (formal) cover*, satisfying:*

  $$\frac{a \in U}{a \lhd_{\mathcal{A}} U}\ \textit{reflexivity},$$

[1]From now on we will omit subscripts when clear from the context.

$$\frac{a \lhd_{\mathcal{A}} U \quad U \lhd_{\mathcal{A}} V}{a \lhd_{\mathcal{A}} V} \; transitivity,$$

$$\frac{a \lhd_{\mathcal{A}} U \quad a \lhd_{\mathcal{A}} V}{a \lhd_{\mathcal{A}} U \cdot V} \; \cdot\text{-}right,$$

$$\frac{a \lhd_{\mathcal{A}} U}{a \cdot b \lhd_{\mathcal{A}} U} \; \cdot\text{-}left,$$

where $U \lhd_{\mathcal{A}} V$ *denotes a derivation* $x \lhd_{\mathcal{A}} V[x \in U]$ *of* $x \lhd_{\mathcal{A}} V$ *from the assumption* $x \in U$ *and* $U \cdot V \equiv \{a \cdot b : a \in U, b \in V\}$*;*

- $Pos_{\mathcal{A}}$ *is a predicate, called* positivity predicate *satisfying:*

$$\frac{Pos_{\mathcal{A}}(a) \qquad a \lhd_{\mathcal{A}} U}{Pos_{\mathcal{A}}(U)} \; monotonicity,$$

$$\frac{Pos_{\mathcal{A}}(a) \to a \lhd_{\mathcal{A}} U}{a \lhd_{\mathcal{A}} U} \; positivity,$$

where $Pos_{\mathcal{A}}(U) \equiv (\exists b \in U) Pos_{\mathcal{A}}(b)$*.*

Formal opens are subsets of S, up to the equivalence $U =_{\mathcal{A}} V \equiv U \lhd_{\mathcal{A}} V \& V \lhd_{\mathcal{A}} U$. Morphisms between formal topologies are the formal counterpart of the inverse of continuos maps, and thus they are defined as follows:

Definition 2.2 *A morphism between formal topologies* $\mathcal{A} \equiv (S, \cdot, 1, \lhd_{\mathcal{A}}, Pos_{\mathcal{A}})$ *and* $\mathcal{B} \equiv (T, \cdot, 1, \lhd_{\mathcal{B}}, Pos_{\mathcal{B}})$ *is an application* $f : S \to \mathcal{P}T$*, where* $\mathcal{P}T$ *denotes the power set of* T*, such that:*

1. $f(1) =_{\mathcal{B}} 1$*;*
2. $f(a \cdot b) =_{\mathcal{B}} f(a) \cdot f(b)$*;*
3. $a \lhd_{\mathcal{A}} U \to f(a) \lhd_{\mathcal{B}} f(U)$*, where* $f(U) \equiv \cup_{b \in U} f(b)$*;*
4. $Pos_{\mathcal{B}}(f(a)) \to Pos_{\mathcal{A}}(a)$*.*

Equality of morphisms $f, g : \mathcal{A} \to \mathcal{B}$ *is defined by* $f = g \equiv (\forall a \in S)(f(a) =_{\mathcal{B}} g(a))$*.*

It is easy to prove that formal topologies and morphisms of formal topologies, with composition given by $(f \circ g)(a) \equiv f(g(a))$ and identity defined by $1(a) \equiv \{a\}$, form a category, called the category of formal topologies and denoted with **FTop**.

In the sequel we will be mostly concerned with a particular full subcategory **SFTop** of **FTop** whose objects are Stone formal topologies, defined below:

Definition 2.3 *A formal topology* $\mathcal{A}$ *is said to be* Stone *if* $\lhd$ *is a* Stone cover*, i.e. whenever* $a \lhd U$ *there exists a finite subset* K *of* U *such that* $a \lhd K$*.*

The notation $\mathcal{P}_\omega S$ will be used to denote the collection of finite subsets of S as defined in [SV]. If S is a set (or a type) in the ITT framework, then so is $\mathcal{P}_\omega S$.

It is well known (cf. e.g. [S] for details) that a cover $\lhd_\mathcal{A}$ on S is associated with a closure operator $\mathcal{A}$ on $\mathcal{P}S$, defined by $\mathcal{A}U \equiv \{a \in S : a \lhd_\mathcal{A} U\}$. Here, we say that a subset U is $\mathcal{A}$-saturated if $\mathcal{A}U = U$. A bijective correspondence holds between covers on S and closure operators on $\mathcal{P}S$ satisfying the equality $\mathcal{A}(U \cdot V) = \mathcal{A}U \cap \mathcal{A}V$. Thus Stone formal topologies can be characterized in terms of closure operators by the following:

Lemma 2.4 *For any formal topology $\mathcal{A}$, $\mathcal{A}$ is Stone iff $\mathcal{A}U = \cup\{\mathcal{A}K : K \subseteq_\omega U\}$.*

Proof: $\mathcal{A}$ Stone means by definition that $a \lhd U \rightarrow (\exists K \subseteq_\omega U)(a \lhd K)$, which is equivalent to $a \lhd U \leftrightarrow (\exists K \subseteq_\omega U)(a \lhd K)$, which is just another way of expressing the equality $\mathcal{A}U = \cup\{\mathcal{A}K : K \subseteq_\omega U\}$. □

The interest of the above characterization of Stone topologies is that closure operators satisfying the corresponding condition are a well known object of study: they are just *algebraic closure operators*. By a well known result, which can be provided with a constructive proof, a closure operator $\mathcal{C}$ is algebraic if and only if it is *inductive* i.e. the union of directed families of $\mathcal{C}$- saturated subsets is $\mathcal{C}$-saturated.

One of the aims of this paper is to develop a strictly finitary treatment of Stone formal topologies. For any cover $\lhd$ on S, we define the *finite trace* $\prec_\lhd$ of $\lhd$ as the restriction of $\lhd$ to elements and finite subsets of S:

$$a \prec_\lhd K \equiv a \lhd K \ (a \in S, K \in \mathcal{P}_\omega S).$$

As we shall see, any Stone formal topology is uniquely determined by its behaviour on finite subsets, that is by its finite trace.

Our basic definition is obtained by abstraction on finite traces:

Definition 2.5 *A* Stone base *is a quintuple* $\mathcal{S} \equiv (S, \cdot, 1_\mathcal{S}, \prec_\mathcal{S}, Pos_\mathcal{S})$ *where:*

- $(S, \cdot, 1_\mathcal{S})$ *is a (formal) base;*
- $\prec_\mathcal{S}$ *is a* finitary cover, *i.e. a relation between elements and finite subsets of S formally satisfying the same conditions of covers, i.e. for any* $a, b \in S$ *and* $K, L \in \mathcal{P}_\omega S$:

  $$\frac{a \in K}{a \prec K}\ reflexivity,$$
  $$\frac{a \prec K \quad K \prec L}{a \prec L}\ transitivity,$$
  $$\frac{a \prec K}{a \cdot b \prec K}\ \cdot\text{-}left,$$
  $$\frac{a \prec K \quad a \prec L}{a \prec K \cdot L}\ \cdot\text{-}right;$$

- $Pos_{\mathcal{S}}$ *is a predicate on* S *satisfying monotonicity and positivity w.r.t.* $\prec$, *i.e.:*

  $$\frac{Pos_{\mathcal{S}}(a) \quad a \prec K}{(\exists b \in K) Pos_{\mathcal{S}}(b)} \text{ monotonicity,}$$

  $$\frac{Pos_{\mathcal{S}}(a) \rightarrow a \prec K}{a \prec K} \text{ positivity.}$$

For any formal topology $\mathcal{A}$, the finite trace of $\lhd_{\mathcal{A}}$ determines a Stone base, by the following:

Proposition 2.6 *Let* $\mathcal{A} \equiv (S, \cdot, 1, \lhd, Pos)$ *be a formal topology and let* $\prec_{\lhd}$ *be the finite trace of* $\lhd$. *Then the structure* $\mathbf{S}(\mathcal{A}) \equiv (S, \cdot, 1, \prec_{\lhd}, Pos)$ *is a Stone base.*

Proof: Straightforward from definitions. As for $\cdot R$, observe that $U \cdot V$ is finite whenever U and V are finite. □

Conversely, we have the following:

Proposition 2.7 *Given a Stone base* $\mathcal{S} \equiv (S, \cdot, 1, \prec, Pos)$, *let* $\lhd_{\prec}$ *be the minimal cover with trace* $\prec$, *that is the cover inductively generated by taking* $a \lhd_{\prec} K$ *as axioms whenever* $a \prec K$ *and by closing under the rules for cover, i.e. reflexivity, transitivity,* $\cdot$*-left,* $\cdot$*-right. Then for any* $a \in S$ *and* $U \subseteq S$*:*

$$a \lhd_{\prec} U \text{ iff there exists } K \subseteq_{\omega} U \text{ such that } a \prec K. \tag{1}$$

Therefore $\mathbf{A}(\mathcal{S}) \equiv (S, \cdot, 1, \lhd_{\prec}, Pos)$ *is a Stone formal topology.*

Proof: Trivially, $\lhd_{\prec}$ is a cover.

Suppose there exists $K \subseteq_{\omega} U$ such that $a \prec K$. Then $a \lhd_{\prec} K$ by the axioms; from $K \subseteq_{\omega} U$ it follows by reflexivity $K \lhd_{\prec} U$ and hence $a \lhd_{\prec} U$ by transitivity.

The converse is also simple, but it illustrates the method of induction on covers, in this case on the generation of $\lhd_{\prec}$. For axioms and reflexivity the claim is trivial. Assume $a \lhd_{\prec} V$ is obtained from $a \lhd_{\prec} U$ and $U \lhd_{\prec} V$ by transitivity. By inductive hypothesis, $a \prec K$ for some $K \subseteq_{\omega} U$ and for all $x \in K$, $x \prec L_x$ for some $L_x \subseteq_{\omega} V$. By reflexivity, $L_x \prec \cup_{x \in K} L_x$, hence by transitivity of $\prec$, $x \prec \cup_{x \in K} L_x$ for any $x \in K$, hence $K \prec \cup_{x \in K} L_x$ and finally $a \prec \cup_{x \in K} L_x$ by transitivity. For $\cdot L$ and $\cdot R$ the proof is straightforward.

Since by axioms $a \prec K$ gives $a \lhd_{\prec} K$, $\lhd_{\prec}$ is a Stone cover by (1).

To conclude, it remains only to be shown that Pos satisfies monotonicity and positivity w.r.t. the cover $\lhd_{\prec}$. To prove monotonicity, assume $Pos(a)$ and $a \lhd_{\prec} U$. Then there exists $K \subseteq_{\omega} U$ such that $a \prec K$. By monotonicity of Pos w.r.t. $\prec$, $Pos(K)$ holds, thus a fortiori $Pos(U)$ holds too. Before proving positivity, define, for all $U \subseteq S$, $U^+ \equiv \{a \in U : Pos(a)\}$ and for all $a \in S$ let $a^+ \equiv \{a\}^+$. Then the premiss of positivity, i.e. $Pos(a) \rightarrow a \lhd_{\prec} U$ is equivalent to $a^+ \lhd_\prec U$. Moreover, from $a^+ \prec a^+$, we have $Pos(a) \rightarrow a \prec a^+$, which, by positivity of Pos w.r.t. $\prec$, gives $a \prec a^+$ and therefore $a \lhd_\prec a^+$. This, together with $a^+ \lhd_\prec U$, yields by transitivity $a \lhd_\prec U$. □

Let $\lhd$ and $\lhd'$ be covers on the same base S. We say that $\lhd'$ is a *quotient* of $\lhd$ if for all $a \in S$, $U \subseteq S$,

$$a \lhd U \rightarrow a \lhd' U.$$

This relation corresponds to set-theoretic inclusion between the extensions of $\lhd$ and $\lhd'$, thus we also say that $\lhd'$ is *greater* than $\lhd$, and write $\lhd \leq \lhd'$. Given a cover $\lhd$, let $\prec_\lhd$ be its finite trace. By proposition 2.6 and 2.7, $\prec_\lhd$ generates a cover $\lhd_{\prec_\lhd}$ which is Stone and therefore does not coincide in general with $\lhd$. Indeed $\lhd$ is a quotient of $\lhd_{\prec_\lhd}$. In fact, if $a \lhd_{\prec_\lhd} U$ then there exists $K \subseteq_\omega U$ such that $a \prec_\lhd K$ and therefore, since $\prec_\lhd$ is a restriction of $\lhd$, $a \lhd K$, thus $a \lhd U$. Moreover, $\lhd_{\prec_\lhd}$ is the greatest Stone cover of which $\lhd$ is a quotient. In order to prove this, consider a Stone cover $\lhd'$ of which $\lhd$ is a quotient. Then $a \lhd' U$ implies $a \lhd' K$ for some $K \subseteq_\omega U$, and therefore also $a \lhd K$. Thus, by definition of trace of a cover, $a \prec_\lhd K$ and by definition of cover generated by a trace $a \lhd_{\prec_\lhd} U$.

The cover $\lhd$ coincides with $\lhd_{\prec_\lhd}$ iff $\lhd$ is a Stone cover. In fact, suppose that $\lhd = \lhd_{\prec_\lhd}$. Then $\lhd$ is Stone since $\lhd_{\prec_\lhd}$ is Stone by the above remarks. Conversely, $\lhd_{\prec_\lhd} \leq \lhd$ since $\lhd$ is a quotient of $\lhd_{\prec_\lhd}$. Since $\lhd_{\prec_\lhd}$ is the greatest Stone cover of which $\lhd$ is a quotient, then also the opposite inequality holds and the conclusion follows.

We denote $\lhd_{\prec_\lhd}$ with $\lhd_\omega$ and call it the *Stone compactification* [2] *of* $\lhd$. Then the above discussion proves the following:

Proposition 2.8 *Any cover $\lhd$ admits a Stone compactification $\lhd_\omega$, which is the cover induced by its finite trace, $\lhd_{\prec_\lhd}$. This is the greatest Stone cover of which $\lhd$ is a quotient and it coincides with $\lhd$ iff $\lhd$ is a Stone cover.*

We also have:

Corollary 2.9 *A cover $\lhd$ is a Stone cover iff it is finitarily axiomatizable, i.e. there exists a relation R between elements and finite subsets of S such that $\lhd$ coincides with the cover $\lhd_R$ generated by R, which is obtained from the axioms $a \lhd_R U$ whenever $R(a, U)$ holds, by closing under the rules for covers.*

For any formal topology $\mathcal{A}$, let $\mathcal{A}_\omega \equiv \mathbf{AS}(\mathcal{A})$, that is, by definitions of $\mathbf{A}$, $\mathbf{S}$ and $\lhd_\omega$, $\mathcal{A}_\omega \equiv (S.\cdot, 1, \lhd_\omega, Pos)$. Then, by propositions 2.6 and 2.7, $\mathcal{A}_\omega$ is a Stone formal topology, called the *Stone compactification of* $\mathcal{A}$.

Given two formal topologies with the same base monoid and the same positivity predicate, $\mathcal{A} \equiv (S, \cdot, 1, \lhd, Pos)$ and $\mathcal{A}' \equiv (S, \cdot, 1, \lhd', Pos)$, we say that $\mathcal{A}$ is a quotient of $\mathcal{A}'$ if the cover $\lhd'$ is quotient of the cover $\lhd$.

Summing up we have proved:

[2] Not to be confused with the familiar Stone-Čech compactification!

Theorem 2.10 *Any formal topology $\mathcal{A} \equiv (S, \cdot, 1, \lhd, Pos)$ admits a Stone compactification $\mathcal{A}_\omega \equiv (S, \cdot, 1, \lhd_\omega, Pos)$. This is the greatest Stone formal topology of which $\mathcal{A}$ is a quotient. Moreover $\mathcal{A} = \mathcal{A}_\omega$ iff $\mathcal{A}$ is a Stone formal topology.*

An important corollary of the above results is the fact that any Stone formal topology is determined by its finite trace, which is a relation between elements and finite subsets of S and is therefore completely expressible inside the foundational framework of ITT.

The definition of morphism between Stone bases is obtained in the natural way, similarly to the definition of morphism between formal topologies, except for the fact that an element is always mapped into a finite subset of the base. For any Stone base $\mathcal{S}$, let $=_\mathcal{S}$ denote the equivalence relation on $\mathcal{P}_\omega S$ defined by $K =_\mathcal{S} L \equiv K \prec_\mathcal{S} L \;\&\; L \prec_\mathcal{S} K$. Then we can define:

Definition 2.11 *A morphism between two Stone bases $\mathcal{S} \equiv (S, \cdot, 1_\mathcal{S}, \prec_\mathcal{S}, Pos_\mathcal{S})$ and $\mathcal{T} \equiv (T, \cdot, 1_\mathcal{T}, \prec_\mathcal{T}, Pos_\mathcal{T})$ is an application $f : S \to \mathcal{P}_\omega T$ formally satisfying the same conditions of morphisms between formal topologies, i.e.*

1. $f(1_\mathcal{S}) =_\mathcal{T} 1_\mathcal{T}$;
2. $f(a \cdot b) =_\mathcal{T} f(a) \cdot f(b)$;
3. $a \prec_\mathcal{S} K \to f(a) \prec_\mathcal{T} f(K)$;
4. $Pos_\mathcal{T}(f(a)) \to Pos_\mathcal{S}(a)$.

Likewise, equality of morphisms $f, g : \mathcal{S} \to \mathcal{T}$ is defined by $f = g \equiv (\forall a \in S)(f(a) =_\mathcal{T} g(a))$.

As for formal topologies, composition of morphisms $f : \mathcal{S} \to \mathcal{T}$ and $g : \mathcal{T} \to \mathcal{V}$ is defined by $(g \circ f)(a) \equiv g(f(a))$ and the identity $1 : \mathcal{S} \to \mathcal{S}$ is the morphism mapping each element into its singleton, that is $1(a) \equiv \{a\}$. Thus Stone bases form a category which we denote with **SB**.

Let **SFTop*** be the category of Stone formal topologies with *coherent morphisms* i.e. morphisms from $\mathcal{A}$ to $\mathcal{B}$ induced by applications mapping elements of the base of $\mathcal{A}$ into finite subsets of the base of $\mathcal{B}$. Then we can prove that the bijective correspondence determined by **A** and **S** between Stone bases and Stone formal topologies extends to an equivalence between **SFTop*** and **SB**.

Observe that any coherent morphism f between Stone formal topologies $\mathcal{A}$ and $\mathcal{B}$ *is* a morphism between the Stone bases $\mathbf{S}(\mathcal{A})$ and $\mathbf{S}(\mathcal{B})$. In fact, condition *1* and *2* hold since on finite subsets $=_\mathcal{B}$ and $=_{\mathbf{S}(\mathcal{B})}$ coincide. As for condition *3*, suppose that $a \prec_{\lhd_\mathcal{A}} K$. Then by definition $a \lhd_\mathcal{A} K$. Since f is a morphism in **SFTop***, $f(a) \lhd_\mathcal{B} f(K)$, and since $f(K)$ is finite, $f(a) \lhd_{\prec_{\lhd_\mathcal{B}}} f(K)$. Condition *4* holds trivially since the positivity predicate is the same.

Conversely, any morphism g between Stone bases $\mathcal{S}$ and $\mathcal{T}$ *is* a morphism between the Stone formal topologies $\mathbf{A}(\mathcal{S})$ and $\mathbf{A}(\mathcal{T})$. The proof is trivial, except for the third condition. In fact, let $a \lhd_{\prec_\mathcal{S}} U$. Then $a \prec_\mathcal{S} K$ for some

$K \subseteq_\omega U$ and therefore, since g is a morphism of Stone bases, $g(a) \lhd_{\prec_\mathcal{T}} g(K)$. Since $g(K) \subseteq_\omega g(U)$, then $g(a) \lhd_{\prec_\mathcal{T}} g(U)$.

Thus **A** and **S** extend to functors which are the identity on morphisms between **SB** and **SFTop***. Then the functor **A** is a bijection on morphisms, i.e. is full and faithful. Moreover the functor is dense, since for all Stone formal topologies $\mathcal{A}$, $\mathcal{A} \cong \mathbf{A}(\mathbf{S}(\mathcal{A}))$.

We have therefore proved:

Theorem 2.12 *The category* **SFTop*** *of Stone formal topologies with coherent morphisms is equivalent to the category* **SB** *of Stone bases.*

Observe that the proof of this equivalence, as well as the proofs of the other equivalences throughout the paper, is constructive since we actually define the inverse functor, in this case **S**. Moreover the constructive character of the category **SB** consists in the fact that objects are types and morphisms are functions between types in ITT. Anyway, it would be easy to give a more general notion of morphism between Stone bases, with $f : S \to \mathcal{P}T$, in such a way that a category equivalent to the whole of **SFTop** is obtained.

3 Pointfree representation of coherent frames and distributive lattices

In this section we will show how Stone bases give a constructive representation of distributive lattices and relate this with the usual pointfree version of Stone representation, based on coherent frames (see [J], II.3.4).

Since Stone bases are endowed with an intuitionistic predicate of positivity, we will deal with a richer structure than that of distributive lattices, namely distributive lattices with a predicate which reflects the properties of Pos. We say that $\cdot\,\#0$ is a predicate of apartness from 0 in a distributive lattice L if the following properties hold for $a \in L$ and $K \subseteq_\omega L$:

$$\frac{a\#0 \quad a \leq \vee K}{(\exists b \in K)(b\#0)}\ (m), \qquad\qquad \frac{a\#0 \to a \leq \vee K}{a \leq \vee K}\ (p).$$

The name "apartness" is justified by the fact that the predicate $\cdot\,\#0$ is easily seen to satisfy the following

$$\neg(a\#0) \leftrightarrow a = 0, \tag{2}$$

which is an instance of one of the defining properties of apartness (cf. [TvD]). Thus a lattice with apartness from 0 has a *stable* equality to 0, i.e. $\neg\neg(a = 0)$ holds iff $a = 0$ holds. On the other hand, we can prove *classically* that the predicate defined by $a\#0 \equiv \neg(a = 0)$ is a predicate of apartness from 0. This is the coarsest predicate of apartness from 0 since, for every such predicate $\cdot\,\#0$ and for all $a \in L$, it follows from (2) that $a\#0 \to \neg(a = 0)$.

A morphism between distributive lattices with apartness $f : (L_1, \#_1) \to (L_2, \#_2)$ is a lattice morphism such that for all $a \in L_1$, $f(a)\#_2 0 \to a\#_1 0$. The category of distributive lattices with apartness will be denoted with $\mathbf{DLat}_\#$.

Given a distributive lattice with apartness from 0, $(L, \#)$, let $\mathcal{S}_L$ be the *Stone base associated to* L, defined by $\mathcal{S}_L \equiv (L, \wedge, 1, \prec_L, Pos_L)$, with

$$\begin{array}{rcl} a \prec_L K & \equiv & a \leq \vee K \\ Pos_L(a) & \equiv & a\#0. \end{array}$$

Conversely, given a Stone base $\mathcal{S} \equiv (S, \cdot, 1, \prec, Pos)$, we obtain a distributive lattice with apartness from 0 defining $Sat(\mathcal{S}) \equiv \{\mathcal{S}K : K \subseteq_\omega S\}$, where $\mathcal{S}K \equiv \{a \in S : a \prec K\}$. Then $Sat(\mathcal{S})$, with operations given by

$$\begin{array}{rcl} \mathcal{S}K \vee \mathcal{S}L & \equiv & \mathcal{S}(K \cup L) \\ \mathcal{S}K \wedge \mathcal{S}L & \equiv & \mathcal{S}K \cap \mathcal{S}L \end{array}$$

top and bottom $\mathcal{S}\{1\}$ and $\mathcal{S}\emptyset$ respectively and apartness $\mathcal{S}K\#0 \equiv Pos(K)$, is a distributive lattice with apartness from 0. $Sat(\mathcal{S})$ is called the lattice of *saturated subsets* of the Stone base $\mathcal{S}$.

Then we have:

Proposition 3.1 *Any distributive lattice L is isomorphic to the lattice of saturated subsets of the Stone base $\mathcal{S}_L$.*

Proof: First observe that any saturated subset of $\mathcal{S}_L$ is the saturation of a singleton, since for all $K \subseteq_\omega L$, $\mathcal{S}K = \mathcal{S}\{\vee K\}$. Therefore the map

$$\begin{array}{rccc} \phi : & L & \to & Sat(\mathcal{S}_L) \\ & a & \longmapsto & \mathcal{S}\{a\} \end{array}$$

is surjective. Since $S\{a\} = \{b \in L : b \leq a\}$, which will be denoted with $\downarrow a$ for short, ϕ is clearly injective. This map is a morphism of lattices because $\phi(a \vee b) \equiv \mathcal{S}\{a \vee b\} = \mathcal{S}\{a, b\} \equiv \mathcal{S}\{a\} \vee \mathcal{S}\{b\}$; then ϕ preserves meet since $\downarrow (a \wedge b) = (\downarrow a) \cap (\downarrow b)$. The top is preserved by definition and so does the bottom since $\mathcal{S}\{0\} = \mathcal{S}\emptyset$. Finally ϕ is an isomorphism of lattices with apartness since for all $a \in L$, $\mathcal{S}\{a\}\#0$ iff $Pos_L(a)$ iff $a\#0$. □

Let $f : \mathcal{S} \to \mathcal{T}$ be a morphism of Stone bases. Then it is easy to prove that the map

$$\begin{array}{rccc} Sat(f) : & Sat(\mathcal{S}) & \to & Sat(\mathcal{T}) \\ & \mathcal{S}K & \mapsto & \mathcal{T}(f(K)) \end{array}$$

is a morphism of lattices with apartness, and functoriality is straightforward. By the above proposition, *Sat* is a dense functor. Moreover we have:

Lemma 3.2 *The functor Sat establishes a bijective correspondence between* $\mathbf{SB}(\mathcal{S}, \mathcal{T})$ *and* $\mathbf{DLat}_\#(Sat(\mathcal{S}), Sat(\mathcal{T}))$.

Proof: Let $f, g \in$**SB**$(\mathcal{S}, \mathcal{T})$ and suppose $Sat(f) = Sat(g)$. Then for all $a \in S$, $Sat(f)(\mathcal{S}\{a\}) = Sat(g)(\mathcal{S}\{a\})$, that is $\mathcal{T}f(a) = \mathcal{T}g(a)$, i.e. $f(a) =_{\mathcal{T}} g(a)$. By definition of equality between coherent morphisms, $f = g$, thus the correspondence is 1-1. As for surjectivity, let $h \in$**DLat**$_{\#}(Sat(\mathcal{S}), Sat(\mathcal{T}))$. Since for all $a \in S$, $h(\mathcal{S}\{a\}) \in Sat(\mathcal{T})$, then we know [3] that $h(\mathcal{S}\{a\}) = \mathcal{T}(K_a)$, for a suitable finite subset K_a of T. Define $f : S \to \mathcal{P}_\omega T$ by $f(a) \equiv K_a$. It is then routine to verify that f is a morphism of Stone bases. □

The above lemma and theorem 2.12 prove the following:

Theorem 3.3 *The category* **DLat**$_{\#}$ *of distributive lattices with apartness is equivalent to the category* **SB** *of Stone bases with coherent morphisms and to the category* **SFTop*** *of Stone formal topologies with coherent morphisms.*

Before going on with the representation of distributive lattices by means of coherent frames, we introduce the category of frames with apartness from 0. Due to the presence of arbitrary suprema, the notion of frame cannot be directly formalized in ITT. This is the reason why we base our approach to pointfree topology on the equivalent notion of formal topology, which is expressible in the ITT framework. In fact, we will provide an extension of the representation theorem for frames (cf. [BS]) by showing that the category of frames with apartness from 0 is equivalent to the category of formal topologies.

Apartness from 0 in a frame is defined like apartness from 0 in a lattice, except that arbitrary suprema must be considered, i.e. we require:

$$\frac{a\#0 \quad a \leq \vee U}{(\exists b \in U)(b\#0)}\ (m), \qquad \frac{a\#0 \to a \leq \vee U}{a \leq \vee U}\ (p).$$

Given a predicate $\cdot\ \#0$ of apartness from 0 on a base S of a frame F, we can extend it to the whole frame F by defining, for $a \in F$,

$$a\#_F 0 \equiv (\exists b \in S)(b \leq a \ \&\ b\#0).$$

In fact it is easy to prove that for all $a \in S$, $a\#_F 0$ iff $a\#0$, i.e. $\cdot\ \#_F 0$ extends $\cdot\ \#0$, and that $\cdot\ \#_F 0$ is a predicate of apartness from 0 as well.

We say that a morphisms of frames is a morphism between two given frames with apartness from 0, $(F_1, \#_1)$ and $(F_2, \#_2)$, if for all $a \in S_1$ (base of F_1), $f(a)\#_2 0 \to a\#_1 0$. We will denote with **Frm**$_{\#}$ the category thus obtained and call it the category of frames with apartness.

Let Sat be the functor which associates to a formal topology $\mathcal{A} \equiv (S, \cdot, 1, \lhd, Pos)$ the frame of its saturated subsets, $Sat(\mathcal{A}) \equiv \{\mathcal{A}U : U \subseteq S\}$ with operations of finite meet and arbitrary join given by

$$\begin{aligned} \mathcal{A}U \wedge \mathcal{A}V &\equiv \mathcal{A}U \cap \mathcal{A}V, \\ \vee_{i \in I} \mathcal{A}U_i &\equiv \mathcal{A}(\cup_{i \in I} U_i). \end{aligned}$$

[3] A careful formalization in ITT would substitute $Sat(\mathcal{S})$, which is not a set but a set-indexed family, with $(\mathcal{P}_\omega(S), =_S)$, which is a set together with an equality relation and which is isomorphic to $Sat(\mathcal{S})$. In particular, this step of the proof would be more explicit.

A base of $Sat(\mathcal{A})$ is given by the saturations of singletons from S, i.e. $\{\mathcal{A}\{a\} : a \in S\}$. On such a base a predicate of apartness from 0 defined by:

$$\mathcal{A}\{a\}\#_{Sat}0 \equiv Pos(a).$$

If $f : \mathcal{A} \to \mathcal{B}$ is a morphism of formal topologies, then $Sat(f) : Sat(\mathcal{A}) \to Sat(\mathcal{B})$ is defined by

$$Sat(f)(\mathcal{A}U) \equiv \mathcal{B}(f(U))$$

and it is seen to be a morphism between frames with apartness.

The functor Sat is dense since, for all $(F, \#) \in$**Frm**$_\#$, we can construct a formal topology $\mathcal{A}$ such that $(F, \#) \cong (Sat(\mathcal{A}), \#_{Sat})$. Let S be a base of F and define, for all $a \in S$, $U \subseteq S$:

$$\begin{array}{rcl} a \lhd U & \equiv & a \leq \vee U, \\ Pos(a) & \equiv & a\#0. \end{array}$$

Then $\mathcal{A} \equiv (S, \cdot, 1, \lhd, Pos)$ is a formal topology and the map

$$\begin{array}{ccc} (Sat(\mathcal{A}), \#_{Sat}) & \to & (F, \#) \\ \mathcal{A}U & \mapsto & \vee U \end{array}$$

establishes an isomorphism of frames with apartness. Moreover the assignment

$$Sat : \begin{array}{ccc} \mathbf{FTop}(\mathcal{A}, \mathcal{B}) & \to & \mathbf{Frm}_\#(Sat(\mathcal{A}), Sat(\mathcal{B})) \\ f & \mapsto & Sat(f) \end{array}$$

is a bijection, i.e. Sat is full and faithful. We may thus conclude:

Theorem 3.4 *The category of formal topologies* **FTop** *is equivalent to the category of frames with apartness from* 0 **Frm**$_\#$*.*

We will obtain the equivalence between **SFTop*** and the category of coherent frames as a special case of theorem 3.4. We first recall from the literature some definitions and basic properties concerning coherent frames (cf. [J]).

Definition 3.5 *Let F be a frame. We say that an element a of F is* compact *(*finite*) if for every $U \subseteq F$ such that $a \leq \vee U$ there exists $K \subseteq_\omega U$ such that $a \leq \vee K$.*

If F is the frame of open sets of a topological space, we obtain the usual definition of compact open. Compact elements of a frame form a join semilattice, in fact we have:

Lemma 3.6 *Let F be a frame. Then:*

1. *0 is compact;*
2. *If a and b are compact, then $a \vee b$ is compact.*

In general, compact elements of a frame are not closed under meet. This fact justifies the following definition:

Definition 3.7 *A frame F is* coherent *if:*

1. *The compact elements $K(F)$ of F form a sublattice of F;*
2. *Every element of F is a join of compact elements, i.e. $K(F)$ generates F.*

Coherent frames can be made into a category **CohFrm** whose objects are coherent frames and whose morphism $f : F_1 \to F_2$ are coherent morphisms of frames, i.e. maps preserving finite meets and arbitrary joins and mapping $K(F_1)$ into $K(F_2)$. By endowing coherent frames with a predicate of apartness from 0, we obtain the category $\mathbf{CohFrm}_{\#}$ of coherent frames with apartness.

We are now going to prove that the equivalence stated in 3.4 restricts to an equivalence between $\mathbf{SFTop}^*$ and $\mathbf{CohFrm}_{\#}$. This will permit us to use the notion of Stone formal topology (or equivalently of Stone base) as the version expressible inside ITT of the notion of coherent frame with apartness from 0.

In order to prove that for any Stone formal topology $\mathcal{A}$, $Sat(\mathcal{A})$ is a coherent frame, we start with a preliminary lemma characterizing compact elements of $Sat(\mathcal{A})$ in terms of covers.

Lemma 3.8 *Let $\mathcal{A}$ be a formal topology and let $U \in Sat(\mathcal{A})$. Then U is compact in $Sat(\mathcal{A})$ iff for all $V \in Sat(\mathcal{A})$,*

$$U \lhd V \to U \lhd L$$

for a suitable finite subset L of V.

Proof: Suppose that U is compact in $Sat(\mathcal{A})$ and assume $U \lhd V$. Since $\mathcal{A}V = \vee_{a \in V} \mathcal{A}\{a\}$, this hypothesis can be rephrased by the inequality $U \leq \vee_{a \in V} \mathcal{A}\{a\}$ holding in the frame $Sat(\mathcal{A})$. By compactness of U in $Sat(\mathcal{A})$, there exists $L \subseteq_\omega V$ such that $U \leq \vee_{a \in L} \mathcal{A}(a)$, which is equivalent to $U \lhd L$. Conversely, let $U \leq \vee_{i \in I} V_i$, where $V_i \in Sat(\mathcal{A})$. Then $U \lhd \cup_{i \in I} V_i$, hence by hypothesis there exists $L \subseteq_\omega \cup_{i \in I} V_i$ with $U \lhd L$. Let I_0 be the finite subset of I such that $L \subseteq \cup_{i \in I_0} V_i$. Then $U \lhd \cup_{i \in I_0} V_i$, that is $U \leq \vee_{i \in I_0} V_i$, hence U is compact in the frame $Sat(\mathcal{A})$. □

As a corollary, let U be compact in $Sat(\mathcal{A})$. Then there exists $K \subseteq_\omega U$ such that $U = \mathcal{A}(K)$. In fact, by reflexivity, $U \lhd U$, and thus, by the above lemma, there exists $K \subseteq_\omega U$ such that $U \lhd K$. Since the opposite cover relation holds in general, we have $U =_{\mathcal{A}} K$, that is, as U is $\mathcal{A}$-saturated, $U = \mathcal{A}(K)$.

Observe however that the converse need not hold, i.e. U may be the saturation of a finite subset of S without being compact.

Anyway, if the formal topology $\mathcal{A}$ is Stone, the saturations of the singletons are compact, by lemma 3.8 and definition of Stone cover. Thus, by lemma 3.6, the saturation of any finite subset of S is compact. We may thus conclude:

Proposition 3.9 *Let $\mathcal{A}$ be a Stone formal topology. Then U is compact in $Sat(\mathcal{A})$ iff there exists $K \subseteq_\omega U$ such that $U = \mathcal{A}(K)$.*

We define $Sat_\omega(\mathcal{A}) \equiv \{\mathcal{A}K : K \subseteq_\omega S\}$. Then the following result is easily obtained:

Proposition 3.10 *Let $\mathcal{A}$ be a formal topology. Then $\mathcal{A}$ is a Stone formal topology iff the compact elements of $Sat(\mathcal{A})$ coincide with $Sat_\omega(\mathcal{A})$.*

We are now in the position to prove:

Theorem 3.11 *For any Stone formal topology $\mathcal{A}$, $Sat(\mathcal{A})$ is a coherent frame.*

Proof: In order to prove that $Sat_\omega(\mathcal{A})$ is a sublattice of $Sat(\mathcal{A})$, we have to check closure under finite meet. Of course the top element of $Sat(\mathcal{A})$, $\mathcal{A}\{1\}$, is compact. Then suppose U, $V \in Sat_\omega(\mathcal{A})$. By proposition 3.9, there exist $K \subseteq_\omega U$ and $L \subseteq_\omega V$ such that $U = \mathcal{A}K$ and $V = \mathcal{A}L$. Recall that in any formal topology $\mathcal{A}$ the equality $\mathcal{A}U \cap \mathcal{A}V = \mathcal{A}(U \cdot V)$ holds for all $U, V \subseteq S$. Then $U \wedge V \equiv U \cap V = \mathcal{A}(U \cdot V)$, and also $U \wedge V = \mathcal{A}(K \cdot L)$, where $K \cdot L \subseteq_\omega U \cdot V$. Therefore, by proposition 3.9 again, $U \wedge V$ is compact. Finally, by lemma 2.4, $Sat_\omega(\mathcal{A})$ generates $Sat(\mathcal{A})$. $\square$

Observe that $\mathcal{S}$-saturation and $\mathbf{A}(\mathcal{S})$-saturation coincide on finite subsets of S, that is for all $K \subseteq_\omega S$, $\mathcal{S}K = (\mathbf{A}(\mathcal{S}))K$. Thus the compact elements of the coherent frame $Sat(\mathbf{A}(\mathcal{S}))$ are precisely the elements of $Sat(\mathcal{S})$. Since $Sat(\mathcal{S})$ generates (by closure under directed unions) the coherent frame with apartness $Sat(\mathbf{A}(\mathcal{S}))$, proposition 3.1 together with theorem 3.11 gives:

Corollary 3.12 Pointfree Stone's representation theorem for distributive lattices. *Any distributive lattice (with apartness) is isomorphic to the compact elements of a coherent frame (with apartness).*

The reader should notice that the proof of the above pointfree Stone representation theorem is completely constructive, and no prime ideal theorem or equivalents are involved.

We are now in the position to prove the converse of 3.11, i.e. that any coherent frame with apartness is isomorphic to $Sat(\mathcal{A})$ for a suitable Stone formal topology $\mathcal{A}$. Let F be a coherent frame with apartness $\cdot \# 0$, and consider the Stone base $\mathcal{S}_L$ associated to the sublattice L of compact elements of F. By the results in section 1, a Stone formal topology $\mathcal{A}$ is obtained by defining $\mathcal{A} \equiv \mathbf{A}(\mathcal{S}_L)$. Observe that for all $a \in L$, $U \subseteq L$, $a \lhd_{\mathcal{A}} U$ holds iff there exists $K \subseteq_\omega U$ such that $a \leq \vee K$ that is iff $a \in \mathcal{I}(U)$, where $\mathcal{I}(U)$ denotes the ideal generated by U in L. Thus $Sat(\mathcal{A})$ coincides with the frame of ideals on L, denoted $Idl(L)$. Consider now the map

$$\begin{array}{rccc} \phi: & F & \rightarrow & Sat(\mathcal{A}) \\ & a & \mapsto & \downarrow_L a \equiv \{b \in L : b \leq a\}. \end{array}$$

Since L generates F, ϕ is injective. As for surjectivity, observe that for all $\mathcal{A}$-saturated subsets $\mathcal{A}U$, $\mathcal{A}U = \downarrow \vee U$ since for any $b \in L$, $b \in \mathcal{A}U$ iff there exists $K \subseteq_\omega U$ such that $b \leq \vee K$ iff (by compactness of b) $b \leq \vee U$.

The bijection ϕ is a frame morphism since it is an order preserving map with order preserving inverse, given by $\psi(I) \equiv \vee I$ for all $I \in Idl(L)$. Indeed ϕ is a morphism of frames with apartness since for all $a \in K(F)$, $\phi(a) \#_{Sat} 0$ iff $\downarrow_L a \#_{Sat} 0$ iff $a \#_F 0$. We can summarize the above result with the following:

Proposition 3.13 *For any coherent frame with apartness F there exists a Stone formal topology $\mathcal{A}$ such that $F \cong Sat(\mathcal{A})$. Such formal topology $\mathcal{A}$ is given by the topology of ideals on the distributive lattice of compact elements of F.*

If F is the coherent frame of saturated subsets of a Stone formal topology, by using the characterization of compact elements of $Sat(\mathcal{A})$ provided in 3.10, we obtain:

Corollary 3.14 *If $\mathcal{A}$ is a Stone formal topology, the map*

$$\begin{array}{rlcl} \phi & : Sat(\mathcal{A}) & \to & Idl(Sat_\omega(\mathcal{A})) \\ & U & \mapsto & \downarrow U \equiv \{\mathcal{A}K : K \subseteq_\omega U\} \end{array}$$

is a frame isomorphism with inverse $\psi(I) \equiv \vee I$.

Proposition 3.13 together with theorem 3.11 gives (by observing that coherence is preserved under frame isomorphism):

Theorem 3.15 *Let F be a frame with apartness from 0. Then F is coherent iff there exists a Stone formal topology $\mathcal{A}$ such that $F \cong Sat(\mathcal{A})$.*

By proposition 3.13, $Sat : \mathbf{SFTop}^* \to \mathbf{CohFrm}_\#$ is a dense functor; moreover it is easy to see that the bijection on morphisms given by theorem 3.4 restricts to a bijection between $\mathbf{SFTop}^*(\mathcal{A}, \mathcal{B})$ and $\mathbf{CohFrm}_\#(Sat(\mathcal{A}), Sat(\mathcal{B}))$. Therefore we have:

Theorem 3.16 *The category $\mathbf{CohFrm}_\#$ of coherent frames with apartness is equivalent to the category $\mathbf{Sftop}^*$ of Stone formal topologies with coherent morphisms.*

The above result gives, together with theorem 2.12 and theorem 3.3:

Corollary 3.17 *The category $\mathbf{CohFrm}_\#$ of coherent frames with apartness is equivalent to the category $\mathbf{SB}$ of Stone bases and to the category $\mathbf{DLat}_\#$ of distributive lattices with apartness from 0.*

Finally, we observe that in all the categories introduced so far, morphisms between two given objects can be partially ordered pointwise and moreover the functors yielding the equivalences preserve this order. This means that all the equivalences are indeed equivalences of **2**-categories.

4 Stone formal spaces and topological representation

In order to connect our pointfree approach to representation theory with the traditional one, the notion of point has to be recovered. Since we reverse the usual conceptual order between points and opens, and take the opens as *primitive*, points will be defined as particular, well behaved, collections of opens. We recall here the definition of (formal) point on a formal topology, and specialize it to the Stone case.

Definition 4.1 *Let $\mathcal{A} \equiv (S, \cdot, 1, \lhd, Pos)$ be a formal topology. A subset α of S is said to be a* formal point *if for all $a, b \in S$, $U \subseteq S$ the following conditions hold:*

1. $1 \in \alpha$;
2. $\dfrac{a \in \alpha \quad b \in \alpha}{a \cdot b \in \alpha}$;
3. $\dfrac{a \in \alpha \quad a \lhd U}{(\exists b \in U)(b \in \alpha)}$;
4. $\dfrac{a \in \alpha}{Pos(a)}$.

Observe that this definition connects to the usual intuition on points by reading "$a \in \alpha$" as "α is a point of a", 1 as the whole space, product as intersection, "$a \lhd U$" as "a is included in the union of U", and "$Pos(a)$" as "a is inhabited", that is a positive way of saying that a is not empty.[4]

Now let $\mathcal{A}$ be a Stone formal topology. Then, by definition, $a \lhd U$ holds iff there exists a finite subset K of U such that $a \lhd K$. Then condition *3* above can be weakened to the following:

3'. $\dfrac{a \in \alpha \quad a \lhd K}{(\exists b \in K)(b \in \alpha)}$ for all $a \in S,\ K \subseteq_\omega S$.

and thus points can be defined directly in terms of the Stone base $\mathbf{S}(\mathcal{A})$.

The definition of formal point of a Stone formal topology can be rewritten in such a way that it results in the more familiar notion of prime filter on a suitable lattice.

The notion of filter in a lattice is defined as usual. In a lattice L endowed with a predicate of apartness from 0, the property that a filter F is proper is expressed in a stronger form by

$$a \in F \to a\#0.$$

[4] For a more exhaustive treatment of this topic cf. [S2].

As usual, a filter F is said to be *prime* if for all $a, b_i \in L$

$$a \in F, a \leq b_1 \vee \ldots \vee b_n \rightarrow (\exists i \leq n)(b_i \in F).$$

As announced we have:

Proposition 4.2 *There exists an order preserving bijection between formal points on a Stone formal topology $\mathcal{A}$ and prime filters on $Sat_\omega(\mathcal{A})$.*

Proof: Let α be a formal point on $\mathcal{A}$ and define $\hat{\alpha} \equiv \{\mathcal{A}\{a\} : a \in \alpha\}$. Then $\hat{\alpha}$ is a prime filter on $Sat_\omega(\mathcal{A})$. In fact $\mathcal{A}\{1\} \in \hat{\alpha}$ since $1 \in \alpha$. It is closed under meet since $\mathcal{A}\{a\} \wedge \mathcal{A}\{b\} = \mathcal{A}\{a \cdot b\}$ and α is closed under product. Then suppose $\mathcal{A}\{a\} \in \hat{\alpha}$ and $\mathcal{A}\{a\} \leq \mathcal{A}\{b_1\} \vee \ldots \vee \mathcal{A}\{b_n\}$, that is $a \in \alpha$ and $a \lhd \{b_1, \ldots, b_n\}$. Since α is a formal point, there exists $i \leq n$ such that $b_i \in \alpha$, that is such that $\mathcal{A}\{b_i\} \in \hat{\alpha}$. Finally, if $\mathcal{A}\{a\} \in \hat{\alpha}$, then $a \in \alpha$, hence $Pos(\{a\})$ holds and therefore $\mathcal{A}\{a\} \#_{Sat} 0$.

Similarly one proves that if F is a prime filter on $Sat_\omega(\mathcal{A})$, then $F^\circ \equiv \{a \in S : \mathcal{A}\{a\} \in F\}$ is a formal point on $\mathcal{A}$. The correspondence is clearly order preserving and it is bijective since, for all formal points α on $\mathcal{A}$ and for all filters F on $Sat_\omega(\mathcal{A})$, the equalities $(\hat{\alpha})^\circ = \alpha$ and $\hat{F^\circ} = F$ hold by definitions. □

For any formal topology $\mathcal{A}$, the formal space $Pt(\mathcal{A})$ of formal points on $\mathcal{A}$ can be endowed with a topology, called the *extensional topology*. Define, for $a \in S$

$$ext(a) \equiv \{\alpha \in Pt(\mathcal{A}) : a \in \alpha\}.$$

The family $\{ext(a)\}_{a \in S}$ is a base for a topology on $Pt(\mathcal{A})$. In fact, from definition of formal point we have $ext(1) = Pt(\mathcal{A})$, thus the whole space is a basic open, and $ext(a) \cap ext(b) = ext(a \cdot b)$, thus the family is closed under intersection. If we denote $\cup_{b \in U} ext(b)$ with $ext(U)$, then the generic open is of the form $ext(U)$ for $U \subseteq S$.

Let $\Omega Pt(\mathcal{A})$ be the topology so obtained. Then the map

$$\begin{array}{cccc} \phi : & Sat(\mathcal{A}) & \rightarrow & \Omega Pt(\mathcal{A}) \\ & U & \longmapsto & ext(U) \end{array}$$

is clearly a surjective frame homomorphism, therefore it is a frame isomorphism iff it is injective. Injectivity of ϕ amounts to the condition

$$ext(U) = ext(V) \rightarrow U = V$$

which is equivalent to $ext(U) \subseteq ext(V) \rightarrow U \subseteq V$, which in turn is equivalent to

$$ext(a) \subseteq ext(V) \rightarrow a \in V. \tag{3}$$

When condition (3) holds, we say that the formal topology is *extensional* or *has enough points*, since it is classically equivalent to

$$a \notin V \rightarrow (\exists \alpha \in Pt(\mathcal{A}))(a \in \alpha \ \& \ V \cap \alpha = \emptyset)$$

which intuitively says that formal points are enough to "separate" different opens.

If $\mathcal{A}$ is a Stone formal topology extensionality can be proved by admitting classical logic and the prime filter theorem. Indeed, by corollary 3.14 and by proposition 4.2, respectively characterizing saturated subsets of $\mathcal{A}$ as ideals on $Sat_\omega(\mathcal{A})$ and points on $\mathcal{A}$ as prime filters on $Sat_\omega(\mathcal{A})$, the above condition holds iff for all ideals I of $Sat_\omega(\mathcal{A})$ and for all $a \in Sat_\omega(\mathcal{A})$, $a \notin I$, there exists a prime filter F such that $a \in F$ and $F \cap I = \emptyset$. The reader may here recognize a well known non constructive principle (cf. e.g. [G]):

Theorem 4.3 The Prime Filter Theorem. *Let I be an ideal of a distributive lattice L and let a be an element of L such that $a \notin I$. Then there exists a prime filter F of L such that $I \subseteq L$ and $F \cap I = \emptyset$*

With the above discussion we have proved:

Theorem 4.4 *Any Stone formal topology $\mathcal{A}$ is extensional iff the Prime Filter Theorem holds.*

We say that a topological space $(X, \Omega X)$ is *sober* if it is homeomorphic to the space of formal points on its frame of opens, that is if $X \cong Pt(\Omega X)$ where the right hand side is provided with the extensional topology. A sober space is said to be *coherent* (cf. [J]) if the collection $K(\Omega X)$ of compact opens of X is closed under intersection and forms a base for ΩX. Thus, simply by "adding points" to our pointfree results of section 3, we obtain:

Theorem 4.5 Stone representation theorem for distributive lattices. *Any distributive lattice with apartness is isomorphic to the lattice of compact opens of a coherent topological space.*

Proof: By corollary 3.12 any distributive lattice with apartness L is isomorphic to the compact elements of a coherent frame with apartness, thus, by proposition 3.10, to the compact saturated subsets of a Stone formal topology $\mathcal{A}$, that is

$$L \cong K(Sat(\mathcal{A})).$$

By theorem 4.4 such an $\mathcal{A}$ is extensional and therefore there exists a frame isomorphism between $Sat(\mathcal{A})$ and $\Omega Pt(\mathcal{A})$. Since compactness is preserved under frame isomorphisms (and compact opens of a space X are just the compact elements of the frame ΩX) the claim is proved. □

The above result can be formulated in a categorical language. Given two coherent spaces $(X, \Omega X)$ and $(Y, \Omega Y)$, a map $f : X \to Y$ is called a *coherent* map if f^{-1} is a coherent frame morphism between ΩY and ΩX. Coherent spaces with coherent maps form a category which we denote with **CohSpa**. By a basic result in pointfree topology, the category of sober spaces is dual to the category of *spatial* frames, that is frames arising from extensional formal topologies. Thus, by restriction of the equivalence, the category of coherent spaces with coherent maps is dual to the category of coherent frames with coherent morphism, which is spatial by 4.4. Then, by composing this duality with the equivalence given by theorem 3.17, we have:

Theorem 4.6 *The category* $\mathbf{DLat}_{\#}$ *of distributive lattices with apartness is dual to the category* **CohSpa** *of coherent spaces with coherent maps.*

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References

[BS] G. Battilotti, G. Sambin. *Pretopologies and a uniform presentation of sup-lattices, quantales and frames*, to appear.

[FG] M. P. Fourman, R.J. Grayson. *Formal Spaces*, in "The L. E. J. Brouwer Centenary Symposium", A. S. Troelstra and D. van Dalen (eds), 107-122, North Holland, Amsterdam, 1982.

[G] G. Grätzer. "Lattice theory", Birkhauser, Basel, 1978.

[J] P. T. Johnstone. "Stone Spaces", Cambridge University Press, 1982.

[MCL] S. Mac Lane. "Categories for the Working Mathematician", Graduate Texts in Mathematics, Springer Verlag, 1971.

[ML] P. Martin-Löf. "Intuitionistic type theory", notes by Giovanni Sambin of a series of lectures given in Padua, June 1980,Bibliopolis, Napoli, 1984.

[S] G. Sambin. *Intuitionistic formal spaces – a first communication*, in Mathematical logic and its applications, D. Skordev ed., Plenum, 187-204, 1987.

[S1] G. Sambin. *Intuitionistic formal spaces and their neighbourhood*, in "Logic Colloquium 88", Ferro, Bonotto, Valentini, Zanardo (eds), 261-285, North Holland, Amsterdam, 1989.

[S2] G. Sambin. "Notes for a course in Intuitionistic Pointfree Topology", Università di Padova, 1990.

[S3] G. Sambin. *The semantics of pretopologies*, in "Substructural Logics", eds. P. Schroeder-Heister, K. Došen, Clarendon Press, 293-307, 1993

[SV] G. Sambin, S. Valentini. *Building up a toolbox for Martin-Löf intuitionistic type theory*, to appear.

[SVV] G. Sambin, S. Valentini, P. Virgili. *Constructive Domain Theory as a branch of Intuitionistic Pointfree Topology*, to appear.

[ST] M. H. Stone. *Topological representation of distributive lattices and Brouwerian logics*, Casopis pro pěstování matematiky a fysiky **67**, 1-25, 1937.

[TvD] A. S. Troelstra, D. van Dalen. "Constructivism in mathematics, an introduction", vol. II, Studies in logic and the foundations of mathematics, Springer Verlag, 1988.