Proof theory for modal logic

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Abstract

The axiomatic presentation of modal systems and the standard formulations of natural deduction and sequent calculus for modal logic are reviewed, together with the difficulties that emerge with these approaches. Generalizations of standard proof systems are then presented. These include, among others, display calculi, hypersequents, and labelled systems, with the latter surveyed from a closer perspective.

1 Introduction

In the literature on modal logic, an overall skepticism on the possibility of developing a satisfactory proof theory was widespread until recently. This attitude was often accompanied by a belief in the superiority of model-theoretic methods over proof-theoretic ones.

Standard proof systems have been shown insufficient for modal logic: Natural deduction presentations of even the most basic modal logics present difficulties that have been resolved only partially, and the same has happened with sequent calculus. These traditional proof system have failed to meet in a satisfactory way such basic requirements as analyticity and normalizability of formal derivations. Therefore alternative proof systems have been developed in recent years, with a lot of emphasis on their relative merits and on applications.

We review first the axiomatic presentations of modal systems and the standard formulations of natural deduction and sequent calculus for modal logic, and the difficulties that emerge with these approaches. We then move to the generalizations of standard proof systems. These include, among others, display calculi, hypersequent, and labelled systems. The last ones are presented from a closer perspective; we discuss their methodological, computational, and metatheoretical properties: the way they answer the challenge of extending proof-theoretic semantics to non-classical logics, their expressive power, analyticity, applicability to proof search, the possibility to obtain direct completeness proofs without artificial Henkin-set constructions, and their use in the solution of problems that usually involve complex model-theoretic constructions such as negative results in correspondence theory and modal embeddings among different logics. All these results can be obtained in an elementary way through methods of proof analysis of labelled sequent calculi. These calculi cover most modal and non-classical logics that permit a relational semantics.

2 Axiom systems

The language of propositional modal logic is obtained by adding to the language of propositional logic the two modal operators $\Box$ and $\Diamond$, to form from any given formula

A the formulas $\Box A$ and $\Diamond A$. These are read as "necessarily A" and "possibly A," respectively.

An axiom system for modal logic can be an extension of intuitionistic or classical propositional logic. In the latter, the notions of necessity and possibility are interdefinable by the equivalence $\Box A \supset\subset \neg\Diamond\neg A$.

It is seen that necessity and possibility behave analogously to the quantifiers: In one interpretation, the necessity of A means that A holds in all circumstances, and the possibility of A means that A holds in some circumstances. The definability of possibility in terms of necessity is analogous to the classical definability of existence in terms of universality.

The first axiomatization of modal logic is usually attributed to Lewis and Langford (1932) but it seems that the historically correct attribution is MacColl (1906), as documented in a series of paper appeared in Astroh and Read (1998) and in Rahman and Redmond (2008). The axiomatization was based on "strict implication," a notion introduced to solve the paradoxes of material implication, so that the way the system was specified was not through an extension of classical logic. The axiomatization in use nowadays, given in Lemmon (1957), is instead directly an extension of classical logic. Part III of the book by Hughes and Cresswell (1968) contains a detailed historical account on the development of axiomatic modal logic.

The system of basic modal logic, denoted by K in the literature, adds to the axioms of classical propositional logic and modus ponens the following schemes:

1. Axiom: $\Box(A \supset B) \supset (\Box A \supset \Box B)$,
2. Rule of necessitation: From A to infer $\Box A$.

The axiom states that the necessity operator distributes over implication. The rule of necessitation, in turn, can be applied only to premisses that are derivable in the axiomatic system, i.e., it states that if A is a theorem, also $\Box A$ is a theorem. The condition is automatically satisfied in traditional axiomatic systems but not in their extensions that permit inference from assumptions, and for this reason the rule has caused some confusion in the literature; we shall return to this issue in the next section.

Sometimes axioms are given using propositional variables and a rule of substitution is given that permits to substitute any formula for them. We shall instead regard axioms as schematic, so that substitution is superfluous and indeed admissible.

The early study of modal logic, to the late 1950s, consisted mainly of suggested axiomatic systems based on an intuitive understanding of the basic notions. Certain axiomatizations became standard and are collected here in the form of a table. All of them start with the axioms of classical propositional logic and the axiom and rule of basic modal logic K. The systems obtained in this way are called normal modal logics.

Name	Axiom
T	$\Box A \supset A$
4	$\Box A \supset \Box\Box A$
E	$\Diamond A \supset \Box\Diamond A$
B	$A \supset \Box\Diamond A$
3	$\Box(\Box A \supset B) \lor \Box(\Box B \supset A)$
D	$\Box A \supset \Diamond A$
2	$\Diamond\Box A \supset \Box\Diamond A$
W	$\Box(\Box A \supset A) \supset \Box A$

Well-known extensions of basic modal logic are obtained through the addition of one or more of the above axioms to system K, for instance K4 is obtained by adding axiom 4, S4 by adding axiom T and 4, S5 by adding axioms T, 4, and E (or axioms T, 4, and B), and deontic S4 and S5 by replacing axiom T with axiom D in S4 and S5, respectively. The addition of W gives what is known as the Gödel-Löb system GL. Axiom 2, also known as axiom M, gives the extensions of K4 and S4 known as K4.1 and S4.1, respectively. Axiom 3 is used for instance in the extension S4.3 of system S4.

Of these systems, GL has been of particular interest for mathematical logic, because it encodes logical properties of the notion of provability. It can be shown that the notion of derivability in a formal system of arithmetic, when internalized in arithmetic as in Gödel's incompleteness theorem, is captured by GL: An arithmetic provability predicate $\exists n Pr(n, m)$ expresses that there exists a Gödel number n of a formal derivation of the formula A with the Gödel number m. To the arithmetic notion of provability corresponds a modal operator $\Box A$ that expresses the provability of A. Solovay's theorem, one of the highlights in research on modal logic, proves that the logic GL is complete with respect to arithmetic provability in a precisely definable sense (cf. Solovay 1976).

An axiomatic proof starts from instances of the axiom schemes of propositional logic or of the specific modal logic in question and proceeds by application of the rules of modus ponens and necessitation. Whereas the presentation of a logic in axiomatic terms and the recognition of what counts as a proof in an axiomatic system is unproblematic, the actual finding of proofs is painstaking. Typically one has to start from complex instances of the axioms even to get to obvious conclusions such as $A \supset A$. For this reason, in presentations that limit themselves to axiomatic systems, as a preliminary to the actual use of such systems, a number of rules are shown admissible: If A is derivable and B is derivable, then $A\&B$ is derivable, if A is derivable, then $A \vee B$ is derivable, and so on. These results are not only of practical value; they are also needed for proving the equivalence of axiomatic systems and systems of natural deduction or sequent calculus. Another necessary move in proving such equivalence is the extension of the notion of derivability in an axiomatic system to the notion of derivability from assumptions and the proof of the deduction theorem. The latter states that if B is derivable from A and some other assumptions Γ, then $A \supset B$ is derivable from the assumptions Γ alone. This roughly amounts to the fact that implication internalizes in the object language the notion of derivability. The rule of necessitation then requires a restriction, otherwise we could conclude $\Box A$ from any given assumption A. Since $A \supset \Box A$ is not derivable in basic modal logic, this argument has been widely used to sustain the claim that the deduction theorem fails in modal logic, and alternative statements of the deduction theorem have been proposed to circumvent the apparent problem. By imposing the restriction that the necessitation rule be applied only to derivations without assumptions, the deduction theorem holds in its usual formulation also for modal logic. If one thinks of the analogy between necessity and universal quantification, it appears that the restriction is analogous to the variable condition in the rule for introducing the universal quantifier. For a survey on the debate around the deduction theorem in modal logic, see Hakli and Negri (2011).

3 Natural deduction

Axiomatic systems for modal logics are easy to formulate–one just needs to add certain axioms to a given stock of other axioms and rules–but they are inadequate both for proof search and for the study of the structure of formal derivations. Once a proof has been found, there is no possibility to transform it into some standard form that would make its structure transparent. Natural deduction was introduced by Gentzen as a formalism with strong structural properties that permit to establish decidability and consistency results through a combinatorial analysis of derivations. This analysis is made possible by the conversion, called *normalization*, of a derivation to a normal form that does not contain redundant parts and that satisfies the subformula property, a basic requirement of analytic calculi.

Modal logic is an extension of classical logic, and the difficulties in developing a normalizing system of natural deduction for classical logic, that led Gentzen to the definition of another formalism, sequent calculus, are well known. To obtain a normalizing system of natural deduction for classical logic with the subformula property, Prawitz limited the rule of indirect proof (if from $\neg A$ follows $\bot$, then A) to atomic formulas. The rule is then admissible for all formulas in the fragment without disjunction and the existential quantifier. Since in classical logic these are definable from the other connectives, one obtains in this way a normalizing system for classical logic. It remained, however, an open problem to extend the result to the full language of classical logic, and several proposals have been made in the literature. In addition to these difficulties, modal logic has a context-dependent rule (the rule of necessitation) that cannot be applied to arbitrary derivations. It is therefore not surprising that the search for a normalizing system for the full language of modal logic, with a subformula property without exceptions, is still a subject of active investigation.

Prawitz (1965, pp. 74–80) gives systems of natural deduction for S4 and S5 based on classical, intuitionistic, and minimal logic. The first system for S4 allows application of the necessitation rule to derivations of formulas (without assumptions) and to derivations with only modal assumptions (formulas with $\Box$ as outermost sign). For S5 also negations of modal formulas are allowed among the assumptions. The necessitation rule is regarded as the introduction rule for $\Box$, whereas the elimination rule is simply $\Box A/A$. Although "natural," the system is not normalizable, as there are non-eliminable detours. Prawitz introduces for this reason a liberalization on the restriction to the rule of necessitation and allows as assumptions *essentially modal formulas*. These are defined inductively as modal formulas, their disjunctions, conjunctions, and falsity (and their negations for S5, and the clause for implication for the classical base). Another example shows that it is not always possible to eliminate detours from derivations in this system. Prawitz' third system makes a further liberalization on the applicability of the necessitation rule: Necessitation on A can be applied whenever every branch of the derivation of A contains an essentially modal formula that does not not depend on any assumptions on which A does not depend (we say that the derivation has an *essentially modal bar*). Medeiros (2006), however, gives an example of a derivation in Prawitz' third system for classical S4 in which the conversion for the rule of indirect proof produces an illicit application of necessitation, because a formula on which the essentially modal formula depends is discharged before the step of necessitation. The move to obtain a normalizable system in the work of Medeiros, as well as in Martins and Martins (2005) for system S5, consists in generalizing the rule of indirect proof to arbitrary formulas. They show that a normalizing system of natural deduction is obtained for the full language of S4 (resp. S5), but without the subformula property.

The situation is simpler for modal logic based on intuitionistic logic, as there is no rule of indirect proof. A system with a rule of $\Box$-introduction that formalizes Prawitz' modal bar condition is presented by Bierman and de Paiva (2000) and it is shown that the subformula property for normal derivations can be recovered with the addition of suitable permutation conversions to the standard ones. This is also the system upon which Medeiros (2006) builds her classical systems. Another system for intuitionistic S4 was presented by von Plato (2005). The basic idea of this work lies instead in the use of general elimination rules. These are E-rules written in the way of $\vee$-elimination, so that what are known as permutative conversions apply uniformly to all E-rules, with the effect that normality can be defined as:

All major premisses of E-rules are assumptions (see von Plato 2001 or Negri and von Plato 2001). It is shown that the resulting system is closed under composition and that all derivations can be transformed into the said kind of normal form. The propositional rules are the standard introduction and general elimination rules and the rules for $\Box$ are as follows

$$\dfrac{\begin{matrix}\Box B_1, \ldots, \Box B_n \\ \vdots \\ A\end{matrix}}{\Box A}\,\Box I \qquad\qquad \dfrac{\begin{matrix} \\ \vdots \\ \Box A\end{matrix} \quad \begin{matrix}A \\ \vdots \\ C\end{matrix}}{C}\,\Box E$$

Von Plato shows that the use of general elimination rules together with $\Box I$ and $\Box E$ blocks the problematic counterexample to normalizability of Prawitz' first system, so that there is no need to use either of the two liberalizations (essentially modal assumptions or essentially modal bar) for the application of $\Box I$.

We observe that intuitionistic S4 can also be obtained as a fragment of intuitionistic linear logic. A well-behaved system of natural deduction has been presented in Negri (2002), where references to earlier literature on natural deduction for intuitionistic linear logic can be found. This work followed the same powerful methodology of general elimination rules and showed normalization indirectly, through a translation to sequent calculus, cut elimination, and translation back to natural deduction. The procedure works because general elimination rules preserve the correspondence, otherwise lost, between cut-free derivations in sequent calculus and normal derivations in natural deduction. Useful insights from the study of natural deduction for intuitionistic linear logic for a proof-theoretical treatment of intuitionistic S4 closer to Prawitz' original formulation can instead be found in Troelstra (1995), where additional permutative conversions are used to obtain normal derivations with the subformula property.

4 Sequent systems

Sequent calculus was introduced by Gentzen to overcome the difficulties encountered with normalization of natural deduction for classical logic. Cut elimination, the analogue of normalization for natural deduction, holds with no exceptions for the full language of classical logic. In sequent calculus, unlike in natural deduction, all the rules are local, to the effect that all the open assumptions at each step of inference are explicitly listed in the *antecedent context* of the rules. Rules such as necessitation, then, do not have a special status in sequent calculus, and their applicability does not depend on any global condition that would involve a survey of the whole derivation; They are simply rules with a particular condition on the context of their premisses, in this case that the context be empty.

Sequent calculi for the modal logics K, T, S4, and S5 are presented in Ono (1998). A sequent calculus GK for the logic K is obtained by adding to the (propositional part) of the standard sequent system LK the single rule

$$\frac{\Gamma \to A}{\Box\Gamma \to \Box A}\,\Box$$

The rule of necessitation is the special case of the rule with Γ empty, and the axiom $\Box(A \supset B) \supset (\Box A \supset \Box B)$ is derivable by rule $\Box$ together with the rules for implication.

A sequent system for T is then obtained by adding to GK the rule

$$\frac{A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\,\Box\!\to$$

and a sequent system for S4 by adding also the rule

$$\frac{\Box\Gamma \to A}{\Box\Gamma \to \Box A}\ \Box \to_1$$

Cut elimination is established for each of the systems thus obtained (cf. Ohnishi and Matsumoto, 1957, and the survey by Ono, 1998).

Instead of starting with the original Gentzen system LK, one can take the calculus G3c as a base. The calculus, introduced by Ketonen and successively improved and refined by Kleene, Dragalin, Troelstra, and extended by Negri and von Plato has, besides cut, the structural rules of weakening and contraction admissible (cf. Negri and von Plato 2001 for a thorough presentation and historical references). Contraction can have an effect as bad as cut for the purpose of proof search because there is no a priori bound on the number of duplications of formulas needed, so G3-calculi have been preferred to the original Gentzen sequent calculi in recent proof-theoretic literature, especially for applications to automated deduction (cf. e.g. Beeson 1991) and to extensions of proof analysis to mathematical theories (von Plato 2008, Negri and von Plato 2011).

With G3c as a base, a cut-free sequent system for K is obtained by the rule

$$\frac{\Gamma \to A}{\Phi, \Box\Gamma \to \Box A, \Psi}\ LR\Box$$

A detailed proof of the structural properties of the calculus thus obtained and of its equivalence with an axiomatic system for K is presented in Hakli and Negri (2011). This equivalence was used to prove that $\Box A$ is not derivable from A in the axiomatic system, and to show that the standard argument of failure of the deduction theorem in modal logic rests therefore on an untenable assumption.

A cut- and contraction-free sequent system for S4 is presented in Troelstra and Schwichtenberg (Ch. 9, 2000). It is obtained as an extension of G3c by the following rules for necessity and possibility, where both $\Box$ and $\Diamond$ are retained as primitive to maintain the symmetry of the system:

$$\frac{\Gamma, A, \Box A \to \Delta}{\Gamma, \Box A \to \Delta}\ L\Box \qquad \frac{\Box\Gamma \to A, \Diamond\Delta}{\Gamma', \Box\Gamma \to \Box A, \Diamond\Delta, \Delta'}\ R\Box$$

$$\frac{\Box\Gamma, A \to \Diamond\Delta}{\Gamma', \Box\Gamma, \Diamond A \to \Diamond\Delta, \Delta'}\ L\Diamond \qquad \frac{\Gamma \to A, \Diamond A, \Delta}{\Gamma \to \Diamond A, \Delta}\ R\Diamond$$

The calculus is then used to give a syntactic proof of faithfulness of a variant of Gödel's (1933) embedding of intuitionistic logic into the modal logic S4. Another proof that takes advantage of G3-style calculi as a common ground is given in Egly (2001).

Difficulties in the Gentzen-style formalization of modal logic were, however, encountered at a very elementary level, for instance in the search of an adequate cut-free sequent calculus for the modal logic S5. In 1957 Ohnishi and Matsumoto presented sequent calculi with cut elimination for various modal logics, but showed in a subsequent paper in 1959 that cut elimination failed for the system for S5 they had presented. A decision procedure was instead established indirectly by the use of a translation that bounds the number of modalities in an S5-formula.

Mints (1968) gave a sequent calculus for S5 with quantifiers that enjoys cut elimination but not the subformula property. The same limitation is encountered in Sato (1980). A more elegant treatment is presented in Shvarts (1989), who gave an indirect proof of cut elimination: A is provable in S5 if and only if $\Box A$ is provable in a suitable cut-free calculus. A similar idea, translated in terms of tableaux systems, was exploited in Fitting (1999). Braüner (2000) proved cut elimination for a calculus for S5 with global design constraints, thus departing from what is usually accepted as an orthodox system of sequent calculus, where all the rules are local.

Another logic that receives an unproblematic sequent system presentation is the deontic logic D. A cut-free sequent calculus was obtained by Valentini (1993) by the addition, to a previously formulated system for K, of a rule that corresponds to the axiom $\neg\Box\perp$

$$\frac{\Gamma \rightarrow}{\Box\Gamma \rightarrow}$$

We observe that Valentini used a system with an explicit rule of weakening and a contraction rule implicit in the treatment of "sequents as sets." As discussed in Chapter 6 of Negri and von Plato (2011) this approach, when completely formalized, would bring contraction back into the system among other structural rules, so it is not genuinely contraction free.

A sequent system for GL was obtained by the addition of the rule

$$\frac{\Box\Gamma, \Gamma, \Box A \rightarrow A}{\Box\Gamma \rightarrow \Box A}$$

to a system for classical propositional logic (Sambin and Valentini 1980, Leivant 1981). Sambin and Valentini (1982) and Avron (1984)[1] presented semantic proofs of closure with respect to cut for the resulting system, and Valentini (1983) presented a syntactic proof of cut elimination. In the early 1980s there was also another competing approach to the proof theory of GL by Bellin (1985) who gave a proof of normalization for a system of natural deduction for provability logic.

There has been recently a renewed interest in the sequent calculus proof theory of GL, in particular a quest for a fully convincing and simple syntactic proof of cut elimination after a criticism of the previous approaches. Goré and Ramanayake (2008) used an argument from von Plato (2001a) to tackle the problematic case that arises in Valentini's proof if an explicit rule of contraction is used in place of the implicit "sequents as sets" treatment.

It seems that the difficulties with the proof theory of provability logic arise from the general design of its sequent calculus rules: They have a single rule (both left and right) for $\Box$ that results in a lack of harmony. Semantically originated rules for the constants of modal logic should result in harmonious pairs of rules that ensure cut-elimination, as emphasized in Negri 2005. Harmony is achieved by the labelled approach which permits, in particular, a very simple proof of cut elimination for provability logic (Negri 2005).

5 Generalizations of Gentzen systems

As we have seen in the previous sections, traditional Gentzen systems have proved inadequate even for the simplest propositional modal logics. Alternative proof systems have therefore been proposed in recent years for modal logic and a variety of non-classical logics.

Systems that internalize the semantics of the logical constants, either implicitly through a more structured language, or explicitly through the use of labels, have been an intense object of enquiry, with emphasis on their range of applicability and on their relative merits.

[1] Avron presented, in the same paper, a cut-free sequent system for the logic of provability Grz, characterized by reflexive, transitive, and well founded Kripke frames, obtained by adding the rule

$$\frac{\Box(A \supset \Box A), \Box\Gamma \rightarrow A}{\Box\Gamma \rightarrow \Box A}$$

to an LK-based system for T.

5.1 Display calculi, hypersequents, etc.

Display calculi are justified by the Galois correspondence that can be used to explain the rules of logic. In particular, if P is the modality of possibility in the past and $\Box$ the modality of necessity in the future, one can observe that they form a residuated pair, in the sense that $PA \to B$ is valid if and only if $A \to \Box B$ is. The basic idea of display logic, introduced by Belnap (1982), is the exploitation of such correspondences and an extension of the set of "structural connectives." Whereas in standard sequent systems there is only one such connective, the multivalent comma, which is interpreted as conjunction on the left and disjunction on the right, in display systems there are monovalent binary and unary connectives. The rules of display systems are grouped into "logical rules," the initial sequents and the cut rule (usually admissible), "structural rules," which are rules that govern the interrelations of the structural connectives, and "operational rules." These last rules correspond to the usual rules that introduce the logical connectives and modalities, but have the full variety of structural connectives in their premisses rather than just commas. Display systems obey a principle according to which different logical systems should be obtained by modifying only the structural rules, while maintaining the logical part unchanged (Došen, 1988, p. 352). In contrast, labelled systems, such as those presented by Negri (2005) have, as we shall see, a common core of logical rules and also admissibility of the usual structural rules. "Logical" and "structural" are both intended here in the traditional Gentzen sense: the former refers to the rules for the logical constants and the latter to the rules of weakening, contraction, and cut. Different systems are obtained through a modification of the "mathematical rules," that is, of the rules that express the frame properties of Kripke semantics. An exception to this general picture has been made for logics such as GL and Grz. These have frame properties that are not expressible by a first-order sentence but that can nevertheless be internalized through a simple modification of the standard rules for the modality (Negri, 2005, 2006, Dyckhoff and Negri 2009a).

An introduction to display calculi in modal logic can be found in Wansing (2002). This survey also presents a synthesis of alternative approaches to the sequent calculus proof theory of modal logic. All these systems are shown to lack some of the properties presented in a preliminary list of desiderata (expressive power, modularity, cut-elimination, etc.) and the display systems argued to be preferable. The alternative systems include Došen's higher level sequent systems for S4 and S5 (1985), Martini and Masini's 2-sequents (1996), multiple-sequent systems, that is, sequent systems with more than one sequent arrow, by Indrzejczak (1998), and hypersequents (see Avron 1996 for an overview of this method). Hypersequents are, roughly, lists of sequents interpreted disjunctively. They have great flexibility, have been shown to capture wide families of non-classical logics, for example, many-valued logics as in Baaz, Ciabattoni, and Montagna (2004), and can be generated in an automated way from the semantics, as in the case of logics characterized by non-deterministic matrices (Avron and Konikowska 2005). The calculi are also analytic, but the presence of internal and external structural rules, usually non-eliminable, make them less suitable for the purposes of automated deduction.

In recent years sequent calculi for modal logic have been generalized to systems in which the semantics becomes an implicit part of a more structured syntax, such as a nested structuring of rules in tree-sequents (as done in Kashima 1994 for temporal logic and in Cerrato 1996 for modal logic) and in tree hypersequents (Poggiolesi 2009) and to systems that employ the method of deep inference (Stewart and Stouppa 2005, Stouppa 2007, Brünnler 2009), but we shall not enter into the details of these developments.

5.2 Labelled calculi

The study of modal logic underwent a great conceptual unification with the introduction of Kripke, or relational, semantics (see Kripke 1963, for a first systematic presentation and Copeland 2002, for a historical survey on the precursors of the idea), to the extent that semantical methods have often become the privileged approach to modal logic of philosophers, mathematicians, and computer scientists, whereas proof-theoretic methods have been often marginalized.

We shall not explain in detail the semantics in this survey on the proof theory of modal logic but just recall that a Kripke frame is a set the elements of which are called possible worlds, with an accessibility relation. A model on a frame is defined by stating for every world which atomic formulas are true in it. The evaluation is then extended to arbitrary propositional formulas by inductive clauses: a conjunction is true at a world if both conjuncts are, a disjunction if either disjunct is, an implication if the succedent is true whenever the antecedent is, and a $\Box$-modalized formula is true at a world whenever the formula is true at all worlds accessible from that world. The accessibility relation can be required to be reflexive, transitive, symmetric, or serial, just to list a few elementary first-order properties of frames. Then axioms T, 4, E, D, respectively, are validated in any model built on a frame with those properties. Also the converse holds, namely, if those axioms are valid, the frame has to satisfy those properties. The basic insight of Kripke semantics, that the specific axioms of various modal systems correspond to specific frame properties, and the subsequent results by Sahlqvist in the seventies have given birth to the field of correspondence theory. Precise statements of the central results and references to the vast literature can be found in the recent survey by Blackburn and van Benthem (2007), in Blackburn et al. (2001), and in van Benthem (1984).

The idea underlying what are nowadays commonly designated as labelled systems is to internalize in the calculus, be it a natural deduction, a tableau or a sequent calculus, the explanation of modalities in terms of relational semantics. This cannot be regarded as semantics in disguise since the treatment is completely syntactic and no trace of external semantic reasoning is left.

This idea, with early precursors as far as in Kanger (1957), has been developed in several forms. Inference systems have been presented that incorporate possible worlds in the form of tableaux (Fitting 1983, Catach 1991, Nerode 1991, Goré 1999, Massacci 2000), in the form of natural deduction (Fitch 1966, Simpson 1994, Basin, Matthews, Viganó 1998), and in the form of sequents (Mints 1997, Viganó 2000, Kushida and Okada 2003, Castellini and Smaill 2002, Castellini 2005). The use of a syntax that includes the relational semantics has been central also in the work on first-order encodings of modal logic (Ohlbach 1993, Schmidt and Hustadt 2003) and in what is called hybrid logic (Blackburn 2000). Internalization of the algebraic - rather than relational - semantics into a natural deduction style presentation is instead mainly used in Labelled Deductive Systems (Gabbay 1996, Russo 1995). Labelled inference is often at the basis of automated proof systems, such as the LoTREC prover developed by Herzig et al.[2] An up-to-date presentation of various labelled proof systems is given in the recent monograph by Indrzejczak (2010).

Orlowska's relational proof systems for non-classical logics are also based on relational semantics, but this is not to be confused with what is commonly called nowadays relational semantics, that is, Kripke semantics. It is instead inspired by Tarski's calculus of relations (1941) in the tradition established by Rasiowa and Sikorski (1963), with the semantics based on Boolean algebras of relations with a monoid of designated elements. Given an assignment for propositional variables, arbitrary formulas are translated into terms of the algebra. The propositional connectives are interpreted into the usual corresponding boolean operations, and the

[2]See `http://www.irit.fr/Lotrec/`.

modalities through composition with designated relations in (the monoid) of the algebra. Proofs in relational proof systems rely on completeness of relational semantics: they start from the translation of the formula to be proved into the relational semantics and proceed by rules that decompose it into smaller expressions until the proof tree closes. The survey by Orlowska (1996) gives an overview of the method and references to its applications in several sub-branches of non-classical logic. A comprehensive algebraic study of the relational semantics of non-classical logics is presented in the recent monograph by Bimbo and Dunn (2008).

Rather than presenting in detail the results of the labelled systems mentioned above, we shall give a compact outline of the labelled sequent calculi proposed by the author of this survey. This method stems from a program initiated in Negri (1999) of extending basic logical calculi such as sequent calculus and natural deduction by rules for specific theories while keeping all the structural properties unaffected, and, in general, by following the guiding principles of harmony and the criteria for a good definition of a logical system, as presented in Chapter 1 of Negri and von Plato (2001).

Read (2008) discusses the labelled natural deduction systems of Simpson (1994) in the context of harmony and inferentialism. Our approach gives a satisfactory solution, for classical modal systems, to the inferentialist challenge of extending proof-theoretic semantics to non-classical logics. The meaning explanation of logical constants, and at the same time their formulation as rules of inference, is obtained as usual in labelled systems through the inductive definition of validity in terms of Kripke semantics. From

$$x \Vdash \Box A \;\; \textit{iff} \;\; \textit{for all} \;\; y, \;\; xRy \;\textit{implies}\; \; y \Vdash A$$

one obtains

If $y : A$ can be derived for an an arbitrary y accessible from x, then $x : \Box A$ can be derived

that is formalized as the rule

$$\frac{xRy, \Gamma \to \Delta, y : A}{\Gamma \to \Delta, x : \Box A}\, {\scriptstyle R\Box}$$

Arbitrariness of y becomes the variable condition y not in Γ, Δ.

From the introduction rule and through the inversion principle, as stated in Negri and von Plato (2001) in the form *"Whatever follows from a proposition must follow from the direct grounds for asserting that proposition,"* one obtains the rule

$$\frac{\Gamma \to \Delta, xRy \quad y : A, \Gamma \to \Delta}{x : \Box A, \Gamma \to \Delta}\, {\scriptstyle L\Box}$$

The rule can be equivalently given as a one-premiss rule in the following form

$$\frac{y : A, x : \Box A, xRy, \Gamma \to \Delta}{x : \Box A, xRy, \Gamma \to \Delta}\, {\scriptstyle L\Box}$$

The rules for the modality of possibility $\Diamond$ are obtained similarly from the semantic explanation

$$x : \Diamond A \;\; \textit{iff} \;\; \textit{for some} \;\; y, \;\; xRy \; \textit{and} \; \; y : A$$

Systems for various modal logics are obtained in a modular way by adding rules that correspond to their characterizing frame properties. To achieve that, first these properties are converted into normal forms consisting of conjunctions of geometric implications or, more simply, of universal closures of quantifier-free axioms in conjuctive normal form[3]. Each of these conjuncts can then be converted to a rule that follows a general scheme for axioms of this form, called the *geometric rule scheme*, or the *regular rule scheme*, respectively. For example, the rules to be added to the basic systems to obtain a system for S4 are reflexivity and transitivity

$$\frac{xRx, \Gamma \to \Delta}{\Gamma \to \Delta}\, {\scriptstyle Ref} \qquad \frac{xRz, xRy, yRz, \Gamma \to \Delta}{xRy, yRz, \Gamma \to \Delta}\, {\scriptstyle Trans}$$

and to obtain a system for deontic logic, the axiom of seriality $\forall x \exists y.xRy$ is turned into the rule

$$\frac{xRy, \Gamma \to \Delta}{\Gamma \to \Delta}\, {\scriptstyle Ser}$$

where y is a fresh variable.

By the general results proved in Negri and von Plato (1998) and in Negri (2003), extensions of G3c with rules maintain the structural properties of the basic G3 system. The same holds if the extension is made for a labelled system, as presented in Negri (2005). The resulting labelled calculi enjoy, besides a full cut elimination, strong structural properties such as height-preserving invertibility of all the rules, height-preserving admissibility of weakening and contraction, and the possibility to apply certain rules in a restricted way to obtain decision procedures directly from the sequent calculus. These properties were expressed as a desideratum in Simpson (1994) and were achieved in full for the first time in Negri (2005).

Labels for possible worlds and accessibility relations provide a conservative extension of the syntax and permit a methodologically uniform investigation of a vast class of non-classical logics. Specific systems for modal and other non-classical logics, such as intermediate and substructural logics, are obtained by the addition of rules that correspond to frame properties (Negri 2005, Negri 2006, Dyckhoff and Negri 2009).

The addition of rules that correspond to frame properties is common to most labelled and hybrid approaches mentioned above. It is an idea that goes back to the syntactic characterization of frame properties as rules for tableau construction (Kripke 1963, Catach 1991, Nerode 1991).

Traditional completeness proofs for modal logic use an adaptation of Henkin's method of maximal consistent sets of formulas. Completeness for labelled calculi can, instead, be achieved in a direct way, which is closer to Kripke's original constructive completeness proof: For any given sequent either a proof in the given logical system or a countermodel in the corresponding frame class is found (cf. Negri 2009). The proof search method also gives a heuristic for finding frame correspondents of modal axioms, as exemplified in Hakli and Negri (2011a). Failed correspondences, such as the result stating that there is no modal axiom that corresponds to the frame property of irreflexivity, are established as simple consequences of a combinatorial proof analysis (Negri 2005).

The expressive power of the class of calculi generated by the approach encompasses what are known as *properly displayable logics* (Kracht 1996, Wansing 1998), which are captured by the extension with rules for geometric implications. In addition, the approach covers also the treatment of quantifiers in modal logic, of systems with frame conditions that are not expressible as first-order sentences, such as Gödel-Löb or Grzegorczyk logic (Negri 2005, Dyckhoff and Negri 2009a), of systems with transitive closure of accessibility relations, such as the logic of Priorean linear time (Boretti and Negri 2010), and it can be used directly in the multi-modal

[3] References to earlier literature on geometric theories, in particular their representation as rules that extend natural deduction (Simpson 1994), can be found in Negri and von Plato (2005) and in the notes to Chapter 8 of Negri and von Plato (2011).

setting of applications in formal epistemology and logics of social interaction (Hakli and Negri 2008, 2011a) and in dynamic epistemic logic (Maffezioli and Negri 2010). The method developed by Negri (2005) was also applied to formulate analytic calculi and decision procedures for conditional logics in Olivetti et al. (2007) and subsequent work.

Although these systems do not have a full subformula property, analyticity is nevertheless guaranteed by the admissibility of the structural rules together with a *subterm property*: all terms (labels) in the derivation are either eigenvariables, introduced by rules that remove a modality, or are terms found in the conclusion. (A similar property would have been difficult to establish for other labelled systems in the literature, such as the calculi of Castellini and Smaill 2002, where all relational formulas disappear from premisses to conclusion.) Whereas admissibility of the structural rules is established once and for all in a modular way for all systems that belong to the class of geometric extensions, the subterm property usually needs to be established through a case-by-case analysis. Uniformity in the calculi obtained provides a valuable tool for all results concerned with the comparison and interaction of different logics. One such result is a uniform proof of soundness and faithfulness of the embedding of intuitionistic logic and classical logic into S4 and S5 respectively, and of a large family of intermediate logics into their corresponding modal logic (Dyckhoff and Negri 2009). A similar approach had been used by Kushida and Okada (2003) to prove faithfulness of the first-order translation of S4, extending an earlier result by Mints (1968a). An important difference is that in the system used by Kushida and Okada, the frame properties become axioms, rather than rules, of the labelled system. An advantage of the purely syntactic proof of these embedding results over the existing model-theoretic ones is that the correspondences obtained are on the level of proofs rather than on the level of provability. In addition, the method based on rules rather than axioms can be extended to logics that are not characterized by first-order frame properties (Dyckhoff and Negri 2009a).

When comparing different proof systems, some aspects emerge that are superior in some and inferior in others, but no proof system is unanimously recognized as superior in all aspects and the choice of which system to use is dictated by the intended applications. In order to transfer results from one system to another, methods have been developed for moving among different proof systems: These include translations from Kripke's original labelled tableaux to labelled sequents and from labelled sequents with reflexive, transitive and symmetric accessibility relations to display calculi (Mints 1997), translations from hypersequents to display sequents (Wansing 1998a), and translations from display systems to labelled systems (Restall 2006). The relationship between hypersequent and labelled calculi for intermediate logics with geometric frame conditions is explored in Rothenberg (2010). It remains to be seen whether labelled systems can be interpreteted in a general way in terms of display or hypersequent formalisms.

The use of a labelled calculus has been sometimes criticized, as mixing semantic elements into what should be a purely syntactic proof system. To this it can be said that even the logical rules of a calculus are motivated by semantical considerations, they reflect the intuitive meaning of the logical constants. The aversion to labels resembles the criticisms that were leveled against the use of complex numbers in the 19th century. Complex numbers permitted the solution of problems, such as the integration of real valued functions without a primitive, with the use of powerful and general methodologies. In the same way, the use of labels permits an extension of the scope of syntactic methods to logics and classes of problems that could not be treated before.

For lack of space, we could not even survey all the approaches to the proof theory of modal logic. A rather detailed presentation of Fitch-style proof systems, of semantic and labelled (or prefixed) tableaux, as well as of hypersequents for modal logic can be found in Fitting (2007). We have omitted also the proof theory of first-order modal logic (for which a labelled sequent calculus along the method presented here can be found in Chapter 12 of Negri and von Plato 2011) and of infinitary proof systems, such as those arising in the treatment of the epistemic operator of common knowledge (see Jäger et al. 2007) and in the logic of time (see e.g. Boretti and Negri 2010 where a finitization within a labelled approach is provided). Also the Curry-Howard isomorphism for modal logic (see Pfenning and Wong 1995) and resolution-based calculi, popular in the automated reasoning community, have had to be omitted from this short survey.

Acknowledgements

I wish to thank Roy Dyckhoff, Paolo Maffezioli, Alberto Naibo, Peter Pagin, and Joseph Vidal-Rosset for useful comments.

References

Astroh, M. and S. Read (eds) (1998) *Proceedings of the Conference: Hugh MacColl and the Tradition of Logic*, *Nordic Journal of Philosophical Logic*, vol. 3.

Avron, A. (1984) On modal systems having arithmetical interpretations, *The Journal of Symbolic Logic*, vol. 49, pp. 935–942.

Avron, A. (1996) The method of hypersequents in the proof theory of propositional non-classical logics, in W. Hodges et al. (eds) *Logic: from foundations to applications*, pp. 1-32, Oxford Science Publications.

Avron, A. and B. Konikowska (2005) Multi-valued calculi for logics based on non-determinism, *Logic Journal of the IGPL*, vol. 13, pp. 365–387.

Baaz, M., A. Ciabattoni, and F. Montagna (2004) Analytic calculi for monoidal t-norm based logic, *Fundamenta Informaticae*, vol. 59, 315–332.

Basin, D., S. Matthews and L. Viganò (1998) Natural deduction for non-classical logics, *Studia Logica*, vol. 60, pp. 119–160.

Bellin, G. (1985) A system of natural deduction for GL, *Theoria*, vol. 51, pp. 89–114.

Belnap, N. (1982) Display logic, *Journal of Philosophical Logic*, vol. 11, pp. 375–417.

Beeson, M. (1991) Some applications of Gentzen's proof theory in automated deduction, *Lecture Notes in Computer Science*, vol. 475, pp. 101–156.

Bierman, G. and V. de Paiva (2000) On an intuitionistic modal logic, *Studia Logica*, vol. 65, pp. 383–416.

Bimbó, K. and M. Dunn (2008) *Generalized Galois Logics: Relational Semantics Of Non-classical Logical Calculi*, CSLI Publications, Stanford.

Blackburn, P. (2000) Representation, reasoning, and relational structures: a hybrid logic manifesto, *Logic Journal of the IGPL*, vol. 8, pp. 339–365.

Blackburn, P. and J. van Benthem (2007) A semantic introduction to modal logic, in P. Blackburn, J. van Benthem, and F. Wolter (eds) *Handbook of Modal Logic*, Elsevier, pp. 1–84.

Blackburn, P., M. de Rijke, and Y. Venema (2001) *Modal Logic*, Cambridge University Press.

Boretti, B. and S. Negri (2010) On the finitization of Priorean linear time, in M. D'Agostino et al. (eds) *New Essays in Logic and Philosophy of Science*, College Publications, London, pp. 67–79.

Braüner, T. (2000) A cut-free Gentzen formulation of the modal logic S5, *Logic Journal of the IGPL*, vol. 8, pp. 629–643.

Brünnler, K. (2009) Deep sequent systems for modal logic, *Archive for Mathematical Logic*, vol. 48, pp. 551–577.

Castellini, C. (2005) *Automated Reasoning in Quantified Modal and Temporal Logic*, Ph.D. thesis, School of Informatics, University of Edinburgh.

Castellini, C. and A. Smaill (2002) A systematic presentation of quantified modal logics, *Logic Journal of the IGPL*, vol. 10, pp. 571–599.

Catach, L. (1991) TABLEAUX: A general theorem prover for modal logic, *Journal of Authomated Reasoning*, vol. 7, pp. 489–510.

Cerrato, C. (1996) Modal sequents, in Wansing (1996), pp. 141–166.

Copeland, B.J. (2002) The Genesis of Possible Worlds Semantics, *Journal ofPhilosophical Logic*, vol. 31, pp. 99–137.

Došen, K. (1988) Sequent-systems and groupoid models I, *Studia Logica*, vol. 47, pp. 353-385.

Dyckhoff, R. and S. Negri (2009) Proof analysis in intermediate propositional logics, ms.

Dyckhoff, R. and S. Negri (2009a) Proof analysis in Grzegorczyk logic, ms.

Egly, U. (2001) On different intuitionistic calculi and embeddings from Int to S4, *Studia Logica*, vol. 69, pp. 249–277.

Fitch, F. B. (1966) Tree proofs in modal logic, *The Journal of Symbolic Logic*, vol. 31, p. 152.

Fitting, M. (1983) *Proof Methods for Modal and Intuitionistic Logics*, Reidel Publishing Co.

Fitting, M. (1999) A simple propositional S5 tableau system, *Annals of Pure and Applied Logic*, vol. 96, pp. 107–115.

Fitting, M. (2007) Modal proof theory, in P. Blackburn, J. van Benthem, and F. Wolter (eds) *Handbook of Modal Logic*, Elsevier, pp. 85–138.

Gabbay, D. (1996) *Labelled Deductive Systems*, Oxford University Press.

Goré, R. (1999) Tableau methods for modal and temporal logis, in D'Agostino, M. *et al.* (eds) *Handbook of Tableau Methods*, Kluwer.

Goré, R. and R. Ramanayake (2008) Valentini's cut-elimination for provability logic resolved, in C. Areces and R. Goldblatt (eds) *Advances in Modal Logic*, vol. 7, College Publications, London, pp. 67–86.

Gödel, K. (1933) Eine Interpretation des intuitionistischen Aussagenkalküls, *Ergebnisse eines mathematischen Kolloquiums*, vol. 4, pp. 39–40. Engl. tr. in Gödel's *Collected Works* (1986), vol. I, pp. 300–303.

Hakli, R. and S. Negri (2008) Proof theory for distributed knowledge, in F. Sadri and K. Satoh (eds) *Computational Logic in Multi-Agent Systems*, Springer Lecture Notes in Artificial Intelligence, vol. 5405, pp. 100–116.

Hakli, R. and S. Negri (2011) Does the deduction theorem fail for modal logic? *Synthese*, in press.

Hakli, R. and S. Negri (2011a) Reasoning about collectively accepted group beliefs, *Journal of Philosophical Logic*, vol. 40, pp. 531–555.

Hughes, G. E. and M. J. Cresswell (1968) *An Introduction to Modal Logic*, Methuen, London.

Indrzejczak, A.(1998) Cut-free double sequent calculus for S5, *Logic Journal of the IGPL*, vol. 6, pp. 505–516.

Indrzejczak, A. (2010) *Natural Deduction, Hybrid Systems and Modal Logics*, Springer.

Jäger, G., M. Kretz, and T. Studer (2007) Cut-free common knowledge, *Journal of Applied Logic*, vol. 5, pp. 681–689.

Kanger, S. (1957) *Provability in Logic*, Almqvist & Wiksell, Stockholm.

Kashima, R. (1994) Cut-free sequent calculi for some tense logics, *Studia Logica*, vol. 53, pp. 119–135.

Kracht, M. (1996) Power and weakness of the modal display calculus, in Wansing (1996), pp. 93–121.

Kripke, S. (1963) Semantical analysis of modal logic I. Normal modal propositional calculi. *Zetschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 9, pp. 67–96.

Kushida, H. and M. Okada (2003) A proof-theoretic study of the correspondence of classical logic and modal logic, *The Journal of Symbolic Logic*, vol. 68, pp. 1403–1414.

Leivant, D. (1981) On the proof theory of the modal logic for arithmetic provability, *The Journal of Symbolic Logic*, vol. 46, pp. 531–538.

Lemmon, J. (1957) New foundations for Lewis modal systems, *The Journal of Symbolic Logic*, vol. 22, pp. 176–186.

Lewis, C. and C. Langford (1932) *Symbolic Logic*, Appleton-Century-Crofts, New York. Reprinted Dover Publications, New York, 1951, 1959.

MacColl, H. (1906) *Symbolic Logic and Its Applications*, Longmans, Green, and Co.

Maffezioli, P. and S. Negri (2010) A Gentzen-style analysis of public announcement logic, in X. Azzarola and M. Ponte (eds) *Proceedings of the International Workshop on Logic and Philosophy of Knowledge, Communication and Action*, pp. 293–313, University of the Basque Country Press, 2010.

Martini, S. and A. Masini (1996) A computational interpretation of modal proofs, in Wansing (1996), pp. 213–241.

Martins, A. T. and L. R. Martins (2008) Full classical S5 in natural deduction with weak normalization, *Annals of Pure and Applied Logic*, vol. 152, pp. 132-147.

Massacci, F. (2000) Single step tableaux for modal logics: methodology, computations, algorithms, *Journal of Automated Reasoning*, vol. 24, pp. 319–364.

Medeiros, P. N. M. (2006) A new S4 classical modal logic in natural deduction, *The Journal of Symbolic Logic*, vol. 17, pp. 249–265.

Mints, G. (1968) Cut-free calculi of type $S5$. (in Russian) Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) vol. 8, pp. 166–174. Translated into English in *Studies in Constructive Mathematics and Mathematical Logic*, vol. 8, Part II, A. O. Slisenko (ed), Leningrad, 1970.

Mints, G. (1968a) Some calculi of modal logic, *Proceedings of the Steklov Institute of Mathematics*, vol. 98, pp. 97–124.

Mints, G. (1997) Indexed systems of sequents and cut-elimination, *Journal of Philosophical Logic* vol. 26, pp. 671–696.

Negri, S. (1999) Sequent calculus proof theory of intuitionistic apartness and order relations, *Archive for Mathematical Logic*, vol. 38, pp. 521–547.

Negri, S. (2002) A normalizing system of natural deduction for intuitionistic linear logic, *Archive for Mathematical Logic*, vol. 41, pp. 789–810.

Negri, S. (2003) Contraction-free sequent calculi for geometric theories, with an application to Barr's theorem, *Archive for Mathematical Logic*, vol. 42, pp. 389–401.

Negri, S. (2005) Proof analysis in modal logic, *Journal of Philosophical Logic*, vol. 34, pp. 507–544.

Negri, S. (2006) Proof analysis in non-classical logics. In C. Dimitracopoulos, L. Newelski, D. Normann, and J. Steel (eds)*Logic Colloquium '05*, ASL Lecture Notes in Logic, vol. 28, pp. 107–128.

Negri S. (2009) Kripke completeness revisited, in G. Primiero and S. Rahman (eds), *Acts of Knowledge - History, Philosophy and Logic*, pp. 247–282, College Publications, London.

Negri, S. and J. von Plato (1998) Cut elimination in the presence of axioms, *The Bulletin of Symbolic Logic*, vol. 4, pp. 418–435.

Negri, S. and J. von Plato (2001) *Structural Proof Theory*, Cambridge University Press.

Negri, S. and J. von Plato (2005) The duality of classical and constructive notions and proofs. In L. Crosilla and P. Schuster (eds) *From Sets and Types to Topology and Analysis: Towards Practicable Foundations for Constructive Mathematics*, Oxford University Press, pp. 149–161.

Negri, S. and J. von Plato (2011) *Proof Analysis: A Contribution to Hilbert's Last Problem*, Cambridge University Press, in press.

Nerode, A. (1991) Some lectures on modal logic, in F. L. Bauer (ed) *Logic, Algebra, and Computation*, NATO ASI Series, vol. F79, Springer, pp. 281–334.

Ohlbach, H. J. (1993) Translation methods for non-classical logics – An overview, *Bulletin of the IGPL*, vol. 1, pp. 69–91.

Ohnishi, M. and K. Matsumoto (1957) Gentzen method in modal calculi, *Osaka Mathematical Journal*, vol. 9, pp. 113–130.

Ohnishi, M. and K. Matsumoto (1959) Gentzen method in modal calculi II, *Osaka Mathematical Journal*, vol. 11, pp. 115–120.

Olivetti, N., G. Pozzato, and C. Schwind (2007) A sequent calculus and a theorem prover for standard conditional logics, ACM Transactions on Computational Logic, vol. 8, no. 4.

Ono, H. (1998) Proof-theoretic methods in nonclassical logic. An Introduction. In M. Takahashi, M. Okada, and M. Dezani-Ciancaglini (eds.) *Theories of Types and Proofs*, Mathematical Society of Japan Memoirs 2, pp. 207–254.

Orlowska, E. (1996) Relational proof systems for modal logics, in Wansing (1996), pp. 55–78.

von Plato, J. (2001) Natural deduction with general elimination rules. *Archive for Mathematical Logic*, vol. 40 (2001), pp. 541–567.

von Plato, J. (2001a) A proof of Gentzen's Hauptsatz without multicut, *Archive for Mathematical Logic*, vol. 40, pp. 9–18.

von Plato, J. (2005) Normal derivability in modal logic, *Mathematical Logic Quarterly*, vol. 51, pp. 632–638.

von Plato, J. (2008) The development of proof theory, entry in *Stanford Encyclopedia of Philosophy* [Online] Retrieved on 25 April 2011 from http://plato.stanford.edu/entries/proof-theory-development/.

Pfenning, F. and H. Wong (1995) On a modal lambda-calculus for S4, in S. Brookes and M. Main (eds) *Electronic Notes in Theoretical Computer Science*, vol. 1, Elsevier.

Poggiolesi, F. (2009) The tree-hypersequent method for modal propositional logic, in D. Makinson, J. Malinowski and H. Wansing (eds) *Trends in Logic: Towards Mathematical Philosophy* , pp. 31–51, Springer.

Prawitz, D. (1965) *Natural Deduction: A Proof-Theoretical Study.* Almqvist & Wicksell, Stockholm.

Rahman, S. and J. Redmond (2008) Hugh MacColl and the birth of logical pluralism, in D. Gabbay and J. Woods (eds) *Handbook of the History of Logic*, vol. 4, Elsevier, pp. 533–604.

Rasiowa, H. and R. Sikorski (1963) *The Mathematics of Metamathematics*, Polish Science Publishers, Warsaw.

Read, S. (2008) Harmony and modality, in C. Dégremont et al. (eds) *Dialogues, Logics and Other Strong Things: Essays in Honour of Shahid Rahman*, College Publications, London, pp. 285–303.

Restall, G. (2006) Comparing modal sequent systems, ms.

Rothenberg, R. (2010) On the Relationship between Hypersequent Calculi and Labelled Sequent Calculi for Intermediate Logics with Geometric Kripke Semantics, Ph.D. thesis, University of St Andrews.

Russo, A. (1995) Modal labelled deductive systems, Dept. of Computing, Imperial College, London, Technical Report 95(7).

Sambin, G. and S. Valentini (1980) A modal sequent calculus for a fragment of arithmetic, *Studia Logica*, vol. 39, pp. 245–256.

Sambin, G. and S. Valentini (1982) The modal logic of provability. The sequential approach, *Journal of Philosophical Logic*, vol. 11, pp. 311–342.

Sato, M. (1980) A cut-free Gentzen-type system for the modal logic S5, *The Journal of Symbolic Logic*, vol. 45, pp. 67–84.

Schmidt, R. and U. Hustadt (2003) A principle for incorporating axioms into the first-order translation of modal formulae, in F. Baader (ed), *Automated Deduction-CADE-19*, *Lecture Notes in Artificial Intelligence* 2741, Springer, pp. 412–426.

Shvarts, G. (1989) Gentzen style systems for K45 and K45D. Logic at Botik '89 (Pereslavl′-Zalesskiy), *Lecture Notes in Computer Science*, vol. 363, pp. 245–256, Springer, Berlin.

Simpson, A. (1994) *Proof Theory and Semantics of Intuitionistic Modal Logic*, Ph.D. thesis, School of Informatics, University of Edinburgh.

Solovay, R. M. (1976) Provability interpretations of modal logic, *Israel Journal of Mathematics*, vol. 25, pp. 287–304.

Stewart, C. and P. Stouppa (2005) A systematic proof theory for several modal logics, in R. Schmidt et al. (eds) *Advances in modal logic*, vol. 5, King's College Publication, London, pp. 309–333.

Stouppa, P. (2007) A deep inference system for the modal logic S5, *Studia Logica*, vol. 85, pp. 199–214.

Tarski, A. (1941) On the calculus of relations, *The Journal of Symbolic Logic*, vol. 6, pp. 73–89.

Troelstra, A. (1995) Natural deduction for intuitionistic linear logic, *Annals of Pure and Applied Logic*, vol. 73, pp. 79–108.

Troelstra, A. and H. Schwichtenberg (2000) *Basic Proof Theory*, Cambridge University Press, second edition.

Valentini, S. (1983) The modal logic of provability: cut-elimination, *Journal of Philosophical Logic*, vol. 12, pp. 471–476.

Valentini, S. (1993) The sequent calculus for the modal logic D, *Bollettino dell'Unione Matematica Italiana*, vol. 7, pp. 455–460.

Van Benthem, J. (1984) Correspondence theory, in D. Gabbay and F. Guenther (eds) *Handbook of Philosophical Logic*, vol. 2, Kluwer, pp. 167–247.

Viganò, L. (2000) *Labelled Non-Classical Logics*, Kluwer.

Wansing, H. (1996) *Proof Theory of Modal Logic*, Proceedings of the workshop held at the University of Hamburg, Hamburg, November 19–20, 1993. Edited by Heinrich Wansing. Applied Logic Series, 2. Kluwer Academic Publishers, Dordrecht.

Wansing, H. (1998) *Displaying Modal Logic*, Trends in Logic Series, Studia Logica Library, Kluwer Academic Publishers, Dordrecht.

Wansing, H. (1998a) Translation of hypersequents into display sequents, *Logic Journal of the IGPL*, vol. 6, pp. 719–733.

Wansing, H. (2002) Sequent systems for modal logics, in D. Gabbay and F. Guenther (eds) *Handbook of Philosophical Logic*, second edition, vol. 8, Kluwer, pp. 61–145.