Proofs and countermodels in non-classical logics

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Abstract. Proofs and countermodels are the two sides of completeness proofs, but, in general, failure to find one does not automatically give the other. The limitation is encountered also for decidable non-classical logics in traditional completeness proofs based on Henkin's method of maximal consistent sets of formulas. A method is presented that makes it possible to establish completeness in a direct way: For any given sequent either a proof in the given logical system or a countermodel in the corresponding frame class is found. The method is a synthesis of a generation of calculi with internalized relational semantics, a Tait-Schütte-Takeuti style completeness proof, and procedures to finitize the countermodel construction. Finitizations for intuitionistic propositional logic are obtained through the search for a minimal derivation, through pruning of infinite branches in search trees by means of a suitable syntactic counterpart of semantic filtration, or through a proof-theoretic embedding into an appropriate provability logic. A number of examples illustrates the method, its subtleties, challenges, and present scope.

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1. Introduction

The duality of proofs and counterexamples, or more generally, refutations, is ubiquitous in science, but involves distinctions often blurred by the rhetoric of argumentation. More crisp distinctions between proofs and refutations are found in mathematics, especially in well defined formalized fragments.

Every working mathematician knows that finding a proof and looking for a counterexample are two very different activities that cannot be carried on simultaneously. Usually the latter starts when the hope to find a proof is fading away, and the failed attempts will serve as an implicit guide to chart the territory in which to look for a counterexample. No general recipe is, however, gained from the failures, and a leap of creativity is required to find a counterxample, if such is at all obtained.

In logic, things are more regimented because of the possibility to reason within formal *analytic calculi* that reduce the proving of theorems to *automatic* tasks. Usually one can rest upon a completeness theorem that guarantees a perfect duality between proofs and countermodels. So in theory. In practice, we are encountered with obstacles: completeness proofs are often non-effective (non-constructive) and countermodels are artificially built from Henkin sets or Lindenbaum algebras, and thus far away from what we regard as counterexamples. Furthermore, the canonical countermodels provided by traditional completeness proofs may fall out of the intended classes and need a model-theoretic fine tuning with such procedures as *unravelling* and *bulldozing*.

The question naturally arises as to whether we can find in some sense "concrete" countermodels in the same automated way in which we find proofs. *Refutation calculi* (as those in [Ferrari, Fiorentini and Fiorino, 2012], [Goranko, 1994], [Pinto and Dyckhoff, 1995], [Skura, 2011]) produce refutations rather than proofs and can be used as a basis for building countermodels. These calculi are separate from the direct inferential systems, their rules are not invertible (root-first, the rules give only sufficient conditions of non-validity) and sometimes the decision method through countermodel constructions uses a pre-processing of formulas into a suitable normal form (as in [Larchey-Wendling, 2002]). As pointed out in [Goré and Postniece, 2010] in the presentation of a combination of a derivation and a refutation calculus for bi-intuitionistic logic, these calculi often depart from Gentzen's original systems, because the sequent calculus **LI** or its contraction-free variant **LJT** [Dyckhoff, 1992] have rules that are not invertible; thus, while preserving validity, they do not preserve refutability. *Prefixed tableaux* in the style of Fitting, on the other hand, restrict the refutations to relational models, and countermodels can be read off from failed proof search. As remarked in [Fitting, 2012], the tree structure inherent in these calculi makes them suitable to a relatively restricted family of logics and, furthermore, the non-locality of the rules makes the extraction of the countermodel not an immediate task.

We shall present a method for unifying proof search and countermodel construction that is a synthesis of a generation of calculi with internalized semantics (as presented in [Negri, 2005] and in chapter 11 of [Negri and von Plato, 2011]), a Tait-Schütte-Takeuti style completeness proof [Negri, 2009] and, finally, a procedure to finitize the countermodel construction. This final part is obtained either through the search for a minimal, or irredundant, derivation (a procedure employed to establish decidability of basic modal logics in [Negri, 2005] and formalized in [Galmiche and Salhi 2011] for a hybrid sequent system for intuitionistic logic), a pruning of infinite branches in search trees through a suitable syntactic counterpart of semantic filtration (a method employed in [Boretti and Negri, 2009] for Priorean linear time logic and in [Garg, Genovese and Negri, 2012] for multi-modal logics) or through a proof-theoretic embedding into an appropriate provability logic that internalizes finiteness in its rules, as in [Dyckhoff and Negri, 2013].

The emphasis here is on the *methodology*, so we shall present the three stages in detail for the case of intuitionistic logic. Our starting point is **G3I**, a labelled contraction- and cut-free intuitionistic multi-succedent calculus in which *all rules are invertible*. The calculus is obtained through the internalization of Kripke semantics for intuitionistic logic: the rules for the logical constants are obtained by unfolding the inductive definition of truth at a world and the properties of the accessibility relation are added as rules, following the method of "axioms as rules" to encode axioms into a sequent calculus while preserving the structural properties of the basic logical calculus [Negri and von Plato, 1998, Negri, 2003]. The structural properties guarantee a root-first determinism, with the consequence that there is no need of backtracking in proof search. Notably for our purpose, all the rules of the calculus not only preserve the existence of countermodels because of invertibility, but are such that the countermodel defined on any suitable terminal node in a failed proof search gives a Kripke countermodel to the endsequent.

The methodology of generation of complete analytic countermodel-producing calculi will be also illustrated for the following (families of) logics and extensions: Intermediate logics and their modal companions; intuitionistic multi-modal logics; provability logics; knowability logic; logics with frame properties beyond geometric theories that cover all the Sahlqvist fragment. We shall conclude with a discussion on open problems and further directions.

2. Formal Kripke semantics in contraction-free sequent calculi

As said above, our purpose is to determine a calculus that allows the automatization not only of proof search, but also of countermodel constructions when the search for a proof fails. Our essential building block is the sequent calculus for classical propositional logic **G3c**, introduced in [Ketonen, 1944] and successively improved and extended by Kleene, Dragalin, Troelstra (cf. the notes to chapter 6 of [Negri and von Plato, 2011] for a detailed historical discussion). Here, as well as in the rest of the paper, sequents are expression of the form $\Gamma \to \Delta$ where Γ, Δ are *multisets* of formulas.

Initial sequents:

$P, \Gamma \to \Delta, P$

Logical rules:

$$\frac{A, B, \Gamma \to \Delta}{A\&B, \Gamma \to \Delta}\,{\scriptstyle L\&} \qquad \frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A\&B}\,{\scriptstyle R\&}$$

$$\frac{A, \Gamma \to \Delta \quad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}\,{\scriptstyle L\vee} \qquad \frac{\Gamma \to \Delta, A, B}{\Gamma \to \Delta, A \vee B}\,{\scriptstyle R\vee}$$

$$\frac{\Gamma \to \Delta, A \quad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}\,{\scriptstyle L\supset} \qquad \frac{A, \Gamma \to \Delta, B}{\Gamma \to \Delta, A \supset B}\,{\scriptstyle R\supset}$$

$$\frac{}{\bot, \Gamma \to \Delta}\,{\scriptstyle L\bot}$$

Table 1. The calculus **G3c**

Observe that in initial sequents P denotes an *atomic* formula. This detail is essential in establishing the following property:

Property 1. All the rules of **G3c** are invertible, with height-preserving inversion.

Height-preserving invertibility of a rule means that whenever a sequent that matches the conclusion of a rule is derivable with derivation height n then also the corresponding premisses are derivable with at most the same derivation height. For example, for the case of conjunction we have

If $\vdash_n \Gamma \to \Delta, A\&B$, then $\vdash_n \Gamma \to \Delta, A$ and $\vdash_n \Gamma \to \Delta, B$

and similarly for the other rules. By the property, the calculus can be used to decompose a task, the verification of provability of a sequent, into simpler tasks, the verification of provability of less complex sequents. This is done in a deterministic way with respect to the formulas that appear in the contexts. In calculi with *independent contexts*, the context in the conclusion is split in the premisses, in a way which is arbitrary in root-first proof search. Here the same context of the conclusion appears in both premisses, that is we have:

Property 2. All the rules have shared contexts.

Property 3. The structural rules of weakening and contraction

$$\frac{\Gamma \to \Delta}{A, \Gamma \to \Delta}\,LW \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, A}\,RW$$

$$\frac{A, A, \Gamma \to \Delta}{A, \Gamma \to \Delta}\,LC \qquad \frac{\Gamma \to \Delta, A, A}{\Gamma \to \Delta, A}\,RC$$

are height-preserving admissible in **G3c**.

In the following we will use the abbreviation *hp-invertible* for height-preserving invertible. Similarly *hp-admissible* will stand for height-preserving admissible.

A lot of emphasis has been posed, since Gentzen's work, on the eliminability of cut in sequent calculi to guarantee the subformula property, or more generally, analyticity of the calculus (through reduction of cut to analytic cut, cf. [D'Agostino, 1990]), but eliminability, or admissibility of the other structural rules is just as crucial. The desirability of the above properties is clear from the point of view of root-first proof search: a rule such as weakening, root-first, removes a formula but it is not known which formula may have to be removed, so backtracking would be needed. Contraction instead duplicates formulas, but there is no a priori indication on which formulas the rule should be applied, nor on the number of times a formula has to be duplicated, so a proof search could fail just because of missing applications of the rule. Having the rules admissible avoids these problems altogether. We also have:

Property 4. The rule of cut

$$\frac{\Gamma \to \Delta, A \quad A, \Gamma' \to \Delta'}{\Gamma, \Gamma' \to \Delta, \Delta'}\,Cut$$

is admissible in **G3c**.

Finally, we observe that the rules have *multisuccedent* sequents, with a multiset of formulas on the right hand side of the sequent arrow. This feature allows a uniform treatment of classical and intuitionistic logic, because the latter can be obtained just by modifying the rules of implication as follows:

$$\frac{A \supset B, \Gamma \to A \quad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta} L\supset \qquad \frac{A, \Gamma \to B}{\Gamma \to \Delta, A \supset B} R\supset$$

Table 2. Implication rules for the intuitionistic calculus **G3im**

Formally, the calculus **G3im** has the same structural properties as those of **G3c** listed above, with the exception of invertibility of the rules for implication. For these, we have only invertibility of $L\supset$ with respect to its right premiss, however the rule can be made invertible by retaining the context Δ in the succedent of its left premiss (cf. pf. 3.7.3 of [Troelstra and Schwichtenberg, 2000]). Also for the properties which are maintained, the rules of implication cause drawbacks that hinder their good consequences for proof search: First, even if contraction stays admissible, the repetition of the principal formulas in the left premiss of $L\supset$ is a relic of contraction itself. Second, the removal of the context in the left premiss of $L\supset$ and in $R\supset$ makes backtracking necessary. We refer to chapter 5.3 of [Negri and von Plato, 2001] for an in-depth discussion of the subtleties of the calculus, the motivations for the form of the above rules, the determination of its properties, and the variants in the literature.

An alternative way to use the calculus as a building block to develop other calculi, rather than modifying in an *ad hoc* way its rules, is to develop richer and more general calculi. This can be done is several different ways which can be divided into two main groups, one that maintains the standard syntax of sequents but enriches its language with constants for semantic entities such as accessibility and forcing relations, another that avoids any *explicit* reference to semantics but uses a more structured syntax: standard sequents are enriched by new binders in addition to the usual commas in *display calculi* [Wansing, 2002], or replaced by objects which encode a graph structure, such as generalizations of *hypersequents* [Avron, 1996, Baaz et al., 2003, Ciabattoni et al., 2008] including *nested sequents* [Kashima, 1994], *tree-sequents* [Cerrato, 1996], *tree-hypersequents* [Poggiolesi, 2010] and *deep sequents* [Brünnler, 2009, Stewart and Stouppa, 2005]. For the purpose of a parallel construction of proofs and countermodels, we are more interested in systems in which the semantics is made explicit in the syntax of the calculus. There are further choices that can be made, such as building upon a system of natural deduction, as in *Labelled Deductive Systems* [Gabbay, 1996, Russo, 1995] or in *labelled natural deduction* [Fitch, 1966, Simpson, 1994, Basin, Matthews and Viganò, 1998], of sequent calculus, as in [Mints,1997, Viganò, 2000, Kushida and Okada, 2003, Castellini, 2005], or of tableaux, as in [Fitting, 2012, Catach, 1991, Nerode, 1991, Goré, 1998, Massacci, 2000, Orlowska and Golińska Pilarek, 2011]. There are also choices concerning the extent of the labelling, that can remain external to formulas, as in the previous labelled calculi, or brought deep inside, as in *hybrid logic*

[Blackburn, 2000], where nominals are treated as "first-class citizens", with the scope of modalities extended upon them.

Because of its good proof-search properties, we shall use as a ground logical system the contraction-free sequent calculus **G3c**, to be extended by the relational or Kripke semantics. The extended syntax thus includes labels x, y for worlds of a Kripke frame, expressions of the form $x : A$ for the truth of a proposition in a world, and xRy for relations between worlds. We shall also use $\leqslant$ when the accessibility relation is a preorder. The rules of the calculus will then become a way to perform a systematic check of validity (i.e. truth in every world for every interpretation) of a formula or a sequent. We then obtain the rules for the logical constants by unfolding the inductive definition of forcing, or truth, of a formula at a world.

The truth conditions for conjunction and disjunction just move the explanation of the connective to the meta-level, that is

$$x \Vdash A\&B \iff x \Vdash A \textit{ and } x \Vdash B$$

This gives the rules

$$\frac{x : A, x : B, \Gamma \to \Delta}{x : A\&B, \Gamma \to \Delta}\, L\& \qquad \frac{\Gamma \to \Delta, x : A \quad \Gamma \to \Delta, x : B}{\Gamma \to \Delta, x : A\&B}\, R\&$$

The rules for disjunction are obtained in a similar way. The assumption that falsity is never forced at any world becomes the labelled correspondent of *ex falso quodlibet*, namely the zero-premiss rule

$$\frac{}{x :\bot, \Gamma \to \Delta}\, L\bot$$

For intuitionistic implication, we have

$$x \Vdash A \supset B \iff \textit{for all } y,\ x \leqslant y \textit{ and } y \Vdash A \textit{ implies } y \Vdash B$$

It gives the following right rule with variable condition that y is fresh, i.e., not in the conclusion of the rule:

$$\frac{x \leqslant y, y : A, \Gamma \to \Delta, y : B}{\Gamma \to \Delta, x : A \supset B}\, R\supset$$

The left rule is

$$\frac{x \leqslant y, x : A \supset B, \Gamma \to \Delta, y : A \quad x \leqslant y, y : B, x : A \supset B, \Gamma \to \Delta}{x \leqslant y, x : A \supset B, \Gamma \to \Delta}\, L\supset$$

To specify a system of intuitionistic logic we need to give the properties of the accessibility relation which we know by the semantics to be a preorder, i.e., reflexive and transitive. These properties are included in the calculus in a way that does not interfere with the admissibility of the structural rules, namely as rules of inference of a suitable form (cf. [Negri and von Plato, 1998]). Further, we need to ensure that the forcing relation is monotone with respect to the preorder, or in other words, that truth is persistent. It turns out that it is enough to impose monotonicity on the forcing of atomic formulas to ensure monotonicity of forcing on arbitrary formulas. This property is then expressed as an initial sequent. The resulting system is as follows, with the condition that y must not be in Γ, Δ in rule $R\supset$:

Initial sequents:

$x \leqslant y, x : P, \Gamma \rightarrow \Delta, y : P$

Propositional rules:

$$\frac{x : A, x : B, \Gamma \rightarrow \Delta}{x : A\&B, \Gamma \rightarrow \Delta} \, L\& \qquad \frac{\Gamma \rightarrow \Delta, x : A \quad \Gamma \rightarrow \Delta, x : B}{\Gamma \rightarrow \Delta, x : A\&B} \, R\&$$

$$\frac{x : A, \Gamma \rightarrow \Delta \quad x : B, \Gamma \rightarrow \Delta}{x : A \vee B, \Gamma \rightarrow \Delta} \, L\vee \qquad \frac{\Gamma \rightarrow \Delta, x : A, x : B}{\Gamma \rightarrow \Delta, x : A \vee B} \, R\vee$$

$$\frac{x \leqslant y, x : A \supset B, \Gamma \rightarrow \Delta, y : A \quad x \leqslant y, x : A \supset B, y : B, \Gamma \rightarrow \Delta}{x \leqslant y, x : A \supset B, \Gamma \rightarrow \Delta} \, L\supset$$

$$\frac{}{x : \perp, \Gamma \rightarrow \Delta} \, L\perp \qquad \frac{x \leqslant y, y : A, \Gamma \rightarrow \Delta, y : B}{\Gamma \rightarrow \Delta, x : A \supset B} \, R\supset$$

Order rules:

$$\frac{x \leqslant x, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta} \, Ref \qquad \frac{x \leqslant z, x \leqslant y, y \leqslant z, \Gamma \rightarrow \Delta}{x \leqslant y, y \leqslant z, \Gamma \rightarrow \Delta} \, Trans$$

Table 3. The system **G3I**

The system **G3I** has the same structural properties as **G3c**, namely:

Proposition 2.1. 1. *All the rules of* **G3I** *are height-preserving invertible.*
2. *The rules of* Weakening *and* Contraction *are height-preserving admissible in* **G3I**.
3. *The* Cut *rule is admissible in* **G3I**.

In a labelled system, a rule of substitution of labels is also often useful

$$\frac{\Gamma \rightarrow \Delta}{\Gamma(y/x) \rightarrow \Delta(y/x)} \, Subst$$

We have:

Proposition 2.2. *The rule of substitution of labels is height-preserving admissible in* **G3I**.

For a proof of the above results, see [Dyckhoff and Negri, 2012] or chapter 12 of [Negri and von Plato, 2011].

Before proceeding with the details of our completeness proof, we observe that **G3I** does not have the restriction of a single-succedent premiss in rule $R\supset$, yet the calculus does not become classical because of this liberalization: Consider the following proof search for *tertium non datur*[1]

$$\cfrac{\cfrac{\cfrac{\cfrac{?}{x \leqslant x, y \leqslant y, x \leqslant y, y : P \rightarrow x : P, y : \perp}}{x \leqslant y, y : P \rightarrow x : P, y : \perp} \, Ref^*}{\rightarrow x : P, x : \neg P} \, R\supset, y \textit{ fresh}}{\rightarrow x : P \vee \neg P} \, R\vee$$

[1] Observe that negation is defined in terms of implication and the asterisk is used to denote repeated (here two) applications of a rule.

Clearly, no rule is applicable to $x \leqslant x, y \leqslant y, x \leqslant y, y : P \to x : P, y : \bot$, nor does the sequent match an initial one, therefore proof search stops there. The additional information that decorates the sequent tells more than unprovability, it explicitly provides a Kripke countermodel: it is easy to see that in the structure consisting of two reflexive nodes, x and y, with $x \leqslant y$, with the valuation $x \nVdash P$, and $y \Vdash P$, is such that $x \nVdash P \vee \neg P$ and therefore is a countermodel to $P \vee \neg P$. We shall see in the next section that it is not necessary to perform any such check: By Theorem 3.5 and its finitization in Section 3.1, a terminal node in a failed proof search gives a Kripke countermodel for the formula/sequent at the root of the proof search tree. The calculus is thus a countermodel-producing calculus.

3. Direct Kripke completeness for intermediate logics

The methodology of countermodel-producing calculi doesn't work just for intuitionistic and classical propositional logic, but for a wide variety of logics characterized by suitable frame conditions in their relational semantics. Since our focus is on intuitionistic logic but the method is easily extended in a broader context, we shall present the proof in detail for intermediate logics, which also indicates the way to generalizations or adaptations to other non-classical logics, following the approach of [Negri, 2009] for modal logics. For intermediate logics, we shall use the formalism of labelled sequent calculi presented in [Dyckhoff and Negri, 2012]. We just recall here that the method covers all intermediate logics the frame conditions of which are expressed by *geometric implications*, that is, sentences of the form

$$\forall \overline{z}(A \supset B)$$

where A and B are formulas that don't contain $\supset$, $\neg$, or $\forall$. Geometric implications have a useful normal form consisting of conjunctions of *geometric axioms*, which are formulas of the form

$$\forall \overline{z}(P_1 \& \dots \& P_m \supset \exists \overline{x}(M_1 \vee \dots \vee M_n))$$

where each M_j is a conjunction of atomic formulas, $Q_{j_1}, \dots, Q_{j_{k_j}}$ and $\overline{z}$ and $\overline{x}$ are sequences of bound variables. Assuming for simplicity that the sequence $\overline{x}$ of bound variables has length 1 and, without loss of generality, that no x_i is free in any P_j, geometric axioms are turned into rule schemes, called *geometric rule schemes*

$$\frac{\overline{Q}_1(y_1/x_1), \overline{P}, \Gamma \to \Delta \quad \dots \quad \overline{Q}_n(y_n/x_n), \overline{P}, \Gamma \to \Delta}{\overline{P}, \Gamma \to \Delta}\,GRS$$

Here $\overline{Q}_j$ and $\overline{P}$ indicate the multisets of atomic formulas $Q_{j_1}, \dots Q_{j_{k_j}}$ and $P_1, \dots, P_m$, respectively, and the *eigenvariables* $y_1, \dots, y_n$ of the premisses are not free in the conclusion.

Observe that in order to maintain admissibility of contraction in the extensions with geometric rules, the formulas $\overline{P}$ in the antecedent of the conclusion of the scheme have (as indicated) to be repeated in the antecedent of each of the premisses. In addition, whenever an instantiation of free parameters in atoms produces a duplication (two identical atoms) in the conclusion of a rule instance, say

$P_1, \ldots, P, P, \ldots, P_m, \Gamma \rightarrow \Delta$, there is a corresponding duplication in each premiss. The *closure condition* imposes the requirement that the rule with the duplication P, P contracted into a single P is added to the system of rules. For each axiom system, there is only a bounded number of possible cases of contracted rules to be added, very often none at all, so the condition is unproblematic.

For example, from the frame condition of *connectedness* $\forall xyz(xRy \,\&\, xRz \supset yRz \vee zRy)$ one obtains the rule scheme:

$$\frac{xRy, xRz, yRz, \Gamma \rightarrow \Delta \quad xRy, xRz, zRy, \Gamma \rightarrow \Delta}{xRy, xRz, \Gamma \rightarrow \Delta}$$

The closure condition imposes that the rule

$$\frac{xRy, yRy, \Gamma \rightarrow \Delta \quad xRy, yRy, \Gamma \rightarrow \Delta}{xRy, \Gamma \rightarrow \Delta}$$

or its equivalent

$$\frac{xRy, yRy, \Gamma \rightarrow \Delta}{xRy, \Gamma \rightarrow \Delta}$$

be part of the system, but this is just a special case of the rule of reflexivity, so no rule has to be added to **G3I**.

We remark that the closure condition is part of the uniform extension method that provides complete contraction-free calculi and it will be always assumed. We also notice that it *is not* a way to smuggle contractions on atomic formulas. In fact, the following result holds:

Proposition 3.1. *Let R be a frame rule, c(R) the contracted instance that arises from the closure condition. If c(R) is an instance of contraction, it is hp-admissible in the system extended with those rules arising from the closure condition that are not instances of contraction.*

Proof. Cf. Proposition 3 in [Hakli and Negri, 2011]. QED

We shall indicate with **G3I*** any extension of **G3I** by rules that follow the geometric rule scheme *GRS*. Examples of such extensions are given in the second (frame property) and third column (corresponding rule) of Table 5 in Section 4. More can be found in [Dyckhoff and Negri, 2012] where complete labelled sequent calculi for intermediate logics are obtained by adding to **G3I** the frame properties that correspond to the characterising axioms of each intermediate system.

Definition 3.2. Let K be a frame with an accessibility relation $\mathcal{R}$ that satisfies the geometric properties $*$. Let W be the set of variables (labels) used in derivations in **G3I***. An *interpretation* of the labels W in frame K is a function $[\![\cdot]\!] : W \rightarrow K$. A *valuation* of atomic formulas in frame K is a map $\mathcal{V} : AtFrm \rightarrow \mathcal{P}(K)$ that assigns to each atom P the set of nodes of K in which P holds; the standard notation for $k \in \mathcal{V}(P)$ is $k \Vdash P$.

Valuations for intuitionistic Kripke semantics are requested to satisfy the monotonicity property: $k\mathcal{R}k'$ and $k \Vdash P$ imply $k' \Vdash P$. They are extended to arbitrary formulas by the following inductive clauses:

$k \Vdash \bot$ for no k,
$k \Vdash A\&B$ if $k \Vdash A$ and $k \Vdash B$,
$k \Vdash A \vee B$ if $k \Vdash A$ or $k \Vdash B$,
$k \Vdash A \supset B$ if for all k', from $k\mathcal{R}k'$, from $k' \Vdash A$ follows $k' \Vdash B$.

Definition 3.3. A labelled formula $x : A$ (resp. a relational atom yRz) is *true* for an interpretation $[\![\cdot]\!]$ and a valuation $\mathcal{V}$ in a frame K iff $[\![x]\!] \Vdash A$ (resp. $[\![y]\!]\mathcal{R}[\![z]\!]$ in K) (where $\Vdash$ is based on $\mathcal{V}$). We also say that $[\![\cdot]\!]$ makes the formula (or atom) true for $\mathcal{V}$. A sequent $\Gamma \to \Delta$ is *true* for an interpretation $[\![\cdot]\!]$ and a valuation $\mathcal{V}$ in the frame $(K, \mathcal{R})$ if, whenever for all labelled formulas $x : A$ and relational atoms yRz in Γ it is the case that $[\![x]\!] \Vdash A$ and $[\![y]\!]\mathcal{R}[\![z]\!]$, then, for some $w : B$ in Δ, $[\![w]\!] \Vdash B$. A sequent is *valid* in a frame if it is true for every interpretation and every valuation in the frame.

Theorem 3.4. *If the sequent* $\Gamma \to \Delta$ *is derivable in* **G3I****, then it is valid in every frame with the properties* $*$.

Proof. Let $\mathcal{V}$ be a valuation in a frame K. We prove that each inference rule preserves truth for $\mathcal{V}$, from which the result follows by induction.

For the zero-premise inference rules, we consider the two cases. If $\Gamma \to \Delta$ is an initial sequent, then Γ contains some $x \leqslant y$, $x : P$ and Δ contains $y : P$, so the claim is obvious: whatever the values of $[\![x]\!]$ and $[\![y]\!]$, the monotonicity property ensures the truth of the sequent for $[\![\cdot]\!]$ and $\mathcal{V}$. Likewise, sequents with $x : \bot$ in the LHS are inevitably true for $\mathcal{V}$, since for no interpretation do we have $[\![x]\!] \Vdash \bot$.

If $\Gamma \to \Delta$ is a conclusion of a rule for & or $\vee$, let (for example) the rule be $L\&$ with the premiss $x : A, x : B, \Gamma' \to \Delta$. Assume that, for every interpretation $[\![\cdot]\!]$ and valuation $\mathcal{V}$, this premiss is true. Since $[\![x]\!] \Vdash A\&B$ is equivalent to $[\![x]\!] \Vdash A$ and $[\![x]\!] \Vdash B$, we obtain the truth of the conclusion. A similar way of reasoning covers the other rules for & and $\vee$.

If $\Gamma \to \Delta$ is a conclusion of $R\supset$, with the premiss $xRy, y : A, \Gamma' \to \Delta', y : B$ (with $\Delta \equiv x : A \supset B, \Delta'$ and y not in Γ, Δ), assume by the induction hypothesis that the premiss is true for $\mathcal{V}$. Let $[\![\cdot]\!]$ be any interpretation that makes all labelled formulas and relational atoms in the antecedent Γ of the conclusion true for $\mathcal{V}$. Without loss of generality we can suppose it makes none of those in Δ' true for $\mathcal{V}$, so we can concentrate on $x : A \supset B$. Let k be an arbitrary element of K such that $[\![x]\!]Rk$ holds in K and with $k \Vdash A$; we have to show that $k \Vdash B$ Let $[\![\cdot]\!]'$ be the interpretation identical to $[\![\cdot]\!]$ except possibly on y, where we set $[\![y]\!] \equiv k$. Clearly $[\![\cdot]\!]'$ makes true for $\mathcal{V}$ all formulas and relational atoms in $xRy, y : A, \Gamma$ (because $[\![x]\!]Rk$, $k \Vdash A$ and y is not in Γ). By our assumption (specialised to $[\![\cdot]\!]'$), $[\![\cdot]\!]'$ makes some formula in $\Delta', y : B$ true for $\mathcal{V}$. By our supposition, it cannot be in Δ'; so it must be $y : B$, as required.

If $\Gamma \to \Delta$ is the conclusion of $L \supset$, the argument is routine, not needing to exploit the freshness of any variable.

If the sequent is a conclusion of a rule without eigenvariables, the argument is also routine: we illustrate this with the rule *Trans*:

$$\frac{xRz, xRy, yRz, \Gamma \to \Delta}{xRy, yRz, \Gamma \to \Delta}$$

Let $[\![x]\!]\mathcal{R}[\![y]\!]$ and $[\![y]\!]\mathcal{R}[\![z]\!]$. Since $\mathcal{R}$ satisfies transitivity by assumption, we have $[\![x]\!]\mathcal{R}[\![z]\!]$, so truth of the premiss for $\mathcal{V}$ implies that of the conclusion.

If the sequent is a conclusion of a mathematical rule with eigenvariables, the argument is also routine: we illustrate this with the rule *Directedness*

$$\frac{yRw, zRw, xRy, xRz, \Gamma \to \Delta}{xRy, yRz, \Gamma \to \Delta}$$

in which w is an eigenvariable. Suppose that the premiss is true for $\mathcal{V}$. Suppose that $[\![\cdot]\!]$ is an interpretation making all formulas and atoms in the antecedent of the conclusion true for $\mathcal{V}$. Since (by hypothesis) the frame is directed, if $[\![x]\!]R[\![y]\!]$ and $[\![x]\!]R[\![z]\!]$, there exists d such that $[\![y]\!]Rd$ and $[\![z]\!]Rd$. Let $[\![\cdot]\!]'$ be the interpretation that coincides with $[\![\cdot]\!]$ at all labels except possibly on w, where we set $[\![w]\!]' \equiv d$ (a choice that works since w is an eigenvariable). It follows that $[\![\cdot]\!]'$ makes all formulae and atoms in the antecedent of the premiss true for $\mathcal{V}$, from which also the truth for $\mathcal{V}$ on one of the formulae in Δ. Since w is not in Δ (and so the distinction between the two interpretations is immaterial), that is just as required. QED

Next, for the completeness proof, we follow Takeuti [Takeuti, 1987], who adapted to Gentzen sequent calculus the method of Schütte [Schütte, 1956].

Theorem 3.5. *Let $\Gamma \to \Delta$ be a sequent in the language of* **G3I****. Then either the sequent is derivable in* **G3I*** *or it has a Kripke countermodel with the properties* $*$.

Proof. We define for an arbitrary sequent $\Gamma \to \Delta$ in the language of **G3I*** a reduction tree, by applying the rules of **G3I*** root first in all possible ways. If the construction terminates we obtain a proof, else the tree becomes infinite. By König's lemma an infinite finitely branching tree has an infinite branch that is used to define a countermodel to the endsequent.

1. Construction of the reduction tree: The reduction tree is defined inductively in stages as follows:

Stage 0 has $\Gamma \to \Delta$ at the root of the tree. Stage $n > 0$ has two cases:

Case I: If every topmost sequent is an initial sequent or a conclusion of $L\bot$ or of a zero-premiss mathematical rule, the construction of the tree ends.

Case II: Otherwise, we continue the construction of the tree by writing above those top-sequents that are not initial, nor conclusions of $L\bot$ or of a zero-premiss mathematical rule, other sequents that are obtained by applying root first the rules of **G3I*** whenever possible, in a given order.

There are $6 + r$ different stages, 6 for the rules of **G3I**, r for the frame rules. At stage $n = 6 + r + 1$ we repeat stage 1, at stage $n = 6 + r + 2$ we repeat stage 2, and so on for each n.

We start, for $n = 1$, with $L\&$: Consider all topmost sequents of the form

$$x_1 : B_1\&C_1, \ldots, x_m : B_m\&C_m, \Gamma' \to \Delta$$

Here $B_1\&C_1, \ldots, B_m\&C_m$ are all the formulas in Γ with a conjunction as the outermost logical connective. We write

$$x_1 : B_1, x_1 : C_1, \ldots, x_m : B_m, x_m : C_m, \Gamma' \to \Delta$$

on top of it. This step corresponds to applying root first m times rule $L\&$.

For $n = 2$, we consider all the sequents of the form

$$\Gamma \to x_1 : B_1\&C_1, \ldots, x_m : B_m\&C_m, \Delta'$$

Here $x_1 : B_1\&C_1, \ldots, x_m : B_m\&C_m$ are all the labelled formulas in the succedent with a conjunction as the outermost logical connective. We write on top of them the 2^m sequents

$$\Gamma \to x_1 : D_1, \ldots, x_m : D_m, \Delta'$$

Here D_i is either B_i or C_i and all possible choices are taken. This step is equivalent to applying $R\&$ root first successively with principal labelled formulas $x_1 : B_1\&C_1$, $\ldots, x_m : B_m\&C_m$.

For $n = 3$ and $n = 4$ we consider $L\vee$ and $R\vee$ and define the reductions symmetrically to the cases $n = 2$ and $n = 1$, respectively.

For $n = 5$, we consider each topmost sequent $\Gamma \to \Delta$ that has the labelled formulas $x_1 : B_1 \supset C_1, \ldots, x_m : B_m \supset C_m$ with implication as the outermost logical connective and relational atoms $x_1Ry_1, \ldots, x_mRy_m$ in the antecedent, $\Gamma' \equiv \Gamma - \{x_1Ry_1, \ldots, x_mRy_m\}$, and write on top of it the 2^m sequents

$$y_{i_1} : C_{i_1}, \ldots, y_{i_k} : C_{i_k}, x_1Ry_1, \ldots, x_mRy_m, \Gamma' \to y_{j_{k+1}} : B_{j_{k+1}}, \ldots, y_{j_m} : B_{j_m}, \Delta$$

Here $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\}$ and $j_{k+1}, \ldots, j_m \in \{1, \ldots, m\} - \{i_1, \ldots, i_k\}$. This step, perhaps less transparent because of the double indexing, corresponds to the root-first application of rule $L\supset$ with principal formulas $x_1 : B_1 \supset C_1, \ldots, x_m : B_m \supset C_m, x_1Ry_1, \ldots, x_mRy_m$. Observe that the principal formulas are retained in Γ'.

For $n = 6$, we consider all the labelled sequents that have implications in the succedent, say $x_1 : B_1 \supset C_1, \ldots, x_m : B_m \supset C_m$, and Δ' the other formulas. Let $y_1, \ldots, y_m$ be fresh variables, not yet used in the reduction tree, and write on top of each sequent the sequent

$$x_1Ry_1, \ldots, x_mRy_m, y_1 : B_1, \ldots, y_m : B_m, \Gamma \to y_1 : C_1, \ldots, y_m : C_m, \Delta'$$

So here we apply root first m times rule $R\supset$.

Finally, for $n = 6 + j$, we consider the generic case of a mathematical rule, that is, a rule for the relation R. Because of the subterm property (cf. the discussion on analyticity in section 8 of [Dyckhoff and Negri, 2012]), the mathematical rules need to be instantiated only with terms in the conclusion or with eigenvariables. Thus, if the system contains rule *Ref*, instances of that rule consist in adding to the antecedent all the relational atoms xRx for x in $\Gamma \to \Delta$. Observe that because of height-preserving substitution and height-preserving admissibility of contraction, once a rule with eigenvariables has been considered, it need not be instantiated again on the same principal formulas. If it is a rule such as *Trans*, consider all the sequents with a pair of atoms of the form xRy, yRz in the antecedent and write on top of them the sequents with the atoms xRz added.

For any n, for sequents that are neither initial, nor conclusions of $L\bot$, nor of zero-premiss mathematical rules, nor treatable by any one of the above reductions, we write the sequent itself above them. This repetition is made to treat uniformly the failure of proof search in the following two cases: the case in which the search goes on for ever because new rules always become applicable and the case in which a sequent is reached which is not a conclusion of any rule nor an initial sequent.

If the reduction tree is finite, all its leaves are initial or conclusions of $L\bot$, or of zero-premiss mathematical rules; observe also that all the reduction steps can be seen as simultaneous applications of rules. Once these superpositions are expanded into the system's rules, the tree, read from the leaves to the root, yields a derivation.

2. Construction of the countermodel: If the reduction tree is infinite, it has an infinite branch. Let $\Gamma_0 \to \Delta_0 \equiv \Gamma \to \Delta, \Gamma_1 \to \Delta_1 \dots, \Gamma_i \to \Delta_i, \dots$ be one such branch. Consider the sets of labelled formulas and relational atoms

$$\mathbf{\Gamma} \equiv \bigcup_{i>0} \Gamma_i \qquad \mathbf{\Delta} \equiv \bigcup_{i>0} \Delta_i$$

We define a Kripke model that forces all the formulas in $\mathbf{\Gamma}$ and no formula in $\mathbf{\Delta}$ and is therefore a countermodel to the sequent $\Gamma \to \Delta$.

Consider the frame K the nodes of which are all the labels that appear in the relational atoms in $\mathbf{\Gamma}$, with their mutual relationships expressed by the xRy's in $\mathbf{\Gamma}$. Clearly, the construction of the reduction tree imposes the frame properties of the countermodel, for instance, in the system **G3I**, the constructed frame is reflexive and transitive. The model is defined as follows: For all atomic formulas P such that $x : P$ in $\mathbf{\Gamma}$, or $y \leqslant x$, $y : P$ are in Γ, we stipulate that $x \Vdash P$.

We show now inductively on the weight of formulas that A is forced in the model at node x if $x : A$ is in $\mathbf{\Gamma}$ and A is not forced at node x if $x : A$ is in $\mathbf{\Delta}$. Therefore we have a countermodel to the endsequent $\Gamma \to \Delta$.

If A is $\bot$, it cannot be in $\mathbf{\Gamma}$ because no sequent in the branch contains $x : \bot$ in the antecedent, so it is not forced at any node of the model.

If A is an atomic formula in Γ, the claim holds by the definition of the model. If $y : Q$ is in $\mathbf{\Delta}$, since the sequent is neither initial nor derivable, neither $x \leqslant y$ and $x : Q$ for some x nor $y : Q$ can be in Γ, so $y \nVdash Q$.

If $x : A \equiv x : B\&C$ is in $\mathbf{\Gamma}$, there exists i such that $x : A$ appears first in Γ_i, and therefore, for some $l \geqslant 0$, $x : B$ and $x : C$ are in Γ_{i+l}. By the induction hypothesis, $x \Vdash B$ and $x \Vdash C$, and therefore $x \Vdash B\&C$.

If $x : A \equiv x : B\&C$ is in $\mathbf{\Delta}$, consider the step i in which the reduction for A applies. This gives a branching, and one of the two branches belongs to the branch under consideration, so either $x : B$ or $x : C$ is in $\mathbf{\Delta}$, and therefore by the inductive hypothesis, $x \nVdash B$ or $x \nVdash C$, and therefore $x \nVdash B\&C$.

If $x : A \equiv x : B \vee C$ is in $\mathbf{\Gamma}$, we reason similarly to the case of $x : A \equiv x : B\&C$ in $\mathbf{\Delta}$.

If $x : A \equiv x : B \vee C$ is in $\mathbf{\Delta}$, we argue as with $x : A \equiv x : B\&C$ in $\mathbf{\Gamma}$.

If $x : A \equiv x : B \supset C$ is in $\mathbf{\Gamma}$, we consider all the relational atoms xRy that occur in $\mathbf{\Gamma}$. If there is no such atom, then the condition that for all y accessible from x in the frame, $y \Vdash B$ implies $y \Vdash C$ is vacuously satisfied, and therefore $x \Vdash B \supset C$ in the model. Else, for any occurrence of xRy in $\mathbf{\Gamma}$ we find, by the construction of the reduction tree, either an occurrence of $y : C$ in $\mathbf{\Gamma}$ or of $y : B$ in $\mathbf{\Delta}$. So, by the inductive hypothesis, either $y \Vdash C$ or $y \nVdash B$, and therefore $x \Vdash B \supset C$ in the model.

If $x : A \equiv x : B \supset C$ is in $\mathbf{\Delta}$, consider the step at which the reduction for $x : A$ applies. We then find $y : B$ in $\mathbf{\Gamma}$ and $y : C$ in $\mathbf{\Delta}$ for some y with xRy in $\mathbf{\Gamma}$. By the induction hypothesis, $y \Vdash B$ and $y \nVdash C$, and therefore $x \nVdash A$. QED

Completeness is then obtained as an immediate corollary to the above theorem:

Corollary 3.6. *If a sequent* $\Gamma \to \Delta$ *is valid in every intuitionistic Kripke model with the frame properties* $*$, *then it is derivable in the system* **G3I***.

Observe that the exhaustive proof search described in the proof of Theorem 5.7 is not a decision method nor an effective way of finding countermodels when proof search fails, as it may produce infinite branches and therefore infinite countermodels. As an example, consider the following branch in the search for a proof of the law of double negation (to save space we have omitted derivable premisses of $L\supset$ and formulas of the form $x :\perp$ from the succedents):

$$
\begin{array}{cl}
\vdots & \\
y \leqslant y, x \leqslant y, y \leqslant w, y \leqslant z, z \leqslant w, y : \neg\neg A, z : A, w : A \to y : A, w : \neg A & \\
\hline
y \leqslant y, x \leqslant y, y \leqslant w, y \leqslant z, z \leqslant w, y : \neg\neg A, z : A, w : A \to y : A & L\supset \\
\hline
y \leqslant y, x \leqslant y, y \leqslant z, z \leqslant w, y : \neg\neg A, z : A, w : A \to y : A & \mathit{Trans} \\
\hline
y \leqslant y, x \leqslant y, y \leqslant z, y : \neg\neg A, z : A \to y : A, z : \neg A & R\supset \\
\hline
y \leqslant y, x \leqslant y, y \leqslant z, y : \neg\neg A, z : A \to y : A & L\supset \\
\hline
y \leqslant y, x \leqslant y, y : \neg\neg A \to y : A, y : \neg A & R\supset \\
\hline
y \leqslant y, x \leqslant y, y : \neg\neg A \to y : A & L\supset \\
\hline
x \leqslant y, y : \neg\neg A \to y : A & \mathit{Ref} \\
\hline
\to x : \neg\neg A \supset A & R\supset
\end{array}
$$

Clearly the search goes on forever. To see that it cannot produce a derivation we can either apply a minimality argument or exhibit a finite countermodel which is a suitable truncation of the infinite countermodel provided by the completeness proof.

Minimality argument: If the sequent were derivable, suppose that the topmost sequent in the attempted proof search has a derivation of height n. By the hp-substitution $[z/w]$ we obtain a derivation of the same height of the sequent

$$y \leqslant y, x \leqslant y, y \leqslant z, y \leqslant z, z \leqslant z, y : \neg\neg A, z : A, z : A \to y : A, z : \neg A$$

and thus, by hp-contraction, of

$$y \leqslant y, x \leqslant y, y \leqslant z, z \leqslant z, y : \neg\neg A, z : A \to y : A, z : \neg A.$$

A step of Ref gives a derivation of height $n + 1$ of

$$y \leqslant y, x \leqslant y, y \leqslant z, y : \neg\neg A, z : A \to y : A, z : \neg A.$$

But this sequent had derivation height $n + 3$ so the derivation has been shortened by two steps. It is therefore useless to proceed with steps that lead to duplications of formulas modulo a fresh labelling.

3.1. Finite countermodel construction

We consider the procedure of proof-search construction described in the completeness proof and apply root-first the rules until a suitable *saturation* condition is met. The saturation condition is applied to branches rather than sequents[2]. Intuitively, a branch is saturated when its leaf is not an an initial sequent nor a conclusion of $L\bot$ and when it is closed under all the rules of the calculus with the exception of applications of $R\supset$ that would produce loops modulo a new labelling; to generate the finite countermodel we define a partial order obtained by taking the reflexive and transitive closure of the original partial order together with a relation that witnesses such loops. We make this intuition formal by the definitions below, where we indicate with $\downarrow\Gamma$ ($\downarrow\Delta$) the union of the antecedents (succedents) in the branch from the end-sequent up to $\Gamma \to \Delta$.

For a sequent $\Gamma \to \Delta$ in a proof search tree and a label x, we indicate with $\mathcal{F}_{\Gamma\to\Delta}(x)$ the ordered pair of sets $(\mathcal{F}^1_{\Gamma\to\Delta}(x), \mathcal{F}^2_{\Gamma\to\Delta}(x))$ where

$$\begin{aligned}\mathcal{F}^1_{\Gamma\to\Delta}(x) &\equiv \{A \mid x : A \in\downarrow\Gamma\} \cup \{P \mid y : P, y \leqslant x \in \Gamma\}\\ &\quad\cup\{A \supset B \mid y : A \supset B, y \leqslant x \in \Gamma\}\\ \mathcal{F}^2_{\Gamma\to\Delta}(x) &\equiv \{A \mid x : A \in\downarrow\Delta\}\end{aligned}$$

We then pose $x \preccurlyeq_{\Gamma\to\Delta} y$ iff $\mathcal{F}_{\Gamma\to\Delta}(x) \subseteq \mathcal{F}_{\Gamma\to\Delta}(y)$, i.e. $\mathcal{F}^i_{\Gamma\to\Delta}(x) \subseteq \mathcal{F}^i_{\Gamma\to\Delta}(y)$ for $i = 1, 2$. Subscripts will be omitted when clear from the context.

Definition 3.7. We say that a branch in a proof search up to a sequent $\Gamma \to \Delta$ is *saturated* if the following conditions hold:

1. If x is a label in Γ, Δ, then $x \leqslant x$ is in Γ.
2. If $x \leqslant y$ and $y \leqslant z$ are in Γ, then $x \leqslant z$ is in Γ.
3. If $x : P$ is in Γ, there is no y such that $x \leqslant y$ is in Γ and $y : P$ is in Δ.
4. There is no x such that $x :\bot$ is in Γ.
5. If $x : A\&B$ is in $\downarrow\Gamma$, then $x : A$ and $x : B$ are in $\downarrow\Gamma$.
6. If $x : A\&B$ is in $\downarrow\Delta$, then either $x : A$ or $x : B$ is in $\downarrow\Delta$.
7. If $x : A \vee B$ is in $\downarrow\Gamma$, then either $x : A$ or $x : B$ is in $\downarrow\Gamma$.
8. If $x : A \vee B$ is in $\downarrow\Delta$, then $x : A$ and $x : B$ are in $\downarrow\Gamma$.
9. If $x : A \supset B$ and $x \leqslant y$ are in Γ, then either $y : A$ is in $\downarrow\Delta$ or $y : B$ is in $\downarrow\Gamma$.
10. If $x : A \supset B$ is in $\downarrow\Delta$, then either
    (i) for some y there is $x \leqslant y$ in Γ, $y : A$ is in $\downarrow\Gamma$, and $y : B$ is in $\downarrow\Delta$
    or
    (ii) there is y such that $y \neq x$, $y \leqslant x$ is in Γ and $x \preccurlyeq_{\Gamma\to\Delta} y$.

To see this definition at work, observe that in the example above the saturation condition blocks the search exactly at the point reached by the indicated proof search because of clause $10(ii)$ applied to the formula $w : \neg A$. If we indicate with $\mathcal{S}$ the

[2] We observe that the definition of saturation can be referred to sequents rather than branches if rules are written in a cumulative style, i.e. by always copying the principal formulas in the premisses (a choice that was pursued in [Garg, Genovese and Negri, 2012]), otherwise the definition refers to a branch of sequents up to the given one. This is the reason why we have used the downward closure of Γ and Δ. The choice is irrelevant for the completeness proof and for the definition of the countermodel but permits to keep here the more economic version of the calculus.

top-sequent in the above proof search, we have $\mathcal{F}_{\mathcal{S}}(w) = (\{A, \neg\neg A\}, \{\neg A\}) = \mathcal{F}_{\mathcal{S}}(z)$.

To define the countermodel $\mathcal{M} \equiv (K, \leq, \Vdash)$ we proceed as in the completeness proof but instead of building it on an infinite branch we build it starting from a saturated branch and take the two finite sets $\downarrow\Gamma$, $\downarrow\Delta$ in place of the (potentially) infinite sets $\mathbf{\Gamma}$, $\mathbf{\Delta}$:

1. The set K consists of all the labels in Γ;
2. The preorder $\leq$ is the transitive and reflexive closure of the union of the relations on Γ and the additional relations $x \preccurlyeq y$;
3. The forcing relation is defined on atomic formulas by $x \Vdash P$ if there are $y \leqslant x$ and $y : P$ in Γ and extended to arbitrary formulas following the usual clauses of Kripke semantics for intuitionistic logic.

The following lemma shows that $\mathcal{M}$ is an intuitionistic Kripke model:

Lemma 3.8. *If $x \Vdash P$ and $x \leq y$ then $y \Vdash P$.*

Proof. We show by induction on the length n of a chain from x to y in terms of the relations $\leqslant$ and $\preccurlyeq$ that P is in $\mathcal{F}^1(y)$ and therefore $y \Vdash P$. For $n = 0$ it is clear. Suppose by inductive hypothesis that the claim is true for $n = i$ and let x_i be the element of the chain reached by i steps; thus P is in $\mathcal{F}^1(x_i)$, i.e., for some z $z \leqslant x_i$ and $z : P$ are in Γ. If $x_i \leqslant x_{i+1}$ then by clause 2. of the definition of saturated branch, also $z \leqslant x_{i+1}$ is in Γ and thus P is in $\mathcal{F}^1(x_{i+1})$. If instead $x_i \preccurlyeq x_{i+1}$, the conclusion follows from $\mathcal{F}^1(x_i) \subseteq \mathcal{F}^1(x_{i+1})$. QED

Next we prove that $\mathcal{M}$ is a countermodel to the saturated branch ending with the sequent $\Gamma \to \Delta$. We need to prove that for all $x : A$ in Γ, $x \Vdash A$ and for all $x : A$ in Δ, $x \nVdash A$. This immediately follows from the following proposition:

Proposition 3.9. *The following hold for $\mathcal{M}$:*

1. *If A is in $\mathcal{F}^1(x)$, then $x \Vdash A$.*
2. *If A is in $\mathcal{F}^2(x)$, then $x \nVdash A$.*

Proof. The two claims are proved simultaneously by induction on A and on the tree order of $\leqslant$ in Γ.

If A is an atomic formula P, 1. holds by definition of $\Vdash$ and 2. by the third saturation clause. If A is a conjunction or a disjunction, the claim holds by the corresponding saturation clauses and inductive hypothesis on smaller formulas. If $A \equiv B \supset C$ is in $\mathcal{F}^1(x)$, we need to prove that for all y such that $x \leq y$ in the model, we have that $y \Vdash B$ implies $y \Vdash B$. We have $x \equiv x_0 R \ldots R x_n \equiv y$ where R is either $\leqslant$ or $\preccurlyeq$. We proceed by induction on n. If n is 0, either $x : B \supset C$ is in $\downarrow\Gamma$ (hence in Γ) or for some z we have $z \leqslant x$ and $z : B \supset C$ in Γ. In both cases (for the former we add the saturation clause for reflexivity) we obtain the conclusion using the saturation clause for implication in the antecedent. For the inductive step we proceed in a similar way, using in addition the definition of $\preccurlyeq$. If $A \equiv B \supset C$ is in $\mathcal{F}^2(x)$, then either (i) there are $x \leqslant y$ and $y : B$ are in $\downarrow\Gamma$ and $y : C$ in $\downarrow\Delta$ or (ii) there is y distinct from x such that $y \leqslant x$ is in Γ and $x \preccurlyeq y$. In the first case we have by definition of $\mathcal{F}$, B in $\mathcal{F}^1(y)$, C in $\mathcal{F}^2(y)$ and therefore by induction on smaller formulas $y \Vdash B$ and $y \nVdash C$, which give $x \nVdash B \supset C$. In the second case, by the inclusion $\mathcal{F}^2(x) \subseteq \mathcal{F}^2(y)$, we have $B \supset C$ in $\mathcal{F}^2(y)$, and by inductive hypothesis (y is a smaller label on the tree ordering) we have $y \nVdash B \supset C$, and therefore $x \nVdash B \supset C$. QED

To prove *termination* of the decision procedure, observe that by the subformula property the number of distinct formulas in sequents in an attempted proof is bounded. Since duplications of the same labelled formulas are not possible by hp-admissibility of contraction, it is enough to show that the number of distinct labels that gets generated along the proof search/saturation procedure is finite.

Let F be the set of (unlabelled) subformulas of the end-sequent and consider a chain of labels $x_0 \leqslant x_1 \leqslant x_2 \ldots$ generated by the saturation procedure. Consider, for an arbitrary new label x_i, the values of the sets $\mathcal{F}(x_j)$ for $j < i$ computed at the step in which the label x_i has been introduced. Clearly, $\mathcal{F}(x_i) \nsubseteq \mathcal{F}(x_j)$ (else, by the saturation condition 10., x_i would not have been introduced), so each new label corresponds to a new subset of $F \times F$. Since the number of these subsets is finite, also the length of each chain of labels must be finite.

4. Intermediate logics and their modal companions

As we detailed out in Theorem 3.5, complete, countermodel-producing sequent calculi are obtained for all intermediate logics characterized by frame conditions that obey the scheme of geometric axioms. The procedure is parallel to the extension of the modal systems **S4** by the same conditions. The starting point (cf. [Negri, 2005]) is the labelled sequent calculus for *basic modal logic*:

Initial sequents: $x : P, \Gamma \to \Delta, x : P$

Propositional rules:

$$\frac{x : A, x : B, \Gamma \to \Delta}{x : A\&B, \Gamma \to \Delta}\ L\& \qquad \frac{\Gamma \to \Delta, x : A \quad \Gamma \to \Delta, x : B}{\Gamma \to \Delta, x : A\&B}\ R\&$$

$$\frac{x : A, \Gamma \to \Delta \quad x : B, \Gamma \to \Delta}{x : A \vee B, \Gamma \to \Delta}\ L\vee \qquad \frac{\Gamma \to \Delta, x : A, x : B}{\Gamma \to \Delta, x : A \vee B}\ R\vee$$

$$\frac{\Gamma \to \Delta, x : A \quad x : B, \Gamma \to \Delta}{x : A \supset B, \Gamma \to \Delta}\ L\supset \qquad \frac{x : A, \Gamma \to \Delta, x : B}{\Gamma \to \Delta, x : A \supset B}\ R\supset$$

$$\frac{}{x : \bot, \Gamma \to \Delta}\ L\bot$$

Modal rules:

$$\frac{y : A, x : \Box A, xRy, \Gamma \to \Delta}{x : \Box A, xRy, \Gamma \to \Delta}\ L\Box \qquad \frac{xRy, \Gamma \to \Delta, y : A}{\Gamma \to \Delta, x : \Box A}\ R\Box$$

Table 4. The system **G3K**

Modular extensions of the systems are obtained as indicated in the table below: first the frame properties that correspond to the modal axioms are considered, and then the sequent rules that correspond to the frame properties.

	Axiom	Frame property	Rule
T	$\Box A \supset A$	$\forall x\, xRx$ reflexivity	$\dfrac{xRx, \Gamma \to \Delta}{\Gamma \to \Delta}$
4	$\Box A \supset \Box\Box A$	$\forall xyz(xRy \,\&\, yRz \supset xRz)$ transitivity	$\dfrac{xRz, \Gamma \to \Delta}{xRy, yRz, \Gamma \to \Delta}$
E	$\Diamond A \supset \Box\Diamond A$	$\forall xyz(xRy \,\&\, xRz \supset yRz)$ euclideanness	$\dfrac{yRz, \Gamma \to \Delta}{xRy, xRz, \Gamma \to \Delta}$
B	$A \supset \Box\Diamond A$	$\forall xy(xRy \supset yRx)$ symmetry	$\dfrac{yRx, \Gamma \to \Delta}{xRy, \Gamma \to \Delta}$
3	$\Box(\Box A \supset B) \vee$ $\Box(\Box B \supset A)$	$\forall xyz(xRy \,\&\, xRz \supset yRz \vee zRy)$ connectedness	$\dfrac{yRz, \Gamma \to \Delta \quad zRy, \Gamma \to \Delta}{xRy, xRz, \Gamma \to \Delta}$
D	$\Box A \supset \Diamond A$	$\forall x \exists y\, xRy$ seriality	$\dfrac{xRy, \Gamma \to \Delta}{\Gamma \to \Delta}\, y$
2	$\Diamond\Box A \supset \Box\Diamond A$	$\forall xyz(xRy \,\&\, xRz \supset \exists w(yRw \,\&\, zRw))$ directedness	$\dfrac{yRw, \Gamma \to \Delta \quad zRw, \Gamma \to \Delta}{xRy, xRz, \Gamma \to \Delta}\, w$

Table 5. Correspondence between modal axioms, frame properties, and sequent rules

Observe that in the rules for seriality and directedness, the variable indicated is fresh and that, to save space, the principal atoms in the conclusion are not repeated in the premisses.

A contraction- and cut-free sequent calculus for **S4** is obtained by adding the rules that correspond to reflexivity and transitivity of the accessibility relation of **G3K**. Extensions of **S4** are obtained by adding any combination of rules for frame properties. Completeness for these systems can be established either by the usual means of structural proof theory and, ultimately, by equivalence with an axiomatic system, as in [Negri, 2005], or through a proof of completeness with respect to the standard relational semantics in the style of the one detailed above for extensions of intuitionistic logic, as in [Negri, 2009].

The famous modal translation * of intuitionistic logic was defined in a note by Gödel [Gödel, 1933] together with a syntactic proof that the translation is sound, namely that $\vdash_{Int} A$ implies $\vdash_{S4} A^*$. On the other hand, the proof of faithfulness, that is, the reverse implication, was shown by [McKinsey and Tarski, 1948]. The proof was indirect because of the detour through completeness of intuitionistic logic with respect to Heyting algebras (then called Brouwerian algebras) and of **S4** with respect to topological Boolean algebras (called closure algebras) and a Stone-type representation of Heyting algebras as the opens of topological Boolean algebras. The proof was also non-constructive because of the use of Stone representation of distributive lattices, in particular Zorn's lemma. Similar methods were later employed in [Dummett and Lemmon, 1959] to extend the faithfulness result to intermediate logics **Int**$+A$ and their modal companions **S4**$+A^*$, i.e. to prove that **Int** $+ Ax \vdash A$ if and only if **S4** $+ Ax^* \vdash A^*$.

Among the several slightly different variants of the Gödel translation that have been proposed, we use the following (see [Troelstra and Schwichtenberg, 2000]):

$$
\begin{array}{rcl}
P^{\Box} & := & \Box P \\
\bot^{\Box} & := & \bot \\
(A \supset B)^{\Box} & := & \Box(A^{\Box} \supset B^{\Box}) \\
(A\&B)^{\Box} & := & A^{\Box}\&B^{\Box} \\
(A \vee B)^{\Box} & := & A^{\Box} \vee B^{\Box}
\end{array}
$$

The same frame conditions are imposed as extensions of the sequent calculus **G3I** or of **G3K**. In particular, because of the uniformity of generation of these calculi, the proof of faithfulness of the modal translation between an intermediate logic ant its modal companion is achieved in a modular, simple, and completely syntactic way (induction on the height of derivations): in [Dyckhoff and Negri, 2012] it is shown that given an extension **G3I*** of **G3I** with rules for $\leqslant$ and given the corresponding extension **G3S4*** of **G3S4**, we have

$$\textbf{G3I*} \vdash \Gamma \to \Delta \text{ if and only if } \textbf{G3S4*} \vdash \Gamma^{\Box} \to \Delta^{\Box}$$

We observe that the translation of **Int** to **S4** is of no help for obtaining a terminating proof search procedure because a non-termination similar to the one seen for **G3I** occurs in **G3S4**. For example, we have the following infinite proof search (only the relevant part is considered as derivable premisses are omitted as well as formulas that become inactive)

$$
\begin{array}{cl}
\vdots & \\
xRz, x : \Box(\Box A \supset A), z : \Box A \supset A \to z : \Box A & \\
\hline
xRz, x : \Box(\Box A \supset A), z : \Box A \supset A \to z : A & {\scriptstyle L\supset} \\
\hline
xRz, x : \Box(\Box A \supset A), y : \Box A \supset A \to z : A & {\scriptstyle L\Box} \\
\hline
xRy, yRz, x : \Box(\Box A \supset A), y : \Box A \supset A \to z : A & {\scriptstyle Trans} \\
\hline
xRy, x_0 : \Box(\Box A \supset A), y : \Box A \supset A \to y : \Box A & {\scriptstyle R\Box} \\
\hline
xRy, x : \Box(\Box A \supset A), y : \Box A \supset A \to y : A & {\scriptstyle L\supset} \\
\hline
xRy, x : \Box(\Box A \supset A), x : \Box A \supset A \to y : A & {\scriptstyle L\Box} \\
\hline
x : \Box(\Box A \supset A), x : \Box A \supset A \to x : \Box A & {\scriptstyle R\Box} \\
\hline
xRx, x : \Box(\Box A \supset A), x : \Box A \supset A \to x : A & {\scriptstyle L\supset} \\
\hline
xRx, x : \Box(\Box A \supset A) \to x : A & {\scriptstyle L\Box}
\end{array}
$$

The propositional base of **S4** is classical, in particular, there is no "hidden contraction" in the left implication rule in the form of repetition of the principal formula in the left premiss of $L\supset$ in **G3I**. The difficulty is, however, moved elsewhere, because loops created by those repetitions are replaced by similar loops created by the interplay of transitivity of the accessibility relation with the repetition of the principal formula in $L\Box$. The difficulty is intrinsic to the system, and cannot be avoided through a streamlining of the rules while maintaining completeness of the calculus: It was proved in [Minari, 2013] that, whereas the repetition in rule $L\Box$ can be avoided in **G3K** and in its extension with seriality, it cannot be dispensed with in **G3S4**.

The failure of the modal translation to force intuitionistic logic into a terminating modal system takes us to the next step of internalization in labelled systems, that of the condition of *finiteness* that characterizes provability logics.

5. Provability logics

The provability logic of Gödel-Löb is characterized by irreflexive, transitive, and Noetherian frames (equivalently, transitive and such that every chain is finite). Finiteness is not first-order expressible and thus the method of conversion of frame properties into rules outlined above is not directly applicable. As shown in [Negri, 2005], the finiteness property can be internalized in the syntax of a labelled sequent calculus through a modification of the usual forcing relation for modal formulas, and a consequent modification of the rules for $\Box$. In particular, the method generates a *harmonious* pair of rules that allows a simple syntactic proof of cut elimination.

We recall briefly the procedure of determination of the harmonious pair of rules. In order to make it more transparent, we use as an intermediate step rules of natural deduction. First, the following characterization of forcing is used:

Lemma 5.1. *In irreflexive, transitive, and Noetherian Kripke frames* $x \Vdash \Box A$ *if and only if for all* y*,* xRy *and* $y \Vdash \Box A$ *implies* $y \Vdash A$*.*

The sufficient condition gives the introduction rule for $\Box$, where y is a fresh variable:

$$\dfrac{\begin{matrix}[xRy, y : \Box A], \Gamma \\ \vdots \\ y : A\end{matrix}}{x : \Box A}\ \Box I\text{-}L$$

The rule is then generalized to one with arbitrary multisets as consequences, in a sequent calculus format:

$$\dfrac{xRy, y : \Box A, \Gamma \to \Delta, y : A}{\Gamma \to \Delta, x : \Box A}\ R\Box\text{-}L$$

From the introduction rule and the *inversion principle* (cf. [Negri and von Plato, 2001]) we find the elimination rule:

$$\dfrac{x : \Box A \quad \begin{matrix}\vdots \\ xRy\end{matrix} \quad \begin{matrix}\vdots \\ y : \Box A\end{matrix} \quad \begin{matrix}[y : A] \\ \vdots \\ C\end{matrix}}{C}\ \Box E\text{-}L$$

Again, this is transformed into a sequent calculus rule

$$\dfrac{x : \Box A, xRy, \Gamma \to \Delta, y : \Box A \quad y : A, x : \Box A, xRy, \Gamma \to \Delta}{x : \Box A, xRy, \Gamma \to \Delta}\ L\Box\text{-}L$$

Observe that not only the right rule departs from the usual $R\Box$ rule of the basic system **G3K**, but also the left rule does. The left rule is obtained through the general principle of *harmony* in the design of logical systems that originated in Gentzen's work and the importance of which was stressed in the literature on proof-theoretic semantics to these days (cf. [Schroeder-Heister, 2012]).

With these two rules, however, the sequent $x : \Box A \to x : \Box A$ is not derivable, so, to obtain a complete calculus, sequents of the form $x : \Box A, \Gamma \to \Delta, x : \Box A$ have to be added to the stock of initial sequents. The reduction to atomic initial sequents, hence hp-admissibility of all the rules, therefore does not hold. With these provisos to obtain a complete calculus, the proof of cut-elimination for the system obtained follows the usual pattern of cut-admissibility proofs for G3-style sequent calculi, save for the addition of a third inductive parameter, the *range* of the cut formula, a measure that indicates the position of the label of the cut formula in the relational structure associated to the derivation (cf. [Negri, 2005] for details).

An alternative possibility for presenting GL as a labelled sequent calculus is to use the standard $L\Box$ rule in place of $L\Box$-L, that corresponds to the following elimination rule

$$\dfrac{x : \Box A \qquad \begin{array}{c}\vdots\\ xRy\end{array} \qquad \begin{array}{c}[y : A]\\ \vdots\\ C\end{array}}{C}\;\Box E$$

Then sequents of the form $\Box A, \Gamma \to \Delta, \Box A$ are derivable, with the possibility to restrict initial sequents to atomic form and hp-invertibility of all rules as a consequence. The syntactic proof of cut elimination, however, no longer obtains. The reason for this is best seen using the natural deduction formulation of the rules, keeping in mind that elimination of principal cuts corresponds to detour conversions in natural deduction.

If the major premiss of $\Box E$ is derived by an introduction we have the non-eliminable detour

$$\dfrac{\dfrac{\begin{array}{c}[xRz, z : \Box A]\\ \vdots\\ z : A\end{array}}{x : \Box A}\,\Box I\text{-}L \qquad \begin{array}{c}\vdots\\ xRy\end{array} \qquad \begin{array}{c}[y : A]\\ \vdots\\ C\end{array}}{C}\;\Box E$$

With the harmonious pair of rules we have the following detour conversion, where the substitution $[y/z]$ has been used:

$$\dfrac{\dfrac{\begin{array}{c}[xRz, z : \Box A]\\ \vdots\\ z : A\end{array}}{x : \Box A}\,\Box I\text{-}L \qquad \begin{array}{c}\vdots\\ xRy\end{array} \qquad \begin{array}{c}\vdots\\ y : \Box A\end{array} \qquad \begin{array}{c}[y : A]\\ \vdots\\ C\end{array}}{C}\;\Box E\text{-}L$$

$\rightsquigarrow$

$$\begin{array}{c} \vdots \qquad\quad \vdots \\ xRy \quad , y : \Box A \\ \vdots \\ y : A \\ \vdots \\ C \end{array}$$

Clearly the conversion is not possible if the elimination rule is $\Box E$ because it would leave an extra non-discharged formula.

In the sequel we show how the syntactic proof of cut elimination can be replaced by a direct completeness proof which at the same time provides a decision procedure.

We consider the system **G3KGL** introduced in [Negri, 2005] obtained from **G3K** by replacing rule $R\Box$ with $R\Box$-L and by adding the rules of transitivity *Trans* and irreflexivity *Irref* ($x \leqslant x, \Gamma \rightarrow \Delta$), or, in other words, the system obtained from **G3GL** (*ibid.*) by removing initial sequents of the form $x : \Box A, \Gamma \rightarrow \Delta, x : \Box A$ and replacing rule $L\Box$-L with the standard rule $L\Box$.

All the structural properties that have been established for **G3GL** hold for **G3KGL**, but hp-admissibility of contraction holds with no limitations. However, because of the lack of harmony between the left and the right rules for $\Box$, admissibility of cut could not be established directly for **G3KGL**, but only indirectly through an equivalence with **G3GL**. Except for admissibility of cut, we shall not give the details of the proofs, because they are routine modifications of the proofs given in [Negri, 2005]. The following lemma will be useful:

Lemma 5.2. *Let xR^*y denote a sequence of accessibility relations xRy_1, ..., $y_{n-1}Ry$. Then sequents of the form $xR^*y, x : \Box A, \Gamma \rightarrow \Delta, y : \Box A$ are derivable in* **G3KGL**.

Proof. Root-first, by successive applications of $R\Box$-L, steps of transitivity, and $L\Box$. QED

Next, instead of proving admissibility of cut syntactically, we proceed by showing that the calculus is sound and complete; we shall prove that derivable sequents are valid in irreflexive and transitive Noetherian frames and that for any sequent in the language of **GL**, either a proof in the calculus or a countermodel given by an irreflexive and transitive Noetherian frame can be found. The size of the endsequent gives a bound to the size of the countermodel and to the height of the search tree and therefore a decision procedure through terminating proof search is obtained. We start with the definitions of interpretation in a frame and of validity adapted to the case of **G3KGL**:

Definition 5.3. Let K be a frame with an irreflexive, transitive, and Noetherian accessibility relation $\mathcal{R}$. Let W be the set of variables (labels) used in derivations in **G3KGL**. An *interpretation* of the labels W in frame K is a function $[\![\cdot]\!] : W \rightarrow K$. A *valuation* of atomic formulas in frame K is a map $\mathcal{V} : AtFrm \rightarrow \mathcal{P}(K)$ that assigns to each atom P the set of nodes of K in which P holds; the standard notation for $k \in \mathcal{V}(P)$ is $k \Vdash P$.

Valuations are extended to arbitrary formulas as in Section 3, except for the following clauses:

$k \Vdash A \supset B$ if from $k \Vdash A$ follows $k \Vdash B$,
$k \Vdash \Box A$ if for all k', from $k\mathcal{R}k'$ follows $k' \Vdash A$.

We then have, with validity in a frame as in Definition 2.4:

Theorem 5.4. *If the sequent* $\Gamma \to \Delta$ *is derivable in* **G3KGL***, then it is valid in every irreflexive, transitive and Noetherian frame.*

Proof. By induction on the derivation of $\Gamma \to \Delta$ in **G3KGL**. All the cases are similar to those for extensions of **G3K** considered in [Negri and von Plato, 2011] except for the rule specific to **G3KGL**, namely the right rule for $\Box$.

If $\Gamma \to \Delta$ is a conclusion of $R\Box$-L, with premiss $xRy, y : \Box A, \Gamma' \to \Delta', y : A$, assume by the induction hypothesis that the premiss is true. We claim that the conclusion $\Gamma' \to \Delta', x : \Box A$ is true. Let $[\![\cdot]\!]$ be an arbitrary interpretation in an irrreflexive, transitive, and Noetheriam frame that makes true all the formulas in Γ'. We claim that one of the formulas in Δ' or $x : \Box A$ is true under this interpretation. We reason by contradiction. If none is true, there exists an interpretation k of x and an element k_1 in K such that $k\mathcal{R}k_1$ holds and $k_1 \nVdash A$; let $[\![\cdot]\!]'$ be the interpretation identical to $[\![\cdot]\!]$ except possibly on y, where we set $[\![y]\!]' \equiv k_1$. Since by assumption the premiss is trues and $k_1 \nVdash A$, we must have that $k_1 \nVdash \Box A$. By proceeding as in the proof of Lemma 5.1 of [Negri, 2005], a chain that never becomes stationary is built, in contradiction with the assumption that K is Noetherian. QED

Theorem 5.5. *Let* $\Gamma \to \Delta$ *be a sequent in the language of* **G3KGL**. *Then it is decidable whether the sequent is derivable in* **G3KGL**. *In the negative case, the failed proof search gives an irreflexive, transitive, and Noetherian Kripke countermodel.*

Proof.

1. Construction of the reduction tree: The definition of the reduction tree is similar to that in the proof of Theorem 2.6 with a modification in the case of implication and a new case for modality on the left which are dealt with as in Theorem 11.28 of [Negri and von Plato, 2011]. There are 9 different stages, 8 for the rules of the basic modal systems and one for *Trans*. At stage $n = 9 + 1$ we repeat stage 1, at stage $n = 9 + 2$ we repeat stage 2, and so on for every n until no more rule is applicable or an initial sequent or instance of *Irref* or a looping sequent (defined below) is found. For the stage relative to $R\Box$-L (stage 8), we consider all the formulas with $\Box$ as the outermost connective in the succedent of top-sequents of the tree, $x_1 : \Box B_1, \ldots, x_m : \Box B_m$, Δ' the other formulas, and write on top of each sequent $\Gamma' \to \Delta'$ the sequent

$$x_1 R y_1, \ldots, x_m R y_m, y_1 : \Box B_1, \ldots, y_m : \Box B_m, \Gamma' \to \Delta', y_1 : B_1, \ldots, y_m : B_m$$

where $y_1, \ldots, y_m$ are fresh variables, that is, we apply m times rule $R\Box$-L.

Without loss of generality, because of height-preserving admissibility of contraction, we shall avoid applying a rule whenever it results in a duplication of labelled formulas or relational atoms. Also, once a rule has been considered, it need not be instantiated again on the same principal formulas. This is clear for a rule such as *Trans* because atomic formulas are never discarded, less immediate for $L\Box$. Observe however that by a permutation argument (see Lemma 6.3 and 6.4 in [Negri, 2005]), any two applications of $L\Box$ on the same principal formulas can be made consecutive, so that the conclusion follows again by hp-admissibility of contraction.

A priori, the reduction tree is not finite, but we show that the search for a proof is indeed finite.

First, by inspection of the rules of **G3KGL**, we see that the only formulas that can occur in a reduction tree for a sequent $\Gamma \to \Delta$ are subformulas of Γ, Δ. Duplications of formulas are ruled out by hp-admissibility of contraction, so, if the reduction tree has an infinite branch, there must be an infinite chain of accessibility relations; since the end-sequent is finite, the chain contains infinitely many labels which are introduced by applications of $R\Box$-L, and thus it must necessarily contain a subchain $x_0Rx_1, x_1Rx_2, \ldots, x_{n-1}Rx_n$ such that x_0 and x_n label the same boxed formula. The branch thus contains the following steps:

$$\begin{array}{c} \vdots \\ x_0Rx_1, x_1Rx_2, \ldots, x_{n-1}Rx_n, x_1 : \Box A, \Gamma_n \to \Delta_n, x_n : \Box A \\ \vdots \\ \dfrac{x_0Rx_1, x_1 : \Box A, \Gamma_0 \to \Delta_0, x_1 : A}{\Gamma_0 \to \Delta_0, x_0 : \Box A}\; R\Box\text{-}L \end{array}$$

The upper sequent is of the form $xR^*y, x : \Box A, \Gamma \to \Delta, y : \Box A$, hence derivable by Lemma 5.2, against the assumption of the branch being infinite. Besides obtaining the conclusion that there is no infinite branch, the above argument shows that there is no need to continue the proof search with $R\Box$-L beyond a sequent of the form $xR^*y, x : \Box A, \Gamma \to \Delta, y : \Box A$ that we shall call a *looping sequent*.

Sequents which are not initial, nor conclusions of *Irref*, nor looping sequents, and that are closed under all the available rules will be called *saturated sequents*. The reduction tree is completed when all leaves lead to sequents that are either initial, or instances of *Irref*, or conclusion of $L\bot$, or saturated, or looping sequents.

2. Construction of the countermodel: If the reduction tree is not a derivation, it has at least one leaf which is a saturated sequent. Let $\mathbf{\Gamma}$ and $\mathbf{\Delta}$ be the unions of the antecedents and the succedents, respectively, of all the sequents $\Gamma_i \to \Delta_i$ of the branch up to the saturated sequent. We define a Kripke model that forces all the formulas in $\mathbf{\Gamma}$ and no formula in $\mathbf{\Delta}$ and is therefore a countermodel to the sequent $\Gamma \to \Delta$.

Consider the frame K the nodes of which are all the labels that appear in the relational atoms in $\mathbf{\Gamma}$, with their mutual relationships expressed by the xRy's in $\mathbf{\Gamma}$. Clearly, the construction of the reduction tree imposes the frame properties of the countermodel: For no x do we have xRx, else the sequent would be conclusion of *Irref* which is excluded, so irreflexivity holds; on the other hand, transitivity holds because the sequent is saturated. Noetherianity instead holds because the saturated sequent is a finite object, as guaranteed by its construction. The countermodel is defined as follows: For all atomic formulas $x : P$ in $\mathbf{\Gamma}$, we stipulate that $x \Vdash P$

in the frame. Since the sequent is not initial, it follows that for all atomic formulas $y : Q$ in $\mathbf{\Delta}$ we have $y \nVdash Q$.

We observe that, unlike in the case of intuitionistic logic, the blocking condition does not cause any exception to the general proof of completeness given in [Negri and von Plato, 2011], because looping sequents are derivable and thus the sequents on which countermodels are built are saturated sequents, closed under all the available rules. It follows routinely, with details as in the above-mentioned proof, that A is forced in the model at node x if $x : A$ is in $\mathbf{\Gamma}$ and A is not forced at node x if $x : A$ is in $\mathbf{\Delta}$. Therefore we have a countermodel to the endsequent $\Gamma \to \Delta$. QED

Corollary 5.6. *If a sequent* $\Gamma \to \Delta$ *is valid in every irreflexive, transitive and Noetherian Kripke model, then it is derivable in* **G3KGL**.

We observe that completeness implies in particular closure of our sequent calculus with respect to cut.

One of the original motivations for the interest in a decision procedure for the provability logic **GL** has been the possibility to inherit a more efficient decision procedure for **Int** through a faithful modal embedding. It turned out, however, that the mismatch caused by assuming frame reflexivity on one side and irreflexivity on the other makes the embedding problematic. This is the reason for choosing for that purpose the provability logic of Grzegorczyk (**Grz**) which is instead semantically characterized by *reflexive*, transitive and Noetherian frames.

Although the guiding ideas to develop an analytic labelled sequent calculus for **Grz** are similar to those exploited for **GL**, the formal developments, all detailed in [Dyckhoff and Negri, 2013] are rather different. We shall here outline just the peculiarity of this approach in relation to the generation of countermodels detailed in Section 3 for intuitionistic logic and above for **GL**. We start from the semantic characterization of the forcing relation:

> If R is a transitive reflexive Noetherian relation (i.e., every chain eventually becomes stationary), then, for all x,
>
> $$x \Vdash \Box A \iff \textit{for all } y, xRy \textit{ and } y \Vdash \Box(A \supset \Box A) \textit{ implies } y \Vdash A$$

The forcing condition for the modality in Noetherian frames justifies the following rule, where $G(A) \equiv \Box(A \supset \Box A)$ and y is fresh:

$$\frac{xRy, y : G(A), \Gamma \to \Delta, y : A}{\Gamma \to \Delta, x : \Box A}\; R\Box\text{-}Z$$

The calculus **G3Grz** is simply like **G3K** with $R\Box$ replaced by $R\Box$-Z and *Ref* and *Trans* added.

As we have seen, proof search terminates for **G3KGL** without loop-checking. For **G3Grz** an almost local blocking condition suffices to end the proof search and find finite countermodels. Specifically, in the definition of the reduction tree we do not saturate with respect to $R\Box$-Z with principal formula $x : \Box B$ if the antecedent contains, for some x_0, both x_0Rx and $x_0 : G(B)$ and the succedent of the sequent or of any sequent in the branch up to the sequent contains $x : B$.

The construction of the countermodel on a saturated branch is effected as in the general case and it is shown by induction on the complexity of formulas that the countermodel forces at the worlds that correspond to their labels formulas in the antecedent, and does not force formulas in the succedent. The new case that arises from the blocking condition is then immediate by use of induction hypothesis and reflexivity, thus we establish:

Theorem 5.7. *Let* $\Gamma \to \Delta$ *be a sequent in the language of* **G3Grz**. *Then it is decidable whether the sequent is derivable in* **G3Grz**. *In the negative case, the failed proof search gives a countermodel to the sequent on a reflexive, transitive and Noetherian frame.*

Further, we established in [Dyckhoff and Negri, 2013] a syntactic proof of soundness and faithfulness of the modal translation of intuitionistic logic:

$$\mathbf{G3I} \vdash \Gamma \to \Delta \text{ if and only if } \mathbf{G3Grz} \vdash \Gamma^{\Box} \to \Delta^{\Box}.$$

As a consequence, a constructive decision procedure for **G3I** and thus for **Int** is obtained.

6. Intuitionitic multi-modal logics

The blocking procedure that we have seen at work for intuitionistic logic in Section 3 can be generalised to systems that have *seriality* instead of reflexivity. This frame condition is met for example in *deontic logic* and in *temporal logic*. In addition to the rules for implication there can be other *label producing rules* such as the rules for the modality, and the co-presence of various accessibility relations with rules that relate them, such as the monotonicity rule of intuitionistic modal logic. As above, the procedure requires that the countermodel is built not just from the accessibility relations that are listed in a non-terminating branch of a failed proof search, but also from additional accessibilities which are added between nodes (looping labels) that satisfy the same formulas. The procedure is reminescent of the method of *semantic filtration* for the extraction of a finite countermodel from an infinite one (cf. [Gabbay, 1972, Blackburn, de Rijke and Venema, 2001]) and takes advantage of the semantic decoration along the proof search trees.

The precise definitions of the blocking conditions and of the generated countermodel depend on the system at hand. We recall how the method works for intuitionitic multi-modal logics from [Garg, Genovese and Negri, 2012]. In particular, the method provides a uniform, constructive and directly implementable decision procedure based on backwards search in labelled sequent calculi. The method works in particular for classes of logics, such as multi-modal intuitionistic logics with interaction of modalities covering the intuitionistic and multi-modal generalizations of the standard systems **K**, **T**, **K4**, **S4**, **D** and combinations thereof and logics with interaction axioms such as the axiom $\Box_a\alpha \supset \Box_b\Box_a A$ encountered in authorization logics (see [Garg, 2009], and [Genovese, 2012]) and multi-modal logics of belief (cf. [Goré and Nguyen, 2009]), the unit axiom $A \supset \Box_a A$ of lax logic [Fairtlough and Mendler, 1997] and the subsumption axiom $\Box_a A \supset \Box_b A$.

A sequent calculus for intuitionistic multi-modal logic is obtained extending **G3I** by the following rules (in $R\Box_a$ the label y is fresh):

$$\frac{y : A, x : \Box_a A, xR_a y, \Gamma \to \Delta}{x : \Box_a A, xR_a y, \Gamma \to \Delta} L\Box_a \qquad \frac{xR_a y, \Gamma \to \Delta, y : A}{\Gamma \to \Delta, x : \Box_a A} R\Box_a$$

$$\frac{xR_a z, x \leqslant y, yR_a z, \Gamma \to \Delta}{x \leqslant y, yR_a z, \Gamma \to \Delta} Mon_{R_a}$$

Table 6. G3IM: A sequent calculus for intuitionistic multi-modal logic

Observe that the modal rules are just the multi-agent version of the standard modal rules, and in addition there is rule Mon_{R_a} that corresponds to the monotonicity property $\forall xyz(x \leqslant y \ \& \ yR_a z \supset xR_a z)$. In addition, there may be other frame rules that follow the geometric rule scheme.

All the structural properties of **G3**-sequent calculi as well as soundness with respect to Kripke semantics are established in a routine way. Clearly, as in the case of **G3I**, proof search may not terminate because of loops originated from an unbounded creation of new worlds in rules $R\supset$ and $R\Box$. The goal is thus twofold: to devise a general method that not only detects such loops, but also produces Kripke countermodels that witness the non-validity of the end-sequent when such loops are detected.

The key insight is the following: all worlds in a sequent $\Gamma \to \Delta$ obtained during backward proof search lie on a rooted, directed tree, the edges of which are relations introduced by rules $R\supset$ and $R\Box$, the *label-producing* rules. We denote by $\ll$ the closure of these relations under the rules for the accessibility relations. Next, a suitable notion of a set of formulas $\mathcal{F}_\mathcal{S}(x)$ associated with a label x in a sequent $\mathcal{S}$, is given (see [Garg, Genovese and Negri, 2012] for the precise definition). A condition is then given that connects saturation of the proof tree to the particular form of $\mathcal{F}_\mathcal{S}(x)$ in relation to the tree of labels that has been constructed along the proof search: If there is a world y such that $y \neq x$, $y \ll x$ and $\mathcal{F}_\mathcal{S}(x) \subseteq \mathcal{F}_\mathcal{S}(y)$, then it is *useless* to apply any of the rules R$\supset$ and R$\Box$ on any principal formula labelled by x in the sequent $\mathcal{S}$ in backwards proof search. This fact is seen by structural a proof analysis and is used in the pruning of a proof-search tree and in the countermodel construction.

The saturation condition modulo looping is defined as follows: For every label-producing rule on x, either the branch is closed under the rule, or there is some early label $y \ll x$ with $\mathcal{F}_\mathcal{S}(x) \subseteq \mathcal{F}_\mathcal{S}(y)$. A similar condition applies to seriality. For all other rules, the procedure of saturation is defined by simply closing under the rule, in all possible ways, as in the construction of the proof search tree in the proof of completeness. When a sequent is obtained that is saturated and not initial, nor a conclusion of zero-premiss rules, a countermodel is built by taking as worlds the worlds that occur in the saturated sequent, as relations the closure of the union of the relations in the sequent together with the new relations $x \preccurlyeq y$ that originate from the inclusions $\mathcal{F}_\mathcal{S}(x) \subseteq \mathcal{F}_\mathcal{S}(y)$ witnessing a looping label x, and the forcing is defined as usual. It is then shown that whenever a saturated sequent is found, the procedure defines a countermodel to the endsequent.

Termination is then shown by ruling out the possibility that the procedure generates infinite chains of labels and by using the inclusion of $\mathcal{F}_{\mathcal{S}}(x)$ in the power set of subformulas of the endsequent (cf. [Garg, Genovese and Negri, 2012] for details).

7. Extensions beyond geometric theories

So far the scope of the method of labelled sequent calculi measured in terms of well defined classes of frame conditions has covered in a uniform fashion those expressible through geometric implications. They cover most known non-classical logics, but important exceptions are met when considering *multi-dimensional modal logics*, such as those that merge in their axiomatizations the epistemic and alethic modalities. In fact, the extension of proof analysis beyond geometric theories started with the proof-theoretical investigation of what is known as *knowability logic*. This logic has been in the focus of recent literature on the investigation of paradoxes that arise from the principles of the verificationist theory of truth [Salerno, 2009]. By the methods of proof analysis, it has been possible to pinpoint how the ground logic is responsible for the paradoxical consequences of these principles. A study focused on the well known Church-Fitch paradox brought forward a new challenge to the method of conversion of axioms into rules. The *knowability principle*, which states that whatever is true can be known, is rendered in a standard multi-modal alethic-epistemic language by the axiom $A \supset \Diamond\mathcal{K}A$. This axiom corresponds, in turn, to the frame property

$$\forall x \exists y (xRy \,\&\, \forall z (yR_{\mathcal{K}}z \supset x \leqslant z))$$

Here R, $R_{\mathcal{K}}$, and $\leqslant$ are the alethic, epistemic, and intuitionistic accessibility relations, respectively. This frame property goes beyond geometric implications and therefore the conversion into rules cannot be carried through with the geometric rule scheme. In this specific case, we succeeded with a combination of two rules linked together by a side condition on the eigenvariable. The resulting calculus has all the structural properties of the ground logical system and leads to definite answers to the questions raised by the Church-Fitch paradox by means of a complete control over the structure of derivations for knowability logic [Maffezioli, Naibo and Negri, 2012].

The generalization and systematization of the method of system of rules allows the treatment of axiomatic theories and of logics characterized by frame properties expressible through *generalized geometric implications* that admit arbitrary quantifier alternations and a more complex propositional structure than that of geometric implications [Negri, 2013]. The class of generalized geometric implications is defined as follows: We start from a geometric axiom (cf. Section 3) but we do not require it to be a sentence, i.e., we allow the presence of free variables. Here the P_i range over a finite set of atomic formulas and all the M_j are conjunctions of atomic formulas and the variables y_j are not free in the P_i:

$$GA_0 \equiv \forall \overline{x}(\& P_i \supset \exists y_1 M_1 \vee \cdots \vee \exists y_n M_n)$$

Taking GA_0 as the base case in the inductive definition of a *generalized geometric axiom*, we define

$$GA_1 \equiv \forall \overline{x}(\,\&\, P_i \supset \exists y_1 \,\&\, GA_0 \vee \cdots \vee \exists y_m \,\&\, GA_0)$$

Next we define by induction

$$GA_{n+1} \equiv \forall \overline{x}(\,\&\, P_i \supset \exists y_1 \,\&\, GA_{k_1} \vee \cdots \vee \exists y_m \,\&\, GA_{k_m})$$

Here $\& \, GA_i$ denotes a conjunction of GA_i-axioms and $k_1, \ldots, k_m \leqslant n$.

System of rules for generalized geometric implications are defined inductively. For $n = 1$ we have the following scheme:

$$\dfrac{\begin{array}{ccc} \Gamma'_1 \to \Delta'_1 & & \Gamma'_m \to \Delta'_m \\ \vdots & & \vdots \\ \mathcal{D}^1_0 & & \mathcal{D}^m_0 \\ \vdots & & \vdots \\ \Gamma''_1 \to \Delta''_1 & & \Gamma''_m \to \Delta''_m \\ \vdots & & \vdots \\ \mathcal{D}^1 & & \mathcal{D}^m \\ \vdots & & \vdots \\ z_1 = z_1, \overline{P}, \Gamma \to \Delta & \ldots & z_m = z_m, \overline{P}, \Gamma \to \Delta \end{array}}{\overline{P}, \Gamma \to \Delta}$$

Here z_i are eigenvariables in the last inference step, the derivations indicated with $\mathcal{D}^i_0$ use rules of the form $GRS_0(z_i)$ that correspond to the geometric axioms $GA_0(z_i)$ in addition to logical rules, and the $\mathcal{D}^i$ use only logical rules.

The scheme GRS_{n+1} is defined inductively with the same conditions as above once the schemes GRS_{k_i} have been defined for $k_i \leqslant n$ as follows

$$\dfrac{\begin{array}{ccc} \Gamma'_1 \to \Delta'_1 & & \Gamma'_m \to \Delta'_m \\ \vdots & & \vdots \\ \mathcal{D}^1_{k_1} & & \mathcal{D}^m_{k_m} \\ \vdots & & \vdots \\ \Gamma''_1 \to \Delta''_1 & & \Gamma''_m \to \Delta''_m \\ \vdots & & \vdots \\ \mathcal{D}^1 & & \mathcal{D}^m \\ \vdots & & \vdots \\ z_1 = z_1, \overline{P}, \Gamma \to \Delta & \ldots & z_m = z_m, \overline{P}, \Gamma \to \Delta \end{array}}{\overline{P}, \Gamma \to \Delta}$$

Through an operative conversion to normal form, generalized geometric implications can also be characterized in terms of *Glivenko classes* as those first-order formulas that do not contain implications or universal quantifiers in their negative parts. By this result and Kracht's characterization theorem (cf. Theorem 3.59 in [Blackburn, de Rijke and Venema, 2001]) the method is seen to cover all systems of normal modal logics axiomatized by *Sahlqvist formulas*.

The equivalence, established in [Negri, 2003], between the axiomatic systems based on geometric axioms and contraction- and cut-free sequent systems with geometric rules, is extended by a suitable definition of *systems of rules* for generalized geometric axioms. Here the word "system" is used in the same sense as in linear algebra where there are systems of equations with variables in common, and each equation is meaningful and can be solved only if considered together with the other equations of the system. In the same way, the systems of rules considered in this context consist of rules connected to each other by some variables and subject in addition to the condition of appearing in a certain order in a derivation.

The precise form of system of rules, the structural properties for the resulting extensions of sequent calculus (admissibility of cut, weakening, and contraction), a generalization of Barr's theorem, examples from axiomatic theories and applications to the proof theory of non-classical logics through a proof of completeness of the proof systems obtained, are all detailed in [Negri, 2013].

8. Conclusion

There are three main aspects to be considered in logical investigations: the normative, the descriptive and the deductive one. These are respectively associated to the questions: *What are the axioms? What are the models? What are the proofs?* Answers to these questions are typically given for first-order logic by supplying a logical system with axiomatizations, canonical models, and Gentzen calculi. These aspects are tightly related to each other by metatheorems such as the deduction and the completeness theorem. The same solid and stable picture is not inherited by non-classical logics and a very rich literature has been developed in the past decade to fill gaps in the picture, especially in its greyest area, the deductive corner, to provide richer formalisms for the proof theory of non-classical logics, given the failure of traditional Gentzen systems (cf. the survey [Negri, 2011] and references therein). As for the connections between these aspects, there has been a long debate on the validity of the deduction theorem for modal logic, which has been summarised and clarified with a definite answer in [Hakli and Negri, 2011a]. The present work is specially aimed at filling the gap in the picture between the descriptive and deductive aspects of non-classical logics, i.e. between their proofs and (counter)models.

Building on the framework of labelled sequent calculi, a general method has been presented for the simultaneous search of proofs and countermodels in non-classical logics. The method has been illustrated in detail for the case of intuitionistic logic, but it has also been shown how failed proof search can be used to construct countermodels in several other contexts: intuitionistic and intermediate logics, classical modal logics, provability logics, intuitionistic multi-modal logics, and (more generally) logics with rather complex frame conditions (involving quantifier alternations). The direct completeness proof provides concrete countermodels that by construction already satisfy all the properties of the intended class of models whereas more traditional proofs, based on Henkin sets, do so only through intermediate manipulation stages. Further, finitization methods have been detailed that turn a parallel proof search and countermodels generation into a decision procedure. Such decision procedures work not just for simple logics like intuitionistic logic or **S4**, but also for logics such intuitionistic multi-modal logics

[Garg, Genovese and Negri, 2012], linear temporal logic [Boretti and Negri, 2009], and authorisation logic [Genovese, 2012].

The method introduced in [Negri, 2005] to formulate a labelled sequent calculus for the logic GL has been used and successfully applied to the tree-hypersequent case by in [Poggiolesi, 2009], thus witnessing the possibility to transfer to other formalisms methods and results of the labelled approach. The question has thus been posed as to whether other formalisms for non-classical logics, such as display calculi or hypersequents and their generalisations could be used to achieve results similar to those presented in this paper. First we observe that among the various approaches to the proof theory of non-classical logics, those based on the internalization of the relational semantics are clearly the most adequate for the direct extraction of countermodels from failed proof search. Among these, labelled sequent calculi are preferable as they extend in a uniform way the expressive power of prefixed tableau systems and other labelled approached based on natural deduction. However, besides uniformity and expressive power, an essential ingredient for a successful implementation of the methodology is the possibility to use the calculi in an analytic way in root-first proof search, including the property of admissibility of all the structural rules. Further, and more specifically, rules should be invertible, so no backtracking is required during proof search, and have the stronger property that a countermodel to the premiss(es) of a rule is also a countermodel to the conclusion.

The question on the feasibility of other formalisms can be thus answered directly by the above checklist. It can be also answered indirectly by a suitable embedding of labelled sequent calculi into other formalisms. However, unlike other general embeddings that have been established so far [Restall, 2006, Wansing, 1998, Poggiolesi, 2010, Goré and Ramanayake 2012], embeddings of labelled calculi into display or hypersequent calculi have been obtained only for limited fragments [Mints,1997, Rothenberg, 2010].

References

[Avron, 1996] Avron, A. The method of hypersequents in the proof theory of propositional non-classical logics, in W. Hodges et al. (eds) *Logic: from foundations to applications*, pp. 1-32, Oxford Science Publications, 1996.

[Baaz et al., 2003] Baaz, M., A. Ciabattoni, and C. Fermüller (2003) Hypersequent Calculi for Gödel Logics - a Survey. *Journal of Logic and Computation*, vol. 13, pp. 835–861.

[Basin, Matthews and Viganò, 1998] Basin, D., S. Matthews and L. Viganò. Natural deduction for non-classical logics, *Studia Logica*, vol. 60, pp. 119–160, 1998.

[Blackburn, 2000] Blackburn, P. Internalizing labelled deduction. *Journal of Logic and Computation*, vol. 10, pp. 137–168, 2000.

[Blackburn, de Rijke and Venema, 2001] Blackburn, P. M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge University Press, 2001.

[Boretti and Negri, 2009] Boretti, B. and S. Negri. Decidability for Priorean linear time using a fixed-point labelled calculus. In M. Giese and A. Waaler (eds) *Automated Reasoning with Analytic Tableaux and Related Methods*, Lecture Notes in Computer Science, vol. 5607, Springer, pp. 108–122, 2009.

[Brünnler, 2009] Brünnler, K. Deep sequent systems for modal logic, *Archive for Mathematical Logic*, vol. 48, pp. 551–577, 2009.

[Castellini, 2005] Castellini, C. *Automated Reasoning in Quantified Modal and Temporal Logic*, Ph.D. thesis, School of Informatics, University of Edinburgh, 2005.

[Catach, 1991] Catach, L. TABLEAUX: A general theorem prover for modal logic, *Journal of Authomated Reasoning*, vol. 7, pp. 489–510, 1991.

[Cerrato, 1996] Cerrato, C. Modal sequents, in Wansing (ed) ibid., pp. 141–166, 1996.

[Ciabattoni et al., 2008] Ciabattoni, A., N. Galatos, and K. Terui. From axioms to analytic rules in nonclassical logics, *Proc. Logic in Computer Science*, IEEE Computer Society, pp. 229–240, 2008

[D'Agostino, 1990] D'Agostino, M. Investigations into the Complexity of some Propositional Calculi. Ph.D. thesis, Oxford University Computing Laboratory, 1990.

[Dummett and Lemmon, 1959] Dummett, M.A.E., and E.J. Lemmon. Modal logics between S4 and S5, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* vol. 5, pp. 250–264, 1959.

[Dyckhoff, 1992] Dyckhoff, R. Contraction-free sequent calculi for intuitionistic logic, *The Journal of Symbolic Logic*, vol. 57, pp. 795–807, 1992.

[Dyckhoff and Negri, 2012] Dyckhoff, R. and S. Negri. Proof analysis in intermediate logics. *Archive for Mathematical Logic*, vol. 51, pp. 71–92, 2012.

[Dyckhoff and Negri, 2013] Dyckhoff, R. and S. Negri. A cut-free sequent system for Grzegorczyk logic, with an application to the Gödel-McKinsey-Tarski embedding. *Journal of Logic and Computation*, DOI: 10.1093/logcom/ext036, 2013.

[Fairtlough and Mendler, 1997] Fairtlough, M. and M. Mendler. Propositional lax logic. *Information and Computation*, vol. 137, pp. 1–33, 1997.

[Ferrari, Fiorentini and Fiorino, 2012] Ferrari, M., C. Fiorentini, and G. Fiorino. Contraction-free linear depth sequent calculi for intuitionistic propositional logic with the subformula property and minimal depth counter-models. *Journal of Automated Reasoning*, DOI 10.1007/s10817-012-9252-7, 2012.

[Fitch, 1966] Fitch, F. B. Tree proofs in modal logic, *The Journal of Symbolic Logic*, vol. 31, p. 152, 1966.

[Fitting, 2012] Fitting, M. Prefixed tableaus and nested sequents. *Annals of Pure and Applied Logic*, vol. 163, pp. 291–313, 2012.

[Gabbay, 1972] Gabbay, D. A general filtration method for modal logics. *Journal of Philosophical Logic*, vol. 1, pp. 29–34, 1972.

[Gabbay, 1996] Gabbay, D. *Labelled Deductive Systems*, Oxford University Press, 1996.

[Galmiche and Salhi 2011] Galmiche, D. and Y. Salhi. Sequent calculi and decidability for intuitionistic hybrid logic. *Information and Computation*, vol. 209, pp. 1447–1463, 2011.

[Garg, 2009] Garg, G. Proof Theory for Authorization Logic and Its Application to a Practical File System. PhD thesis, Carnegie Mellon University, 2009.

[Garg, Genovese and Negri, 2012] Garg, G., V. Genovese, and S. Negri. Counter-models from sequent calculi in multi-modal logics. *LICS 2012*, IEEE Computer Society, pp. 315–324, 2012.

[Genovese, 2012] Genovese, V. *Modalities in Access Control: Logics, Proof-Theory and Applications*. PhD thesis, U. of Luxembourg and U. of Torino, 2012.

[Goranko, 1994] Goranko, V. Refutation systems in modal logic. *Studia Logica*, vol. 53, pp. 299–324, 1994.

[Goré, 1998] Goré, R. Tableau methods for modal and temporal logics, in D'Agostino, M. *et al.* (eds) *Handbook of Tableau Methods*, Kluwer, 1998.

[Goré and Nguyen, 2009] Goré, R. and L. Nguyen. Clausal tableaux for multimodal logics of belief. *Fundamenta Informaticae*, vol. 94, pp. 21–40, 2009.

[Goré and Postniece, 2010] Goré, R. and L. Postniece. Combining derivations and refutations for cut-free completeness in bi-intuitionistic logic. *Journal of Logic and Computation*, vol. 20, pp. 233–260, 2010.

[Goré and Ramanayake 2012] Goré, R. and R. Ramanayake. Labelled tree sequents, tree hyperse- quents and nested (deep) sequents. In *Advances in modal logic*, vol 9, College Publications, London, 2012.

[Gödel, 1933] Gödel, K., *Eine Interpretation des intuitionistischen Aussagenkalküls*, Ergebnisse eines mathematischen Kolloquiums vol. 4, pp. 39–40, 1933. English tr. in Gödel's *Collected Works* vol. I, pp. 300–303, 1986.

[Hakli and Negri, 2011] Hakli, R. and S. Negri, Reasoning about collectively accepted group beliefs. *Journal of Philosophical Logic*, vol. 40, pp. 531–555, 2011.

[Hakli and Negri, 2011a] S. Negri and R. Hakli, Does the deduction theorem fail for modal logic? *Synthese*, vol. 187, pp. 849–867, 2012.

[Kashima, 1994] Kashima, R. Cut-free sequent calculi for some tense logics, *Studia Logica*, vol. 53, pp. 119–135, 1994.

[Ketonen, 1944] Ketonen, O. *Untersuchungen zum Prädikatenkalkül*, (Annales Acad. Sci. Fenn, Ser. A.I. vol. 23), 1944.

[McKinsey and Tarski, 1948] McKinsey, J.C., and A. Tarski. *Some theorems about the sentential calculi of Lewis and Heyting*. Journal of Symbolic Logic, vol.13, pp. 1–15, 1948.

[Kushida and Okada, 2003] Kushida, H. and M. Okada. A proof-theoretic study of the correspondence of classical logic and modal logic, *The Journal of Symbolic Logic*, vol. 68, pp. 1403–1414, 2003.

[Larchey-Wendling, 2002] Larchey-Wendling, D. Combining proof-search and countermodel construction for deciding Gödel-Dummett logic. In *Proc. CADE-18*, LNCS 2392, pp. 94–110, 2002.

[Maffezioli, Naibo and Negri, 2012] Maffezioli, P., A. Naibo, and S. Negri. The Church-Fitch knowability paradox in the light of structural proof theory. *Synthese*, Online first DOI 10.1007/s11229-012-0061-7, 2012.

[Massacci, 2000] Massacci, F. Single step tableaux for modal logics: methodology, computations, algorithms, *Journal of Automated Reasoning*, vol. 24, pp. 319–364, 2000.

[Minari, 2013] Minari, P. Labeled sequent calculi for modal logics and implicit contractions. Online first DOI 10.1007/s00153-013-0350-y, *Archive for Mathematical Logic*, 2013.

[Mints,1997] Mints, G. Indexed systems of sequents and cut-elimination, *Journal of Philosophical Logic* vol. 26, pp. 671–696, 1997.

[Negri, 2003] Negri, S. Contraction-free sequent calculi for geometric theories, with an application to Barr's theorem. *Archive for Mathematical Logic*, vol. 42, pp. 389–401, 2003.

[Negri, 2005] Negri, S. Proof analysis in modal logic. *Journal of Philosophical Logic*, vol. 34, pp. 507–544, 2005.

[Negri, 2009] Negri, S. Kripke completeness revisited. In G. Primiero and S. Rahman (eds.), *Acts of Knowledge - History, Philosophy and Logic*, pp. 247–282, College Publications, 2009.

[Negri, 2011] Negri, S. *Proof theory for modal logic*, Philosophy Compass vol. 6, pp. 523–538, 2011.

[Negri, 2013] Negri, S.Proof analysis beyond geometric theories: from rule systems to systems of rules. Journal of Logic and Computation, in press.

[Negri and von Plato, 1998] Negri, S. and J. von Plato. Cut elimination in the presence of axioms. *The Bulletin of Symbolic Logic*, vol. 4, pp. 418–435, 1998.

[Negri and von Plato, 2001] Negri, S. and J. von Plato. *Structural Proof Theory*. Cambridge University Press, 2001.

[Negri and von Plato, 2011] Negri, S. and J. von Plato. *Proof Analysis*. Cambridge University Press, 2011.

[Nerode, 1991] Nerode, A. Some lectures on modal logic, in F. L. Bauer (ed) Logic, Algebra, and Computation, NATO ASI Series, vol. F79, Springer, pp. 281–334, 1991.

[Orlowska and Golińska Pilarek, 2011] Orlowska, E. and J. Golińska Pilarek. *Dual Tableaux: Foundations, Methodology, Case Studies*. Trends in Logic, vol. 33, Springer, 2011.

[Pinto and Dyckhoff, 1995] Pinto, L. and R. Dyckhoff. Loop-free construction of counter-models for intuitionistic propositional logic. In Behara et al. eds., *Symposia Gaussiana*, Conf. A, pp. 225–232, de Gruyter, Berlin, 1995.

[Poggiolesi, 2009] Poggiolesi, F. A purely syntactic and cut-free sequent calculus for the modal logic of provability, *Review of Symbolic Logic*, vol. 2, pp. 593–611, 2009.

[Poggiolesi, 2010] Poggiolesi, F. *Gentzen Calculi for Modal Propositional Logic*, Springer, 2010.

[Restall, 2006] Restall, G. Comparing modal sequent systems. http://consequently.org/ papers/comparingmodal.pdf, 2006

[Rothenberg, 2010] Rothenberg, R. *On the relationship between hypersequent calculi and labelled sequent calculi for intermediate logics with geometric Kripke semantics*. PhD thesis, University of St Andrews, 2010.

[Russo, 1995] Russo, A. Modal labelled deductive systems, Dept. of Computing, Imperial College, London, Technical Report 95(7), 1995.

[Salerno, 2009] Salerno, J., ed. *New Essays on the Knowability Paradox*. Oxford University Press, 2009.

[Schroeder-Heister, 2012] Schroeder-Heister, P. Proof-Theoretic Semantics. Stanford Encyclopedia of Philosophy, online edition, 2012.

[Schütte, 1956] Schütte, K. (1956) Ein System des verknüpfenden Schliessens, *Archiv für mathematische Logik und Grundlagenforschung*, vol. 2, pp. 55–67.

[Simpson, 1994] Simpson, A. Proof Theory and Semantics of Intuitionistic Modal Logic. Ph.D. thesis, School of Informatics, University of Edinburgh, 1994.

[Skura, 2011] Skura, T. Refutation systems in propositional logic. In D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic*, vo. 16, pp. 115–157, Springer, 2011.

[Stewart and Stouppa, 2005] Stewart, C. and P. Stouppa. A systematic proof theory for several modal logics, *Advances in modal logic*, vol. 5, King's College Publication, London, pp. 309–333, 2005.

[Takeuti, 1987] Takeuti, G. *Proof Theory*, 2nd ed. North-Holland, 1987.

[Troelstra and Schwichtenberg, 2000] Troelstra, A., and H. Schwichtenberg. *Basic Proof Theory*, 2nd ed, Cambridge, 2000.

[Viganò, 2000] Viganò, L. *Labelled Non-Classical Logics*, Kluwer, 2000

[Wansing, 1996] Wansing, H. (ed.) *Proof Theory of Modal Logic*, Proceedings of the workshop held at the University of Hamburg, Hamburg, November 19–20, 1993. Applied Logic Series, 2. Kluwer Academic Publishers, Dordrecht, 1996.

[Wansing, 1998] Wansing, H. Translation of Hypersequents into Display Sequents. *Logic Journal of the IGPL*, vol. 6, pp. 719–33, 1998b .

[Wansing, 2002] Wansing, H. Sequent systems for modal logics, in D. Gabbay and F. Guenther (eds) *Handbook of Philosophical Logic*, 2^{nd} ed, vol. 8, Kluwer, pp. 61–145, 2002.

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