Does the deduction theorem fail for modal logic?

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Abstract.
Various sources in the literature claim that the deduction theorem does not hold for normal modal or epistemic logic, whereas others present versions of the deduction theorem for several normal modal systems. It is shown here that the apparent problem arises from an objectionable notion of derivability from assumptions in an axiomatic system. When a traditional Hilbert-type system of axiomatic logic is generalized into a system for derivations from assumptions, the necessitation rule has to be modified in a way that restricts its use to cases in which the premiss does not depend on assumptions. This restriction is entirely analogous to the restriction of the rule of universal generalization of first-order logic. A necessitation rule with this restriction permits a proof of the deduction theorem in its usual formulation. Other suggestions presented in the literature to deal with the problem are reviewed, and the present solution is argued to be preferable to the other alternatives. A contraction- and cut-free sequent calculus equivalent to the Hilbert system for basic modal logic shows the standard failure argument untenable by proving the underivability of $\Box A$ from A.

1. Introduction

The deduction theorem is a metatheorem in mathematical logic that states that if formula B can be derived from the set of formulas Γ extended with formula A, then $A \supset B$ can be derived from Γ. We can put it more briefly as:

$$\textit{If } \Gamma, A \vdash B, \textit{ then } \Gamma \vdash A \supset B.$$

According to Kleene (1952, chapter V), a version of the deduction theorem was proved first by Jacques Herbrand in his Ph.D. thesis (1929–30), for axiomatizations of propositional and first-order logic. Herbrand considers a theory that has a finite number of axioms (hypotheses) and reduces them to a single proposition H without free variables by taking their universal generalization and then their conjunction, and proves:

> Theorem: A necessary and sufficient condition for the truth of a proposition P [in this theory] is that $H \supset P$ be a propositional identity.

The proof is by induction on the proof of P. He can therefore generalize the result to theories with infinitely many axioms, because a proof uses only finitely many of them. Then he states the deduction theorem in its usual formulation, with H an assumption.

Alfred Tarski may have been aware even earlier that the property holds for the propositional calculus (cf. Porte 1982). In fact, he employed it as a primitive notion in Tarski (1930):

> From the sentences of any set X certain other sentences can be obtained by means of certain operations called *rules of inference*. These sentences are called *consequences of the set X*. The set of all consequences is denoted by the symbol '$Cn(X)$'.

The deduction theorem is assumed as a property in the axiomatization of $Cn(X)$, in the following form (where $c(y, z)$ denotes the implication with antecedent y and consequent z):

> Axiom 8. If $X \subseteq S$, $y \in S$, $z \in S$ and $z \in Cn(X + \{y\})$, then $c(y, z) \in Cn(X)$.

Gentzen, on the other hand, does not state the deduction theorem, but in fact proves it as a part of his translation from sequent calculus to a Hilbert-style system for intuitionistic logic in section 5 of Gentzen (1934–35).

Both the work of Tarski and of Gentzen concerning the abstract notion of deduction had remarkable anticipations in the the contributions by Lesniewski (1929) and several papers of Hertz between 1922 and 1929 (cf. von Plato 2009, p. 669), respectively.

The deduction theorem is made explicit and given its name and a proof by Bernays (cf. Hilbert and Bernays 1934, pp. 151–155).

The deduction theorem is one of the most basic properties for a deductive system; for a concise historical survey, see Porte (1982). Sometimes it is proved as an intermediate step in proving the completeness theorem. In some systems such as natural deduction, it is taken as a primitive in the form of an inference rule, the introduction rule for implication. Montague and Henkin (1956) mention the deduction theorem as one of the desiderata the definition of formal deduction should satisfy.

For modal logic, however, there seems to be lack of agreement about the validity of the deduction theorem. The answer to the question whether the deduction theorem fails for modal logic is far from unanimous. Some sources in the literature claim that the deduction theorem holds, whereas others claim that it fails, some give conditions and restrictions for the theorem to hold or argue for the failure of the deduction theorem as a consequence of a certain formulation of the rule of necessitation. According to the standard textbook Chellas (1980, p. 48), the deduction theorem holds in modal logic:

> $\Gamma \vdash_\Sigma A \supset B$ *iff* $\Gamma \cup \{A\} \vdash_\Sigma B$. ... this states the so-called deduction theorem for systems of modal logic ... The proof is rather easy and is left as an exercise.

On the other hand, Smorynski (1984), Fagin et al. (1995) and Chagrov and Zakharyaschev (1997), among others, claim that it does not hold. For instance, Ganguli and Nerode (2004, p. 141) say explicitly: "In its usual form the Deduction Theorem fails for modal logic."

Our aim here is to analyse these claims that seem to contradict each other and to locate the source of disagreement in the literature. We present in Section 2 the arguments that purport to show that the deduction theorem does not hold in modal logic, present our analysis of the apparent problem, and suggest a solution. In Section 3, we show that our solution permits a proof of the deduction theorem in its usual form. In Section 4, we analyse the problem with the help of sequent calculus.

2. The deduction theorem in modal logic

It has been claimed in several sources that the deduction theorem as it is usually formulated does not hold in normal modal logics that contain the rule of necessitation: From A infer $\Box A$. It is said that this rule allows the inference $A \vdash \Box A$, but that $A \supset \Box A$ is not a theorem contrary to what the deduction theorem would require. The reasoning appears odd, because standard textbook presentations of the necessitation rule emphasize that the meaning of the necessitation rule is that $\vdash \Box A$ whenever $\vdash A$ (Chellas 1980, p. 15), or that if A is a theorem, so is $\Box A$ (Hughes and Cresswell 1996, p. 25).

There is a substantial amount of literature where the apparent problem of the failure of the deduction theorem in modal logic is discussed. Most authors concerned with the problem try to find a formulation of the deduction theorem that can be shown to hold in modal logic. Others are not concerned directly with the deduction theorem as such, but only as a means of proving some other properties of the systems under study. For example, Ganguli and Nerode (2004) are interested in effective completeness theorems. For systems of first-order modal logics these theorems state that there is a decidable model for every decidable theory. Such theorems are usually proved with the aid of deduction theorems, but since the authors take modal logic to lack a deduction theorem, they prove an analogous result that suffices for their purposes. In general, the strategies for circumventing the apparent problem with the deduction theorem seem unmotivated if there is a simple way to recover the theorem itself. We shall here restrict focus to works that are directly concerned with re-establishing the deduction theorem rather than replacing it with some other results. Before that, we review the literature on the deduction theorem and the arguments used to support the claims of failure.

2.1. Early controversy

An early discussion about the validity of the deduction theorem in modal logic started from the results of Ruth C. Barcan (1946, see also Barcan Marcus 1953). She proved versions of deduction theorems for the modal system S4 introduced by Lewis and Langford (1932) and showed that they do not hold for the weaker system S2 that lacks the axiom $\Diamond\Diamond A \strictif \Diamond A$. In a later paper, Barcan Marcus (1953) proved that it holds for S5 as well. She also considered whether it would be possible to prove deduction theorems stated in terms of strict implication instead of material implication in these Lewis' systems, that is, in the form:

$$\Gamma, A \vdash B \Rightarrow \Gamma \vdash A \strictif B.$$

She proved that the above fails in S2, and holds in S4 and S5 only under additional assumptions.

In 1957, Lemmon gave a new set of axioms and rules for Lewis' systems S1–S5, in such a way that they are an extension of classical logic, and using this new basis, J. Jay Zeman (1967) proved the deduction theorem for S4 and S5.

Following Barcan Marcus, Robert Feys (1965) stated that the deduction theorem is not valid in S2, no matter whether it is formulated as "concluding to $\vdash P \strictif Q$ whenever $\vdash Q$ is derived from $\vdash P$" or "in the weaker form according to which if $\vdash Q$ derives from $\vdash P$, then $\vdash P \supset Q$." Zeman (1973) criticized Feys for an erroneous statement of the deduction theorem: "Feys seems to take his claim to mean that whenever we have a rule of inference of form 'If $\vdash \alpha$, then $\vdash \beta$', then $C\alpha\beta$ is a theorem. This, of course, is not what the deduction theorem says. If α (and so β) in such a rule of inference is actually a theorem, then $C\alpha\beta$ is rather trivially a theorem, by propositional calculus. On the other hand, if α is taken to be an arbitrary well-formed formula, the suggested meaning of the deduction theorem does not hold even in the propositional calculus." Zeman continued by formulating a proper version of the deduction theorem and claimed that it holds for S2. It is to be observed that Zeman worked with Lemmon's axiomatizations of S1–S4, whereas Barcan Marcus used Lewis' original axiomatizations, which may explain the discrepancy between their results. In any case, this early controversy seems to be different from the present one. Both Zeman and Barcan Marcus maintain that the deduction theorem holds for normal modal logics such as S4 and S5, and that the problem concerns weaker systems S1 and S2 that do not have the necessitation rule. The current controversy instead concerns normal modal logics with the rule of necessitation.

2.2. Claims of failure

The reasoning behind the claims of failure of the deduction theorem in modal logic is presented in detail by Fagin et al. (1995, pp. 51–52) in the context of epistemic logic. They give the following deductive system K_n for knowledge within a group of n agents. (We depart from their notation and adopt the symbol $\supset$ for implication, rather than $\Rightarrow$, and upper case A, B, C, ... for formulas.) The modalities K_i denote knowledge by agent i.

A1. All tautologies of the propositional calculus,

A2. $K_iA \wedge K_i(A \supset B) \supset K_iB$, $i = 1, \ldots, n$ (Distribution Axiom),

R1. From A and $A \supset B$ infer B (Modus ponens),

R2. From A infer K_iA (Knowledge Generalization).

This is a standard Hilbert-type deductive system. A derivation is obtained by applying the rules of modus ponens and knowledge generalization to the tautologies of classical propositional logic and to instances of the distribution axiom.

A standard Hilbert-type system for first-order classical logic can be presented in a similar way:

B1. All tautologies of the propositional calculus,

B2. $\forall x A \supset A[x/t]$, $A[x/t] \supset \exists x A$,

B3. $\forall x(B \supset A) \supset (B \supset \forall y A[x/y])$, $x \notin FV(B)$, $y \equiv x$ or $y \notin FV(A)$,

B4. $\forall x(A \supset B) \supset (\exists y A[x/y] \supset B)$, $x \notin FV(B)$, $y \equiv x$ or $y \notin FV(A)$,

R1. Modus ponens,

R2. From A infer $\forall y A[x/y]$ (Universal Generalization).

Observe that each formula in a derivation in a Hilbert-type system is in itself a theorem. This notion of derivation is insufficient for the formulation of the deduction theorem. A generalization to the notion of *derivability from assumptions* is needed. Thus Fagin et al. (1995) extend the notion of provability in K_n by defining B to be provable from A in the axiom system Ax, written $Ax, A \vdash B$, if there is a sequence of steps ending with B, each of which is either an instance of an axiom of Ax, A itself, or follows from previous steps by an application of an inference rule of Ax. With this terminology the deduction theorem states that if $Ax, A \vdash B$ then $Ax \vdash A \supset B$. The claim of failure of the deduction theorem for K_n is then supported as follows (in the original, Greek lower case was used for formulas and $\Rightarrow$ for implication):

> [B]y an easy application of Knowledge Generalization (R2) we have $K_n, A \vdash K_i A$. However, we do not in general have $K_n \vdash A \supset K_i A$[.]

The reasoning above is in itself correct but rests on a notion of derivation from assumptions that can be questioned. Consider the deduction step from A to $K_i A$. By the step, if A is an assumption, it would be known by agent i. With the necessity modality, this corresponds to the step from A to $\Box A$, an unrestricted rule of necessitation that would, in particular, turn assumptions into necessities. However, when the primitive Hilbert-type system is generalized to a system for hypothetical derivations, the following formulation is usually taken[1]:

R2′ If A is derivable (without assumptions), then $K_i A$ is derivable.

This rule can be compared to the rule of universal generalization in first-order logic: We can derive $\forall x A(x)$ from $A(x)$ with the proviso that $A(x)$ does not depend on any assumption on x. Otherwise $A(x)$ itself could be an assumption and we could derive $\forall x A(x)$ from $A(x)$, and therefore also from $\exists x A(x)$ by the rule of existential elimination.

The rule of knowledge generalization needs to be appropriately tuned when the extended notion of derivability from assumptions is allowed. *It is then not possible to derive* $K_n, A \vdash K_i A$. Since underivability is not so easy to see in a Hilbert-type system, we shall give in Section 4 a formal proof of the latter statement, using sequent calculus.

Also the formulation of deduction in basic modal logic given in Smorynski (1984, p. 454, 2002, p. 12) legitimates the step from A to $\Box A$, with the consequence that one has to give up the simple formulation of the deduction theorem. Smorynski states an unrestricted deduction theorem for derivations that do not use the rule of necessitation, and alternatively suggests dropping the rule of necessitation and adding $\Box A$ as a new axiom for every axiom A. The rule

[1] Cf. the similar definition of the necessitation rule for standard Hilbert systems of modal logic in Hughes and Cresswell (1968, p. 31), and for systems with assumptions in Troelstra and Schwichtenberg (2000, p. 284).

of necessitation becomes a "derived rule of inference but no longer an obstacle to the validity of the deduction theorem" [2].

2.3. Recent approaches

Once the claim of failure had been accepted, several alternative approaches were proposed to have a modified version of the deduction theorem in modal logic. A recent presentation that essentially follows Mints (1974), by Chagrov and Zakharyaschev (1997), gives a formulation of derivation from assumptions with an unrestricted rule of necessitation, but modifies instead the deduction theorem. For system K, they prove the following:

> Assume that $\Gamma, A \vdash_K B$ and that there exists a derivation of B from the assumptions $\Gamma \cup \{A\}$ in which the rule of necessitation is applied $m \geq 0$ times to formulas that depend on A. We then have
> $$\Gamma \vdash_K \Box^0 A \,\&\, \ldots \,\&\, \Box^m A \supset B.$$

One problem with this approach is that the modal formulation of the deduction theorem differs from the standard formulation. Moreover, the formulation of the theorem now depends on the modal system under consideration. We shall see in Section 3 how a neater solution can be achieved by abandoning the unrestricted rule of necessitation, thus by modifying the notion of derivation rather than the formulation of the theorem.

In the recent *Handbook of Modal Logic*, Melvin Fitting (2007) presents the problem as follows:

> Modal logic raises problems for the notion of deduction. Suppose we want to show $X \supset Y$ in some modal axiom system by deriving Y from X. So we add X to our axioms. Say, to make things both concrete and intuitive, that X is "it is raining" and Y is "it is necessarily raining." Since X has been added to the axiom list the necessitation rule applies, and from X we conclude $\Box X$, that is Y. Then the deduction theorem would allow us to conclude that if it is raining, it is necessarily raining. This does not seem right—nothing would ever be contingent. On the other hand, if we are working in the modal logic **K**, and we want to see what happens if we strengthen it to **T** by adding all instances of the scheme $\Box X \supset X$, we certainly want the necessitation rule to apply to these instances. Things are not simple.

Here the problem arises from understanding derivability from a set of formulas in an axiomatic system as an addition of the formulas to the system's axioms. In traditional Hilbert systems, there is no notion of assumption that is not an axiom, and inference rules always have either axioms or theorems as premisses, which guarantees soundness of the necessitation rule. Obviously, if we then consider assumptions as axioms and use the necessitation rule in the unrestricted form, we can infer $\Box X$ from X, and the deduction theorem in its original formulation fails.

Fitting's solution is to make a distinction between two kinds of premisses so that necessitation applies only to one kind. He distinguishes between *global premisses* and *local premisses*: Global premisses are like the axiom scheme T, namely $\Box X \supset X$. These are intended as logical truths whereas contingent truths such as "it is raining" should be taken as local premisses. Necessitation should then be applied only to global premisses, not to local ones.

The notion of derivability has to be modified to accommodate the two kinds of premisses. Derivations are divided into two parts: In the *global part* each formula is an axiom or a global premiss or follows from earlier formulas by modus ponens or necessitation. This global part is followed by the *local part* in which also local premisses can be used but the rule of necessitation is no longer allowed. This solution seems to rescue the deduction theorem, but Fitting has to give two versions of it: One in which the additional assumption is a local premiss and another in which it is a global one.

In the definition of the deducibility relation $S \vdash_L U \to X$ of X from a set S of global premisses and a set U of local premisses, the rule of necessitation is applied only to the former

[2] Smorynski thus recovers the deduction theorem by removing the rule of necessitation, at the cost, however, of adding infinitely many axioms. Clearly, the derivable rule of necessitation will be the restricted one.

and a global and a local part of a derivation proceed in order. Let S and U be sets of formulas, X and Y single formulas, and L a set of axiom schemes extending K. The two versions of the deduction theorem are then as follows:

1. $S \vdash_L U \cup \{X\} \to Y$ iff $S \vdash_L U \to (X \supset Y)$
2. $S \cup \{X\} \vdash_L U \to Y$ iff $S \vdash_L U \cup \{X, \Box X, \Box^2 X, \Box^3 X, \ldots\} \to Y$

Observe that by the two clauses, a statement equivalent to the one by Chagrov and Zakharyaschev (1997) is obtained.

2.4. Semantic argument

Another argument for the failure of the deduction theorem is based on Kripke semantics (see Basin et al. 1998, p. 121). By the completeness theorem, A is provable if and only if A is valid in every world of a suitable Kripke frame. Then, if the deduction theorem holds, validity of $A \supset B$ should follow from validity of the derivation of B from A. According to Basin et al. (1998) the deduction theorem then reads as follows:

$$(\forall w \in W(w \Vdash A) \Rightarrow \forall w \in W(w \Vdash B)) \Rightarrow \forall w \in W(w \Vdash A \supset B).$$

Observe, however, that the antecedent of the above implication states that if A is valid, then B is valid, that is, through completeness, that if A is derivable, then B is derivable and the above becomes

$$\text{If } \vdash A \Rightarrow \vdash B, \text{ then } \vdash A \supset B.$$

This is false in general, as shown in Section 2.1 by the simple counterexample from propositional logic suggested by Zeman's criticism of Feys' account of the deduction theorem: If A and B are distinct propositional atoms P, Q, the antecedent is true (because $\nvdash P$), but it is false that $\vdash P \supset Q$.

In more general terms, the above statement is actually asking more than the deduction theorem, as it hides an inference from *admissibility* to *derivability.* If we have a derivation of B from A, then it follows that whenever A is derivable, also B is derivable, but it is not true that if the derivability of B follows from the derivability of A, then B is derivable form A. An example, again from first-order logic, helps to clarify the matter. Consider Gentzen's sequent calculus LK, or any sequent calculus that admits cut elimination. Whenever the premisses of a cut are derivable, then the conclusion is derivable, but it is certainly not the case that the conclusion of cut is derivable, without cut, from its premisses.

On the basis of Kripke's completeness results (Kripke 1959) one can argue that validity of an inference should be understood in terms of truth preservation rather than validity preservation. This is because that interpretation is the one that allows Kripke's completeness theorem to go through, whereas the other notion of validity validates steps that should not be validated (we shall return to this point in the following section). With a proper interpretation of validity of a sequent $\Gamma \to \Delta$, a completeness result analogous to Kripke's can be proved (Negri 2009). Validity of an inference from A to B should then be equated with the validity of a sequent $\Gamma \to \Delta$, and thus amounts to

$$\forall w \in W(w \Vdash A \Rightarrow w \Vdash B).$$

This is the same as

$$\forall w \in W(w \Vdash A \supset B)$$

so, at least semantically, the deduction theorem seems to be unproblematic.

The semantical explanation based on validity as truth preservation is at the basis of the definition of the labelled system of natural deduction for modal logic, also in Basin et al. (1998). These systems have the deduction theorem in-built in the rule for introduction of implication. In labelled systems, the side condition on assumptions (resp. contexts) for the rule of necessitation in Hilbert (resp. Gentzen) systems for modal logic becomes a condition on labels, similar to the variable conditions of first-order logic, for the right $\Box$-rule, and the rule of necessitation becomes in turn an admissible rule (cf. Negri 2005). However, the more fine-grained syntax of labelled systems is not in itself responsible for the "recovery" of the deduction theorem, but only for a change in the nature of side conditions.

2.5. Varieties of consequence.

The disagreement about the deduction theorem witnesses uncertainty about what should be taken as the right notion of derivation in modal logic and the claimed failure has often been used as an argument in an attitude of defeatism for the proof theory of modal logic.

A recent analysis of the various notions of consequence, and their impact on the deduction theorem, appears in Sundholm (2002), where consequence relations are classified into *semantical* and *syntactical* ones.

The **semantical notion of consequence relation** is relative to a model and involves universal quantification over valuations, assignments, etc., so it splits further into different consequence relations, those that are truth preserving and those that are validity preserving. For first order logic the definitions of the two notions of consequence relations are given by Avron (1991) as follows[3]:

Truth: $A_1, \ldots, A_n \vdash_t B_1, \ldots, B_m$ iff every assignment in a first-order structure which makes all the A_i true does the same to one of the B_j.

Validity: $A_1, \ldots, A_n \vdash_v B_1, \ldots, B_m$ iff if all the A_i are valid, that is, true under all assignments, then so is at least one of the B_j.

Observe that $A(x) \vdash_v \forall x A(x)$ but $A(x) \nvdash_t \forall x A(x)$.

For propositional modal logic, the two notions become:

Truth: $A_1, \ldots, A_n \vdash_t B$ iff given a frame and a valuation in that frame and a world in it, if all the A_i are true in that world, then B is true in that world.

Validity: $A_1, \ldots, A_n \vdash_v B_1, \ldots, B_m$ iff given a frame if all the A_i are true in every world, then B is.

Clearly, $A \vdash_v \Box A$ but $A \nvdash_t \Box A$, so the deduction theorem obtains for $\vdash_t$ but not for $\vdash_v$. If there are no assumptions, the two notions coincide, so both $\nvdash_v A \supset \Box A$ and $\nvdash_t A \supset \Box A$ hold. This explains why the deduction theorem may fail for some notions of logical consequence (cf. the semantical argument of Basin et al.).

As Sundholm observes, with the **syntactical notion of consequence** there is no Tarskian *orgy of definitions* (Sundholm 2002):

> ...derivability from assumptions, that is syntactic consequence, does not share the universal form of semantic consequence. On the contrary, syntactic consequence holds in virtue of the *existence* of a suitable derivation.

In particular, we have:

$$A_1, \ldots, A_n \vdash B \quad \equiv \quad \text{there is a derivation of } B \text{ from the } \textit{assumptions } A_1, \ldots, A_n$$

[3] Here we use the symbol $\vdash$ rather than $\models$ for semantic consequence to adhere to the notation used in the cited articles.

Although the notion of derivation depends on the system considered (Hilbert system, natural deduction, sequent calculus), in first-order logic it is usually understood what a derivation in each system is.

With modal logic things are not so clear. Sundholm (1983) distinguishes between *rules of proof* and *rules of inference*. The former have only theorems as premisses whereas the latter allow premisses that depend on assumptions. Sundholm considers the modal logic S4 and notes that if we define derivability from assumptions in the obvious way by allowing both modus ponens and necessitation, the necessitation rule formulated as a rule of proof

$$\frac{\vdash A}{\vdash \Box A}$$

should be converted into a corresponding rule of inference

$$\frac{\Gamma \vdash A}{\Gamma \vdash \Box A}$$

As Sundholm notes, this rule is unsound, and together with the deduction theorem it would allow the derivation of $\vdash A \supset \Box A$.

Sundholm considers an alternative formulation, quite similar to the one employed e.g. by Goldblatt (1992, p. 17), of derivation from assumptions: $\Gamma \vdash A$ whenever $\vdash A_1 \& \ldots \& A_k \supset A$ for some $A_1, \ldots, A_k$ in Γ. This move saves the deduction theorem but at the cost of losing the possibility, crucial for natural deduction, to start derivations from assumptions.

It thus seems that the divergent views on the validity of the deduction theorem in modal logic are explained by different conceptions concerning the derivability relation. For instance, Fagin et al. (1992) in an earlier article distinguish validity inferences and truth inferences (following Avron 1991) and note that the necessitation rule, as well as the rule of universal generalization, permit validity inferences rather than truth inferences. As Avron (1991) notes, axiomatic systems were designed for proving theorems, and in the case of theorems, there is no difference between validity reasoning and truth reasoning. All the premisses are valid because they are either axioms or theorems. It does not matter whether the inference rules are truth preserving or only validity preserving because every premiss is valid and thus true in every world, and this property is transferred to the conclusion. When axiomatic systems are extended to allow arbitrary assumptions, this is not necessarily the case anymore, and there are choices to be made. With the choice to keep the reasoning system untouched and to treat the assumptions as if they were axioms, one loses the deduction theorem in its usual formulation and violates strong intuitions concerning necessity or knowability. A better approach is instead to modify the reasoning system to take into account the possibility of non-tautologous assumptions. These are not just technical decisions, but they stem directly from one's conception of the derivability relation, that is, whether one understands $\vdash$ as based on truth or validity. The problem is that this understanding is usually left implicit and the choice of the relation one is trying to model is not argued for.

3. A proof of the deduction theorem

In 1967 Zeman stressed the care needed in the formulation of the notion of proof from assumptions as a prerequisite for the statement (and proof) of the deduction theorem:

> In a certain sense, there is no trick to merely stating the deduction theorem for a given system (on the assumption, of course, that it holds for that system). The general statement of the theorem might be, *"If there is a proof from the hypotheses $A_1, \ldots, A_n$ for the formula B, then there is a proof from the hypotheses $A_1, \ldots, A_{(n-1)}$ for the formula $A_n \supset B$."* The problem in formulating the deduction theorem lies not in simply stating it as above, but in

> defining just what we mean by "proof from hypotheses" for the system in question. Once we have such a definition, the statement and proof of the theorem will ordinarily present no real problem.

This basic understanding of the problem has been pursued in detail by Thomas Satre (1972), where a notion of deduction from assumptions in axiomatic presentations for modal logic and a proof of the deduction theorem are given, as a means for formulating systems of natural deduction.

We give here[4] a formal notion of derivation from assumptions and a detailed proof of the deduction theorem for basic modal logic. It is straightforward to modify the proof to other systems of normal modal logic and to epistemic logic.

We consider the following Hilbert system for the basic modal logic K[5]:

A0. $A \supset A$,

A1. $A \supset (B \supset A)$,

A2. $(A \supset (A \supset B)) \supset (A \supset B)$,

A3. $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$,

A4. $(B \supset C) \supset ((A \supset B) \supset (A \supset C))$,

A5. $\neg\neg A \supset A$,

B1. $\Box(A \supset B) \supset (\Box A \supset \Box B)$ (*distribution axiom*),

R1. From A and $A \supset B$ infer B (*modus ponens*),

R2. From A infer $\Box A$ (*necessitation*).

We extend the axiomatic system to a system for derivations from assumptions, with derivability denoted by the symbol $\vdash$, as follows:

DEFINITION 1. *A formula A is* derivable from the multiset of assumptions Γ *in basic modal logic* K*, written $\Gamma \vdash A$, if A is in Γ, or is one of the axioms* A0–A5, B1*, or follows from derivable formulas through applications of the rules of modus ponens and necessitation. Necessitation can be applied only to derivations without assumptions.*

Table 1. The system HK

$$\frac{A \in \Gamma}{\Gamma \vdash A} \qquad \frac{A \in Axiom}{\Gamma \vdash A}$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash A \supset B}{\Gamma, \Delta \vdash B} \qquad \frac{\vdash A}{\Gamma \vdash \Box A}$$

Clearly, we have the following monotonicity of derivations:

PROPOSITION 1. *If $\Gamma \vdash A$, then $\Gamma' \vdash A$ for any Γ' that extends Γ.*

We can now give a proof of the deduction theorem in its usual formulation:

[4] Unlike Satre (1972), who follows Lemmon's axiomatization of modal logic, we use axiom schemata and therefore avoid the use of uniform substitution. The key point, however, already stressed in his article, is that the rule of necessitation applies only to theorems and therefore imposes no restrictions to the deduction theorem.

[5] Axioms A1–A5 are the original axioms from Hilbert (1923). Axiom A0, derivable from A1–A4 and modus ponens (cf. e.g. Negri and von Plato 2001), has been added for convenience.

THEOREM 2. *If* $A, \Gamma \vdash B$, *then* $\Gamma \vdash A \supset B$.

Proof: By induction on the given derivation.

1. If B is among the formulas in A, Γ, we have two cases:

1.1. $B \equiv A$. By A0 and the definition of HK, we have $\Gamma \vdash A \supset B$.

1.2. If B is in Γ, we have $\Gamma \vdash B$ that together with $\vdash B \supset (A \supset B)$ and modus ponens gives $\Gamma \vdash A \supset B$.

2. If B is among the initial axioms we reason as in the case above.

3. If the last rule is necessitation, we have

$$\frac{\vdash C}{A, \Gamma \vdash \Box C}$$

By necessitation we get also $\Gamma \vdash \Box C$ and the conclusion follows by axiom A1 and modus ponens.

4. If the last rule is modus ponens there are two cases:

4.1. Neither premiss of modus ponens is derived by necessitation. Then we have the two subcases:

4.1.1. In the first case, A is among the assumptions of the derivation of the left premiss of modus ponens, with Γ partitioned into Γ', Γ'':

$$\frac{A, \Gamma' \vdash C \quad \Gamma'' \vdash C \supset B}{A, \Gamma', \Gamma'' \vdash B}$$

We apply the inductive hypothesis to the left premiss and obtain a derivation of $A \supset C$ from Γ'; from an instance of A4, $(C \supset B) \supset ((A \supset C) \supset (A \supset B))$ and A3 we get $(A \supset C) \supset ((C \supset B) \supset (A \supset B))$ so by modus ponens we obtain a derivation of $(C \supset B) \supset (A \supset B)$ from Γ', and apply again modus ponens to the latter and $C \supset B$ to obtain $A \supset B$ from Γ.

4.1.2. In the second case, A is among the assumptions of the derivation of the right premiss of modus ponens

$$\frac{\Gamma' \vdash C \quad A, \Gamma'' \vdash C \supset B}{A, \Gamma \vdash B}$$

By the inductive hypothesis we get $\Gamma'' \vdash A \supset (C \supset B)$, and by using A3 and modus ponens we obtain $\Gamma'' \vdash C \supset (A \supset B)$, and the conclusion $\Gamma \vdash A \supset B$ follows by another application of modus ponens.

4.2. If one of the premisses of modus ponens is obtained by necessitation, we have two cases. In the first case we have the derivation

$$\frac{\dfrac{\vdash C}{A, \Gamma' \vdash \Box C} \quad \Gamma'' \vdash \Box C \supset B}{A, \Gamma \vdash B}$$

It is transformed as in 4.1.1 or through the simpler conversion as follows:

$$\frac{\dfrac{\dfrac{\vdash C}{\Gamma' \vdash \Box C} \quad \Gamma'' \vdash \Box C \supset B}{\Gamma', \Gamma'' \vdash B} \qquad \vdash B \supset (A \supset B)}{\Gamma \vdash A \supset B}$$

In the second case we have the derivation

$$\frac{\dfrac{\vdash C}{\Gamma' \vdash \Box C} \quad A, \Gamma'' \vdash \Box C \supset B}{A, \Gamma \vdash B}$$

and we proceed as in case 4.1.2. QED.

Observe that the above proof can be generalized to the cases in which either assumption A does not appear in the premisses or appears with a multiplicity greater than one. This latter generalization is needed in case assumptions are taken as a *set* rather than a multiset.

In the former case, the conclusion is obtained by applying axiom A1, in the latter case by axiom A2. The two axioms correspond in fact to the vacuous and multiple discharge of assumptions in natural deduction.

In the latter case, with two copies of A, from $A, A, \Gamma \vdash B$ we obtain $\Gamma \vdash A \supset (A \supset B)$ by applying Theorem 2 twice. The conclusion $\Gamma \vdash A \supset B$ follows by axiom 2 and modus ponens. We have thus obtained:

COROLLARY 3. *If* $A^n, \Gamma \vdash B$, *then* $\Gamma \vdash A \supset B$ *for all* $n \geq 0$.

As another corollary, we obtain *closure under composition*, that is:

COROLLARY 4. *If* $\Gamma \vdash A$ *and* $A, \Delta \vdash B$, *then* $\Gamma, \Delta \vdash B$.

Proof: By the deduction theorem and modus ponens. QED.

What is called the *principle of detachment*, an inverse of the deduction theorem, is easily proved:

THEOREM 5. *If* $\Gamma \vdash A \supset B$, *then* $A, \Gamma \vdash B$.

Proof: By modus ponens and $A \vdash A$. QED.

The following corollary shows that two assumptions can be contracted into one:

COROLLARY 6. *If* $A, A, \Gamma \vdash B$ *then* $A, \Gamma \vdash B$.

Proof: By Corollary 3 we obtain $\Gamma \vdash A \supset B$ and then use Theorem 5. QED.

Clearly, these results are not specific to the system K, but hold, mutatis mutandis, for all the modal systems with a Hilbert-style axiomatization that follows the pattern of Definition 1.

4. Sequent calculus

An alternative way to formulate a system for basic modal logic is to use sequent calculus. A sequent $\Gamma \rightarrow \Delta$ formalizes the derivability of the cases in Δ from the assumptions in Γ, and therefore no separate notion of hypothetical derivation has to be introduced.

Table 2. The system G3K

Initial sequents:

$P, \Gamma \to \Delta, P$

Propositional rules:

$$\frac{A, B, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta}\, L\wedge \qquad \frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \wedge B}\, R\wedge$$

$$\frac{A, \Gamma \to \Delta \quad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}\, L\vee \qquad \frac{\Gamma \to \Delta, A, B}{\Gamma \to \Delta, A \vee B}\, R\vee$$

$$\frac{\Gamma \to \Delta, A \quad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}\, L\supset \qquad \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}\, R\supset$$

$$\frac{}{\bot, \Gamma \to \Delta}\, L\bot$$

Modal rule:

$$\frac{\Gamma \to A}{\Box\Gamma, \Theta \to \Delta, \Box A}\, LR\Box$$

Observe that the rule of $\Box$-generalization

$$\frac{\to A}{\Gamma \to \Box A, \Delta}\, \Box Gen$$

follows as a special case of rule $LR\Box$ so it need not be assumed as a primitive rule of the system.

To prove the equivalence between the above sequent system and the corresponding Hilbert-type system, we need first to establish results about its structural properties. We shall give only sketches of the proofs and full details for the characteristic cases of the calculus. We refer to Negri and von Plato (2001, chapter 3), for the overall structure of the proof of admissibility of the structural rules for G3-style sequent calculi.

LEMMA 7. *All sequents of the form $A, \Gamma \to \Delta, A$ are derivable in* G3K.

Proof: By induction on the formula A. QED.

LEMMA 8. *All the propositional rules of the system* G3K *are height-preserving invertible.*

Proof: As in Negri and von Plato (2001, theorem 3.1.1) for the system G3c. QED.

PROPOSITION 9. *Left and right weakening are height-preserving admissible in* G3K.

Proof: Weakening is in-built by the presence of an arbitrary context in initial sequents and in the conclusion of each rule. QED.

PROPOSITION 10. *Left and right contraction are height-preserving admissible in* G3K.

Proof: The proof is by simultaneous induction for left and right contraction, with induction on derivation height. We consider the last rule used in the derivation of the premiss of contraction. If the rule is a propositional one, the proof proceeds as in Negri and von Plato (2001, theorem 3.2.2). If it is a modal rule, the cases of both contraction formulas not principal, and of one principal, are treated by re-applying the rule with a context in which one occurrence of the contraction formula is removed. QED.

PROPOSITION 11. *Cut is admissible in* G3K.

Proof: We consider here only the cases that arise from the addition of the modal rules to the calculus G3c. Either the cut formula is principal in both premisses of cut, or it is a side formula in at least one of the premisses. In the former case we have to analyse the case of a cut with premisses derived by $LR\Box$. In the latter, the case that requires attention is the one with rule $LR\Box$ as one of the premisses, because the permutability argument of non-principal cuts has to be modified:

1. The cut formula $\Box A$ is principal in both premisses:

$$\dfrac{\dfrac{\Gamma \to A}{\Box\Gamma, \Theta \to \Delta, \Box A}\,{\scriptstyle LR\Box} \qquad \dfrac{A, \Gamma' \to B}{\Box A, \Box\Gamma', \Lambda \to \Delta', \Box B}\,{\scriptstyle LR\Box}}{\Box\Gamma, \Box\Gamma', \Theta, \Lambda \to \Delta, \Delta', \Box B}\,{\scriptstyle Cut}$$

This is transformed into the following derivation, with a cut of reduced height on a smaller formula

$$\dfrac{\dfrac{\Gamma \to A \qquad A, \Gamma' \to B}{\Gamma, \Gamma' \to B}\,{\scriptstyle Cut}}{\Box\Gamma, \Box\Gamma', \Theta, \Lambda \to \Delta, \Delta', \Box B}\,{\scriptstyle LR\Box}$$

2. If the cut formula is a side formula in the conclusion of $LR\Box$ we can have, for instance, the derivation

$$\dfrac{\begin{matrix}\vdots \\ \Gamma \to \Delta, A\end{matrix} \qquad \dfrac{\Gamma' \to B}{A, \Box\Gamma', \Lambda \to \Box B, \Delta'}\,{\scriptstyle LR\Box}}{\Gamma, \Box\Gamma', \Lambda \to \Delta, \Box B, \Delta'}\,{\scriptstyle Cut}$$

which is transformed into the derivation

$$\dfrac{\Gamma' \to B}{\Gamma, \Box\Gamma', \Lambda \to \Delta, \Box B, \Delta'}\,{\scriptstyle LR\Box}$$

with the cut removed. A similar conversion applies if $LR\Box$ derives the left premiss of cut and the cut formula belongs to the right weakening context. QED.

LEMMA 12. *All the axioms of* K *are derivable in* G3K.

Proof: Straightforward. QED.

We write G3K$\vdash \Gamma \to \Delta$ if the sequent $\Gamma \to \Delta$ is derivable in G3K. The system G3K is equivalent to the system K, in the following sense:

THEOREM 13.

1. HK$\vdash A$ *iff* G3K$\vdash A$.

2. $B_1, \ldots, B_m \vdash A_1 \vee \ldots \vee A_r$ *in* HK *iff* G3K $\vdash B_1, \ldots, B_m \to A_1, \ldots, A_r$.

Proof: We observe that 1 is a special case of 2 and prove the latter.

Assume $B_1, \ldots, B_m \vdash A_1 \vee \ldots \vee A_r$ in HK. We show G3K$\vdash B_1, \ldots, B_m \to A_1, \ldots, A_r$ by induction on the given derivation.

If the assumption holds because $A_1 \vee \ldots \vee A_r$ is one of the B_i, the conclusion follows by Lemma 7.

If it holds because $A_1 \vee \ldots \vee A_r$ is an axiom, we get the claim by Lemma 12 and Proposition 9.

If the last step is modus ponens, then, for some partition Γ, Γ' of the multiset $B_1, \ldots, B_m$, we have $\Gamma \vdash C$ and $\Gamma' \vdash C \supset A_1 \vee \ldots \vee A_r$. By the induction hypothesis, we obtain G3K $\vdash \Gamma \to C$

and G3K $\vdash \Gamma' \rightarrow C \supset A_1, \ldots, A_r$. By the invertibility of $R\supset$ and admissibility of cut, we have the claim.

If the last step is necessitation, we have $A_1 \vee \ldots \vee A_r \equiv \Box A$ and $K \vdash A$, so by the induction hypothesis, G3K $\vdash \rightarrow A$, and by LR$\Box$, G3K $\vdash B_1, \ldots, B_m \rightarrow \Box A$.

Conversely, assume G3K$\vdash B_1, \ldots, B_m \rightarrow A_1, \ldots, A_r$ and proceed to show $B_1, \ldots, B_m \vdash A_1 \vee \ldots \vee A_r$ in HK by induction on the height of the derivation in sequent calculus. If we have an initial sequent, then $A_i \equiv B_j$ for some i, j and therefore we have $B_1, \ldots, B_m \vdash A_i$ that together with modus ponens and $\vdash A_i \supset A_1 \vee \ldots \vee A_r$ gives the desired conclusion.

If the last rule is a propositional rule, say $L\&$ with principal formula $B_1 \equiv B_1' \& B_1''$, by the induction hypothesis, we have $B_1', B_1'', \ldots, B_m \vdash A_1 \vee \ldots \vee A_r$ that gives the claim by applying Corollary 4 together with $B_1 \vdash B_1'$, $B_1 \vdash B_1''$, and Corollary 6.

If the last rule is $LR\Box$, say

$$\frac{\Gamma \rightarrow C}{\Theta, \Box\Gamma \rightarrow \Box C, A_2, \ldots, A_r}$$

then by the inductive hypothesis, we get $\Gamma \vdash C$ in HK, therefore by closure under composition and the deduction theorem we obtain $\vdash \&\Gamma \supset C$, so by the necessitation rule and the distribution axiom and detachment we have $\Box\&\Gamma \vdash \Box C$, and the conclusion follows by using $\Box\Gamma \vdash \Box\&\Gamma$, Proposition 1, and Corollary 4 together with the derivable $\Box C \vdash \Box C \vee A_2 \vee \ldots \vee A_r$. QED.

We conclude with a formal proof of the underivability of $A \vdash \Box A$: It cannot be the conclusion of any rule other than $LR\Box$ because, in general, no logical constant appears in A, that is, A could be an atomic formula, devoid of logical structure. On the other hand, it can be the conclusion of $LR\Box$ only if A is a theorem. But in that case also $K \vdash A \supset \Box A$ would be derivable. Thus, we have not only proved that the deduction theorem holds in modal logic, but we have also proved that *it does not fail* by the standard failure argument.

A sequent system G3K$_n$ for the epistemic logic K_n is obtained by replacing, in system G3K above, the rules for the necessity operator with rules for the knowledge operator K_i relative to agent i. All the results established for G3K hold for G3K$_n$ as well.

5. Conclusions

The answer to the question whether the deduction theorem fails for modal logic depends on what is meant by deduction from assumptions. The problem with the deduction theorem arises if assumptions are taken simply as additional axioms, without any modifications to the rules initially designed for axiomatic systems.

The problem can be understood semantically through the distinction between validity inference and truth inference. We find that inference should be understood as truth inference, as this notion is the one used in proving a completeness theorem for labelled systems of modal logic. It also corresponds to our intuitive notion of valid reasoning: It is generally thought that the distinctive property of deductive reasoning – as opposed to inductive or abductive reasoning – is that deductive reasoning is truth-preserving: If the premisses are true, the conclusion should be true as well. However, the unrestricted rule of necessitation is not truth-preserving, it is only validity-preserving.

A restriction in the rule of necessitation (or knowledge generalization in the epistemic system) maintains the deduction theorem in its original formulation with only a minor modification in the notion of derivability.

We have provided a contraction- and cut-free sequent calculus for basic modal logic and for epistemic logic such that derivability in it is equivalent to derivability from assumptions in the corresponding Hilbert system. The notion of derivability from assumptions is inherent in the notion of a sequent and the rule of necessitation (or knowledge generalization) has no special status, but follows as a special case of a rule of the system. The deduction theorem is just one of the rules of the calculus, namely the right rule for implication.

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