A proof-theoretical perspective on Public Announcement Logic

Paolo Maffezioli *
Sara Negri †

1 Introduction

Knowledge is strictly connected with the practice of communication: obviously, our comprehension of the world depends not only on what is known, but also on what eventually we may come to know in the process of information flow. In this perspective knowledge can change and it is considered as a dynamic rather than a static notion. A satisfactory account to knowledge change was an important task in the last years, and Dynamic Epistemic Logic (DEL) is one of the most prominent and recent approaches to this problem. Actually, DEL is a large family and not a single logic, and in this paper we shall focus on the simplest type of DEL, the logic of public announcements (PAL): the idea is that agents gain new information by announcing publicy some (true) fact. Along with the standard epistemic modal operators $\mathcal{K}_a$ for each agent a and propositional connectives, the language of PAL has formulas for announcement $[A]B$, intuitively read as: once A has been announced, B is true. The announcement of A has the consequence of changing the state of an agent's knowledge in a very simple way: after the announcement of A all the situations in which A does not hold are not any longer considered as possible. The standard presentation of PAL in van Ditmarsch *et al.* (2007) arises from the seminal work of Plaza (1989): despite the binary notation $A + B$ employed there, the operation of announcement is considered as a diamond-like operator $\langle A\rangle B$ which is true whenever A is true and after A is announced B is true; therefore, the dual operator $[A]B$ is satisfied whenever if A is true then after A is announced then B is true. Thus, in the Plaza interpretation (P-interpretation) of announcements a formula can be announced only if it is true and hence announcements are considered as a completely truthful resource of information. In fact, $(A \supset [A]B) \supset [A]B$ is a theorem of PAL. However, this is not the whole picture and alternative interpretations are possible if we drop the requirement that what is announced must be

*Department of Philosophy, University of Firenze, Italy
†Department of Philosophy, University of Helsinki, Finland

true and allow that every formula can be announced, no matter what its truth-value is. In contrast with P-announcements, it may happen that the agents do not assume the truth of what is announced and could correctly esclude as impossible also the world in which the announcement is made. This approach, proposed by Gerbrandy and Groenveled (1997), modifies the original perpective on truthful announcements due to Plaza (1989). For a clear and compact presentation of the Gerbrandy and Groenveled announcements (GG-announcements) see Bucheli *et al.* (2010).

The standard proof system for PAL is obtained as an extension with axioms for announcements of an Hilbert-system for the modal logic **S5**: the system, denoted **PAL**, is proved to be complete by means of a translation argument which reduces the completeness of **PAL** to the completeness of **S5**. For this purpose, the axioms of **PAL** are reduction axioms: Every formula that contains announcements can be rewritten as a formula without announcements. An axiomatic system for **S5** is however of little use for the actual finding of proofs, and the reduction axioms complicate it even further. Our aim here is to develop a Gentzen-style proof system for PAL (**G3PAL**) with GG-announcements. The rules are justified directly in terms of the semantics of announcements and reduction axioms are completely avoided. The completeness theorem can be proved through the equivalence with **PAL** or even directly. Moreover, the admissibility of the structural rules makes it possible to find derivations by applying a proof-search procedure. The system of Section 2 is strictly related with the one presented in Maffezioli and Negri (2010) where instead the P-announcements are considered, so occasionally shall we refer to that work.

2 A Gentzen system for epistemic logic with announcements

We start from the cut-free calculus **G3K** given in Negri (2005), replace the alethic modality $\Box$ with the knowledge operator $\mathcal{K}_a$ and allow an accessibility relation R_a for each agent a, as in Hakli and Negri (2008). The rules for each connective and modality are obtained from their meaning explanation in terms of the Kripke semantics.

Definition 2.1. *Let $\mathcal{P}$ be a set of atomic formulas and $\mathcal{A}$ a set of agents. A (multi-agent)* ***Kripke model*** *is a structure $\mathfrak{M} = \langle W, R_a, \Vdash \rangle$ where W is a non-empty set, for every $a \in \mathcal{A}$, R_a is a binary relation on W, and $\Vdash$ is a binary relation between elements in W and atomic formulas. As usual, $w \Vdash P$ means that P is true at w.*

The relation $\Vdash$ is extended in a unique way to arbitrary formulas by means of inductive clauses. The clauses for the propositional connectives are the standard ones. The inductive step for the knowledge operator is as follows:

$$w \Vdash \mathcal{K}_a A \text{ if and only if for all } v,\ wR_a v \text{ implies } v \Vdash A$$

The left-to-right direction in the explanation above justifies the left rules, the right-to-left direction the right rules. The role of the quantifier is reflected in the variable condition for rule $R\mathcal{K}_a$ that v is the eigenvariable and so it does not appear in the conclusion. The definition thus gives the following rules:

$$\frac{v : A, w : \mathcal{K}_a A, wR_a v, \Gamma \Rightarrow \Delta}{w : \mathcal{K}_a A, wR_a v, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\mathcal{K}_a} \qquad \frac{wR_a v, \Gamma \Rightarrow \Delta, v : A}{\Gamma \Rightarrow \Delta, w : \mathcal{K}_a A}\ {\scriptstyle R\mathcal{K}_a}$$

Systems that extend basic modal logic are handled by suitable rules for the accessibility relation. Following the method of Negri (2005), it is possible to convert the standard properties of R_a as reflexivity, transitivity and symmetry into sequent rules in such a way that the system obtained still satisfies cut-elimination. System **G3S5** is obtained by adding the following rules to **G3K**:

$$\frac{wR_a w, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ {\scriptstyle Ref}$$

$$\frac{wR_a z, wR_a v, vR_a z, \Gamma \Rightarrow \Delta}{wR_a v, vR_a z, \Gamma \Rightarrow \Delta}\ {\scriptstyle Trans} \qquad \frac{wR_a v, vR_a w, \Gamma \Rightarrow \Delta}{wR_a v, \Gamma \Rightarrow \Delta}\ {\scriptstyle Sym}$$

The standard axioms of veridical knowledge and of positive and negative introspection are easily derivable by applying the rules for R_a together with those for $\mathcal{K}_a$.

Announcements are formulas of the form $[A]B$, generated inductively as usual: if A and B are formulas, so is $[A]B$. The intended meaning is: B is true after every public announcement that A. Semantically, the announcement of A yields a restriction on $\mathfrak{M}$, according to the following definition:

Definition 2.2. *Let $\mathfrak{M}$ be a (multi-agent) Kripke model and A a formula. A **restricted (multi-agent) Kripke model** is a (multi-agent) Kripke Model $\mathfrak{M}^A = \langle W^A, R_a^A, V^A \rangle$ where $W^A = W$, $wR_a^A v$ if and only if $wR_a v$ and $v \Vdash A$, and $w \Vdash^A P$ if and only if $w \Vdash P$.*

However, restriction to a single announcement is not enough and we have to take into account the general case of a (possibly empty) list of successive announcements and then of a Kripke model restricted to a (possibly empty) list of formulas. Let φ be a list of formulas $A_1, \ldots, A_n$; we indicate with $\mathfrak{M}^\varphi$ the Kripke model restricted to φ. Restricted forcing $w \Vdash^\varphi B$ coincides with the unrestricted one when $\varphi = \epsilon$, so if φ is the empty list ϵ then $\mathfrak{M}^\epsilon = \mathfrak{M}$; it is extended to arbitrary lists of formulas by induction in the obvious way, so that $wR_a^{\varphi,A} v$ if and only if $wR_a^\varphi v$ and $v \Vdash^\varphi A$. In turn, the forcing on complex formulas is reduced to simpler ones by the following clauses:

$$
\begin{array}{lll}
w \Vdash^{\varphi} B \,\&\, C & \text{if and only if} & w \Vdash^{\varphi} B \text{ and } w \Vdash^{\varphi} C \\
w \Vdash^{\varphi} B \vee C & \text{if and only if} & w \Vdash^{\varphi} B \text{ or } w \Vdash^{\varphi} C \\
w \Vdash^{\varphi} B \supset C & \text{if and only if} & w \Vdash^{\varphi} B \text{ implies } w \Vdash^{\varphi} C \\
w \Vdash^{\varphi} \mathcal{K}_a B & \text{if and only if} & \text{for all } v,\ wR_a^{\varphi} v \text{ implies } v \Vdash^{\varphi} B \\
w \Vdash^{\varphi} [B]C & \text{if and only if} & w \Vdash^{\varphi, B} C
\end{array}
$$

By unfolding the inductive clauses for the restricted accessibility relation we obtain the standard definition for $\mathcal{K}_a$, when φ is empty, and the following definition when the list is non-empty, i.e., of the form φ, A:

$$
w \Vdash^{\varphi, A} \mathcal{K}_a B \quad \text{if and only if} \quad \text{for all } v,\ wR_a^{\varphi} v \text{ and } v \Vdash^{\varphi} A \text{ implies } v \Vdash^{\varphi, A} B
$$

The latter condition embeds into a semantic clause a frame property that does not follow the pattern of the other frames rules, in a way analogous to the definition of the $\Box$-operator in Gödel-Löb provability logic (cf. Negri 2005). Exploiting the semantics of restricted forcing we obtain a Gentzen system of rules. Initial sequents and propositional rules are the same as in the calculus presented in Maffezioli and Negri (2010), p. 299; the rules for the modality with respect to unrestricted forcing are rules $L\mathcal{K}_a$ and $R\mathcal{K}_a$ above; the atomic, modal, and announcement rules are:

$$
\frac{w :^{\varphi} P, \Gamma \Rightarrow \Delta}{w :^{\varphi, A} P, \Gamma \Rightarrow \Delta}\ {\scriptstyle L0:^{\varphi, A}}
\qquad\qquad
\frac{\Gamma \Rightarrow \Delta, w :^{\varphi} P}{\Gamma \Rightarrow \Delta, w :^{\varphi, A} P}\ {\scriptstyle R0:^{\varphi, A}}
$$

$$
\frac{v^{\varphi, A} B, wR_a^{\varphi} v, v :^{\varphi} A, w :^{\varphi, A} \mathcal{K}_a B, \Gamma \Rightarrow \Delta}{wR_a^{\varphi} v, v :^{\varphi} A, w :^{\varphi, A} \mathcal{K}_a B, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\mathcal{K}_a :^{\varphi, A}}
\qquad
\frac{wR_a^{\varphi} v, v :^{\varphi} A, \Gamma \Rightarrow \Delta, v :^{\varphi, A} B}{\Gamma \Rightarrow \Delta, w :^{\varphi, A} \mathcal{K}_a B}\ {\scriptstyle R\mathcal{K}_a :^{\varphi, A}}
$$

$$
\frac{w :^{\varphi, B} C, \Gamma \Rightarrow \Delta}{w :^{\varphi} [B]C, \Gamma \Rightarrow \Delta}\ {\scriptstyle L[\,]:^{\varphi}}
\qquad\qquad
\frac{\Gamma \Rightarrow \Delta, w :^{\varphi, B} C}{\Gamma \Rightarrow \Delta, w :^{\varphi} [B]C}\ {\scriptstyle R[\,]:^{\varphi}}
$$

where v does not appear in the conclusion of $R\mathcal{K}_a :^{\varphi, A}$. Finally, by adding the *announcement composition* rules:

$$
\frac{w :^{\varphi, A, B} C, \Gamma \Rightarrow \Delta}{w :^{\varphi, A \,\&\, [A]B} C, \Gamma \Rightarrow \Delta}\ {\scriptstyle L_{cmp}}
\qquad
\frac{\Gamma \Rightarrow \Delta, w :^{\varphi, A, B} C}{\Gamma \Rightarrow \Delta, w :^{\varphi, A \,\&\, [A]B} C}\ {\scriptstyle R_{cmp}}
$$

we obtain the system **G3PAL**. In the next section we prove that it satisfies all the structural properties required to **G3**-sequent systems.

2.1 Admissibility of the structural rules

G3PAL enjoys all the structural properties of **G3**-systems, in particular height-preserving (hp) admissibility of contraction and admissibility of the cut rule. In order to prove this we need some preliminary results:

Lemma 2.3. *In* **G3PAL** *the following holds:*

i. *Substitution of labels is hp-admissible:* $\Gamma \Rightarrow \Delta$ *implies* $\Gamma[v/w] \Rightarrow \Delta[v/w]$*;*

ii. *Arbitrary initial sequents,* $w :^{\varphi} B, \Gamma \Rightarrow \Delta, w :^{\varphi} B$*, are derivable;*

iii. *All the rules are hp-invertible;*

iv. *Weakening is hp-admissible.*

Proof. By induction on the height h of the derivation defined as the length of its longest branch in its derivation tree. For details involving propositional and modal rules, see Negri (2005); the cases of announcements are analogous to those given in Maffezioli and Negri (2010). □

Now it is possible to prove hp-admissibility of contraction, which is a central ingredient in our proof of cut elimination.

Theorem 2.4. *The rules of contraction*

$$\frac{w :^{\varphi} B, w :^{\varphi} B, \Gamma \Rightarrow \Delta}{w :^{\varphi} B, \Gamma \Rightarrow \Delta}\ {\scriptstyle Ctr} \qquad \frac{\Gamma \Rightarrow \Delta, w :^{\varphi} B, w :^{\varphi} B}{\Gamma \Rightarrow \Delta, w :^{\varphi} B}\ {\scriptstyle Ctr}$$

are hp-admissible in **G3PAL**.

Proof. By simultaneous induction on the height h of the derivation for left and right contraction. The proof proceeds analogously to that in Maffezioli and Negri (2010), with the exception of the announcement rules. The crucial step is to convert a derivation where contraction applies to an announcement formula that is principal in $L[\,] :^{\varphi}$ into a derivation in which only hp-admissible rules and contraction on smaller formulas are applied:

$$\dfrac{\dfrac{w :^{\varphi,B} C, w :^{\varphi} [B]C, \Gamma \Rightarrow \Delta}{w :^{\varphi} [B]C, w :^{\varphi} [B]C, \Gamma \Rightarrow \Delta}\ {\scriptstyle L[\,]}}{w :^{\varphi} [B]C, \Gamma \Rightarrow \Delta}\ {\scriptstyle Ctr} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{w :^{\varphi,B} C, w :^{\varphi} [B]C, \Gamma \Rightarrow \Delta}{w :^{\varphi,B} C, w :^{\varphi,B} C, \Gamma \Rightarrow \Delta}\ {\scriptstyle inv}}{w :^{\varphi,B} C, \Gamma \Rightarrow \Delta}\ {\scriptstyle Ctr}}{w :^{\varphi} [B]C, \Gamma \Rightarrow \Delta}\ {\scriptstyle L[\,]}$$

The case of right-contraction is analogous.

□

We are now in a position to prove the most important result concerning proof analysis for **G3PAL**, that is, admissibility of cut. Admissibility of cut is crucial for delimiting the space of proof search, because it guarantees that no new formulas need be used during the search.

Theorem 2.5. *The rule of cut*

$$\frac{\Gamma \Rightarrow \Delta, w :^{\varphi} B \quad w :^{\varphi} B, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta} \; \mathit{Cut}$$

is admissible in **G3PAL***.*

Proof. The proof has the same structure as the proof of admissibility of cut for the modal systems **G3K** of Negri (2005). We recall that the proof is by induction on the structure of the cut formula with sub-induction on the sum of the heights of the derivations of the premises of cut. The proof is to a large extent similar to the cut-elimination proofs in Negri and von Plato (2001, Theorem 3.2.3) so we shall consider only the case in which the cut formula is $w :^{\varphi} [B]C$ and it is principal in both premisses. A derivation of the form

$$\frac{\dfrac{\Gamma \Rightarrow \Delta, w :^{\varphi,B} C}{\Gamma \Rightarrow \Delta, w :^{\varphi} [B]C} \; R[\,]{:}^{\varphi} \qquad \dfrac{w :^{\varphi,B} C, \Gamma' \Rightarrow \Delta'}{w :^{\varphi} [B]C, \Gamma' \Rightarrow \Delta'} \; L[\,]{:}^{\varphi}}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta} \; \mathit{Cut}$$

is simply converted into one in which cut is applied to smaller formulas

$$\frac{\Gamma \Rightarrow \Delta, w :^{\varphi,B} C \quad w :^{\varphi,B} C, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta} \; \mathit{Cut}$$

□

2.2 Completeness

In this section our aim is to prove that **G3PAL** is complete with respect to the semantics. We recall from Bucheli *et al.* (2010) the axioms of **PAL** and prove their derivability in our system. Along with cut elimination and invertibility of the right rule for implication which prove admissibility of modus pones and the admissibility of necessitation rule (Lemma 2.7), this gives **PAL** $\subseteq$ **G3PAL**. The completeness of **PAL** proved in Gerbrandy and Groenveled (1997) gives immediately an indirect proof of the completeness theorem for **G3PAL**.

A1	All theorems of modal logic **S5**	
A2	$[A]P \supset\subset P$	Atomic Independence
A3	$[A](B \supset C) \supset\subset ([A]B \supset [A]C)$	Normality
A4	$[A]\neg B \supset\subset \neg[A]B$	Functionality
A5	$[A]\mathcal{K}_a B \supset\subset \mathcal{K}_a(A \supset [A]B)$	Update
A6	$[A][B]C \supset\subset [A \,\&\, [A]B]C$	Announcements Composition
R1	From $\Gamma \vdash A \supset B$ and $\Delta \vdash A$ infer $\Gamma, \Delta \vdash B$	Modus Ponens
R2	From $\vdash A$ infer $\vdash \mathcal{K}_a A$	Necessitation

Lemma 2.6. *All the axioms listed above are derivable in* **G3PAL***.*

Proof. By applying a systematic proof-search procedure to the sequent to be derived. We show how **G3PAL** works by giving a derivation of the A5 axiom (left-to-right direction):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{v :^{A} B, wR_a v, v : A, w :^{A} \mathcal{K}_a B \Rightarrow v :^{A} B}{wR_a v, v : A, w :^{A} \mathcal{K}_a B \Rightarrow v :^{A} B}\, L\mathcal{K}_a}{wR_a v, v : A, w :^{A} \mathcal{K}_a B \Rightarrow v : [A]B}\, R[\,]}{wR_a v, w :^{A} \mathcal{K}_a B \Rightarrow v : A \supset [A]B}\, R{\supset}}{w :^{A} \mathcal{K}_a B \Rightarrow w : \mathcal{K}_a(A \supset [A]B)}\, R\mathcal{K}_a}{w : [A]\mathcal{K}_a B \Rightarrow w : \mathcal{K}_a(A \supset [A]B)}\, L[\,]}{\Rightarrow w : [A]\mathcal{K}_a B \supset \mathcal{K}_a(A \supset [A]B)}\, R{\supset}$$

where the top sequent is derivable by Lemma 2.3.

□

Admissibility of Modus Ponens follows from admissibility of cut and invertibility of $R \supset$. Necessitation can be proved admissible by the following

Lemma 2.7. *The rule of Necessitation*

$$\dfrac{\Rightarrow w : A}{\Rightarrow w : \mathcal{K}_a A}$$

is admissible in **G3PAL**.

Proof. Cf. Maffezioli and Negri (2010). □

Finally, the completeness of **G3PAL** follows from the completeness of **PAL** (see Gerbrandy and Groenveled 1997 and Bucheli *et al.* 2010).

3 Conclusion

In this paper we introduced a Gentzen system for PAL and sketched briefly its structural properties and completeness. The system is closely related to the one given in Maffezioli and Negri (2010), with the exception that herein the GG-interpretation of public announcements is considered, whereas the former dealt with the more common notion of P-announcement. If we stick to the usual PAL setting, the difference is simply that P-announcements assume the truth of what is being announced, whereas GG do not. If we allow the possibility of false information in our announcements, there are more situations that can be described, especially the situation in which agents are deceived by misinformations. When PAL is extended the differences between the Plaza and the Gerbrandy and Groenveled interpretation are more relevant and there is some distinction in terms of succinctness of updates; for an extension of PAL with GG-announcements see Kooi and Renne (2010). Finally, the advantage of **G3PAL** with respect to Hilbert-style formulation of PAL goes beyond the simple fact that the former is designed for making explicit to structure of proofs in PAL, whereas the latter is not. In system as **PAL** some remarkable properties cannot be proved schematically: for instance, compositionality (axiom A6) and associativity of public announcements, that is $[A \,\&\, [A]B][C]D \supset\subset [A][B \,\&\, B[C]]D$, are proved to be valid by induction on C and D, respectively. Instead **G3PAL** we can apply a proof-search procedure and find a derivation for each of them without any induction on formulas.

Another system in the literature that takes advantage of the Kripkean semantics for PAL is the tableau system presented in Balbiani *et el.* (2010).

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References

Balbiani, P., H. van Ditmarsch, A. Herzig and T. de Lima (2010) Tableaux for public announcement logic, *Journal of Logic and Computation*, vol. 20, pp. 5576.

Bucheli, S., R. Kuznets, B. Renne, J. Sack, and T. Studer, (2010) *Justified Belief Change*, in X. Arrazola and M. Ponte (eds) *Proceedings of the Second ILCLI International Workshop on Logic and Philosophy of Knowledge, Communication and Action*, The University of the Basque Country Press, pp. 135–155.

van Ditmarsch, H., W. van der Hoek, and B. Kooi (2007) *Dynamic Epistemic Logic*, Synthese Library vol. 337, Springer.

Gerbrandy, J. and Groenveled, W. (1997) Reasoning about information change, *Journal of Logic, Language and Information*, vol. 6, pp. 147–169.

Hakli, R. and S. Negri (2008) Proof theory for distributed knowledge, *Computational Logic in Multi-Agent Systems*, Springer Lecture Notes in Artificial Intelligence, vol. 5405, pp. 100–116.

Kooi, B. and B. Renne, Arrow Update Logic, manuscript, 2010. Available from: `http://bryan.renne.org/docs/rk-aul.pdf`.

Maffezioli, P. and S. Negri (2010) A Gentzen-style analysis of Public Announcement Logic, in X. Arrazola and M. Ponte (eds) *Proceedings of the Second ILCLI International Workshop on Logic and Philosophy of Knowledge, Communication and Action*, The University of the Basque Country Press, pp. 293–313.

Negri S. (2005) Proof analysis in modal logic, *Journal of Philosophical Logic*, vol. 34, pp. 507–544.

Negri, S. and J. von Plato (2001) *Structural Proof Theory*, Cambridge University Press.

Plaza J. (1989) Logics of public communications, in M.L. Emrich et al. (eds), *Proceedings of the 4th International Symposium on Methodologies for Intelligent Systems: Poster Session Program*, pp. 201–216. Reprinted in *Synthese*, vol. 158, pp. 165–179, 2007.