Proof analysis in non-classical logics

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Introduction

The development of sequent systems for non-classical, in particular, modal, logics, started in the 1950s, with the work of Curry (1952) who provided a system with cut elimination and a decision procedure for S4, and Kanger (1957), who gave sequent calculi and decision procedures for T, S4, S5 with the use of "spotted formulas", i.e., formulas indexed by natural numbers.

Difficulties in the Gentzen-style formalization of modal logic were, however, encountered at a very elementary level, for instance in the search of an adequate cut-free sequent calculus for the modal logic S5.[1] These difficulties are well witnessed by the ongoing present interest in the problem, with two more proposals presented in this Colloquium (Restall 2005, Stouppa 2005).

The lack of a general solution has justified an overall pessimistic attitude towards the possibility of applying Gentzen's systems to non-classical logics, as is shown in the following passages:

> Gentzen's methods do not provide anything like a universal approach to logic ... There are certain standard logics to which these methods do not apply in as direct a fashion ... For example, consider the logics B and S5. The Kripke models for these are symmetric ... Such things effectively destroy all possibility of a good, simple cut-free Gentzen system. Fitting (1983, p. 4)

> The other tradition that should be mentioned is that of proof theory. Gentzen methods have never really flourished in modal logic. Bull and Segerberg (1984, p. 7)

[1] In 1957 Ohnishi and Matsumoto presented sequent calculi with cut elimination for various modal logics, but no cut elimination for S5. Mints (1968) gives a sequent calculus for S5 with quantifiers that enjoys cut elimination but not the subformula property. The same limitation is encountered in Sato (1980). Shvarts (1989) gave an indirect proof of cut elimination, showing that A is provable in S5 iff $\Box A$ is provable in a suitable cut-free calculus. A similar idea, translated in terms of tableaux systems, is exploited in Fitting (1999). Braüner (2000) proved cut elimination for a calculus for S5 that cannot be appropriately called a sequent system because of the non-locality of its rules.

> In modal logic the situation is much more delicate; there are significant technical problems ... to be faced when translating one style into another. We step over these problems by choosing ... a Hilbert type proof system. Sally Popkorn (1994, p. 97)

and, in the most recent textbook on modal logic, in the section "What this book is not about"

> The omission of proof theory and automated reasoning techniques calls for a little more explanation. ... as is often the case in modal logic, the proof systems discussed are basically Hilbert-style axiomatic systems. There is no discussion of natural deduction, sequent calculi, labelled deductive systems, resolution, or display calculi. ... Why is this? Essentially because modal proof theory and automated reasoning are still relatively youthful enterprises; they are exciting and active fields, but yet there is little consensus about methods and few general results. Blackburn, de Rijke, and Venema (2001, p. xvi).

Good sequent calculi should satisfy certain design principles. In Wansing (1994, 2002) explicit philosophical, methodological and computational requirements for sequent systems for modal logic have been laid down. In short, they are the following:

1. *Separation*: The rules for each connective/modality should be given through a purely structural account of its meaning, in the sense that they should be independent of any other connective/modality.

2. *Weak symmetry*: Each rule should either be a left or a right rule, introducing the connective either into the left or into the right hand side of the sequent arrow in the conclusion. The requirement can be strengthened to *symmetry* if there are both left and right rules for each connective/modality.

3. *Weak explicitness* (resp. *explicitness*): The connective/modality appears only (resp. only and once) in the conclusion of the rule.

4. The two modalities $\Box$ and $\Diamond$ should be both primitive but interderivable.

5. *Uniqueness*: Each connective should be uniquely characterized by its rules in a given system.

6. Different systems are obtained by changing only the structural rules, while leaving the logical rules unaltered.

7. Cut elimination

8. Subformula property

After reviewing the earlier attempts of defining sequent systems for certain non-classical logics, Wansing is led to the conclusion that

> No uniform way of presenting ... the most important normal modal and temporal propositional logics as ordinary Gentzen calculi is known. Further, the standard approach fails to be *modular*. ... Each of the ordinary sequent systems presented ... fails to satisfy some of the more philosophical requirements mentioned ... there are thus not only technical but also philosophical reasons for investigating generalizations of the notion of a Gentzen sequent.

The generalizations of traditional Gentzen sequent calculi presented include systems such higher-level sequents, higher-dimensional, higher-arity, multiple sequent systems, hypersequents, display logic.

In addition to these generalizations, in recent years an approach based on the internalization of the Kripke semantics into the calculus has gained prominence. This idea, with early precursors as far as in Kanger (1957), has been developed in several forms. Inference systems have been presented that incorporate possible worlds in the form of sequents (Mints 1997, Viganó 2000, Kushida and Okada 2003, Castellini and Smaill 2002, Castellini 2005), in the form of tableaux (Fitting 1983, Catach 1991, Nerode 1991, Goré 1998, Massacci 2000), and in the form of natural deduction (Fitch 1966, Simpson 1994, Basin, Matthews, Viganó 1998). The use of a syntax that includes the relational semantics has been central also in the work on first-order encodings of modal logic (Ohlbach 1993, Schmidt and Hustadt 2003) and in what is called hybrid logic (Blackburn 2000). Internalization of the algebraic - rather than relational - semantics into a natural deduction style presentation is instead mainly used in Labelled Deductive Systems (Gabbay 1996).

Despite their impact, labelled proof systems have been criticized as impure, in contrast to the more traditional proof systems, and difficult to use in practice:

> a deductive treatment congenial to modal logic is yet to be found, for Hilbert systems are not suited for the purpose of actual deductions, and in Hintikka/Kripke systems the alternativeness relation introduces an alien element which, moreover, can become quite unmanageable in special cases. Bull and Segerberg (1984, 2001).

Furthermore, the more goal-oriented labelled tableau systems do not translate into elegant Gentzen sequent calculi.

> [Tableau calculi, translated into sequent system] do not possess all the elegant properties usually demanded of (Gentzen) systems. ... Elegant modal sequent systems respecting the ideals of Gentzen have proved elusive. Goré (1998).

Our aim is to provide a general approach to the proof theory on non-classical logics through labelled sequent calculi that obey all the principles of good design usually required of traditional sequent systems. In particular, the calculi we shall present have all the structural rules–weakening, contraction, and cut–admissible; they support, whenever possible, proof search, and have a simple and uniform syntax that allows easy proofs of metatheoretic results.

These calculi all stem from a systematic development, started with Negri and von Plato (1998), of a method for converting axioms into rules to be added to cut- and contraction-free sequent systems while maintaining all the structural properties in the resulting extension.

In previous work the method has been applied to extensions of logic, that is, to certain mathematical theories such as theories of order (Negri, von Plato, and Coquand 2001), lattice theory (Negri and von Plato 2004, Negri 2005a), linear Heyting algebras (Dyckhoff and Negri 2006), real closed fields (Negri 2001), projective and affine geometry (von Plato 2005a), and to the so-called geometric and cogeometric theories (Negri 2003, Negri and von Plato 2005).

Recently, the method has been applied *inside* logic, for the generation of sequent systems for all those logics that can be characterized in terms of a Kripke-style relational semantics. These include the most standard normal modal logics and provability logic, treated in Negri (2005), and also intermediate logics, relevant logic, and, in general, substructural logics.

In Section 1 we shall review the background on sequent calculus and its extensions with rules. In Section 2, starting from a G3-style labelled sequent calculus for basic modal logic, we shall present the application of the method to modal logics characterized by universal and geometric frame properties. There are certain modal logics, such as the provability logic of Gödel-Löb, that are characterized by frame properties that are not first-order. In Section 3 it is shown how to deal with such extensions through a semantically justified definition of the rules for the modality. In Section 4 we present a sequent calculus with internalized Kripke semantics for intuitionistic logic. It turns out that all the properties characterizing the Kripke frames for the seven interpolable intermediate logics are geometric axioms, and thus fall under the scope of our method. Also relevant, and in general, substructural logics, can be characterized through a suitable relational semantics, with properties following the form of geometric axioms. A uniform proof-theoretic treatment for substructural logic is presented in Section 5. Finally, in the conclusion we indicate how the calculi presented relate to the above requirements for sequent calculi and how they can serve as calculi establishing decidability through terminating proof search.

1 Background

In Negri and von Plato (1998, 2001) and in Negri (2003) a general method was presented for extending sequent calculi with rules for axiomatic theories while preserving all the structural properties of the logical calculus. We recall here the general ideas of the method and the main results.

For extensions of classical predicate logic the starting point is the contraction- and cut-free sequent calculus **G3c**. We recall that all the rules of **G3c** are invertible and all the structural rules are admissible, that is, whenever their premisses are derivable, then so is their conclusion. Weakening and contraction are in addition *height-preserving* admissible, that is, whenever their premisses are derivable with derivation height bounded by n, then also is their conclusion, with the same bound on the derivation height (the *height* of a derivation is its height as a tree, that is, the length of its longest branch). Moreover, the calculus enjoys height-preserving admissibility of substitution. Also, invertibility of the rules of **G3c** is height-preserving (see Chapters 3 and 4 of Negri and von Plato (2001) for detailed proofs).

These remarkable structural properties of **G3c** are maintained in extensions of the logical calculus with suitably formulated rules that represent axioms for specific theories. Universal axioms are first transformed, through the rules of **G3c**, into conjunctive normal form, that is conjunctions of formulas of the form $P_1 \& \dots \& P_m \supset Q_1 \vee \dots \vee Q_n$, where the consequent is $\bot$ if $n = 0$ and all P_i, Q_j are atomic. (Any such formula, universally quantified, is called a *regular* formula.) We abbreviate the multiset $P_1, \dots, P_m$ as $\overline{P}$. Each conjunct is then converted into a schematic rule, called the *regular rule scheme*, of the form

$$\frac{Q_1, \overline{P}, \Gamma \Rightarrow \Delta \quad \dots \quad Q_n, \overline{P}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta} \; Reg$$

By this method, all universal theories can be formulated as contraction- and cut-free systems of sequent calculi.

In Negri (2003), the method is extended to cover also *geometric theories*, that is, theories axiomatized by geometric implications. We recall that a *geometric formula* is a formula not containing $\supset$ or $\forall$ and a *geometric implication* is a sentence of the form

$$\forall \overline{z}(A \supset B)$$

where A and B are geometric formulas. Geometric implications can be reduced to a normal form consisting of conjunctions of formulas, called *geometric axioms*, of the form

$$\forall \overline{z}(P_1 \& \dots \& P_m \; \supset \; (\exists \overline{x}_1 M_1 \vee \dots \vee \exists \overline{x}_n M_n))$$

where each M_j is a conjunction of atomic formulas, $Q_{j_1}, \dots, Q_{j_{k_j}}$. Without loss of generality, no x_i is free in any P_j. Note that regular formulas are geometric implications, with neither conjunctions nor existential quantifications to the right of the implication.

The *left rule scheme* for geometric axioms takes the form

$$\frac{\overline{Q}_1(\overline{y}_1/\overline{x}_1), \overline{P}, \Gamma \Rightarrow \Delta \quad \dots \quad \overline{Q}_n(\overline{y}_n/\overline{x}_n), \overline{P}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta} \; GRS$$

where $\overline{Q}_j$ and $\overline{P}$ indicate the multisets of atomic formulas $Q_{j_1}, \dots Q_{j_{k_j}}$ and $P_1, \dots, P_m$, respectively, and the eigenvariables $\overline{y}_i$ of the premisses are not free in the conclusion.

In order to maintain admissibility of contraction in the extensions with regular or geometric rules, the formulas $P_1, \dots, P_m$ in the antecedent of the conclusion of the scheme have (as indicated) to be repeated in the antecedent of each of the premisses. In addition, whenever an instantiation of free parameters in atoms produces a duplication (two identical atoms) in the conclusion of a rule instance, say $P_1, \dots, P, P, \dots, P_m, \Gamma \Rightarrow \Delta$, there is a corresponding duplication in each premiss and in the conclusion of the rule. The *closure condition* imposes the requirement that the rule with the duplication P, P contracted into a single P is added to the system of rules. For each axiom system, there is only a bounded number of possible cases of contracted rules to be added, very often none at all, so the condition is unproblematic.

The main result for such extensions is the following (Theorems 4 and 5 from Negri 2003):

Theorem 1.1 *The structural rules of weakening, contraction, and cut are admissible in all extensions of* **G3c** *with the geometric rule-scheme and satisfying the closure condition. Weakening and contraction are moreover height-preserving admissible.*

2 Basic modal logic and its extensions

In this section we shall present a sequent system for the basic modal logic K with rules for the modalities $\Box$ and $\Diamond$ obtained through a meaning explanation, in terms of the possible worlds semantics, and an inversion principle. The modal logic K is characterized by arbitrary frames. Restrictions of the class of frames characterizing a given modal logic amounts to adding certain frame properties to the calculus. These properties are added in the form of mathematical rules, following the development outlined in Section 1. All the extensions are thus obtained in a modular way. As a consequence, the structural properties of the resulting calculi can be established in one theorem for all systems.

2.1 Basic modal logic

Basic modal logic is formulated as a labelled sequent calculus through an internalization of the possible worlds semantics into the syntax. The way to achieve this is the following: First we enrich the language so that sequents are expressions of the form $\Gamma \Rightarrow \Delta$ where the multisets Γ and Δ consist of relational atoms xRy and labelled formulas $x : A$ (corresponding to the forcing $x \Vdash A$ of Kripke models), with x, y ranging in a set W of labels/possible worlds and A any formula in the language of propositional logic extended with the modal operators of necessity and possibility, $\Box$ and $\Diamond$.

The rules for each connective/modality are obtained from their meaning explanation in terms of the relational semantics: From the inductive definition of forcing for a modal formula

$$x \Vdash \Box A \ \textit{ iff for all } \ y, \ xRy \textit{ implies } \ y \Vdash A$$

we obtain

If $y : A$ can be derived for an arbitrary y accessible from x, then $x : \Box A$ can be derived

that is formalized into the rule

$$\frac{xRy, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}\, R\Box$$

where arbitrariness of y becomes the variable condition y not in Γ, Δ.

Through the inversion principle[2] we obtain the rule

$$\frac{xRy, \Gamma \Rightarrow \Delta \quad y : A, \Gamma \Rightarrow \Delta}{x : \Box A, \Gamma \Rightarrow \Delta}\, L\Box$$

that can be equivalently given as a one-premiss rule in the following form

$$\frac{y : A, x : \Box A, xRy, \Gamma \Rightarrow \Delta}{x : \Box A, xRy, \Gamma \Rightarrow \Delta}\, L\Box$$

[2]There are several formulations of the inversion principle. Here we follow the inversion principle in the form *Whatever follows from a proposition must follow from the direct grounds for asserting that proposition.* This form allows to uniquely determine the elimination rules in natural deduction and the left rules in sequent calculus, as shown in detail in Negri and von Plato 2001.

The rules for $\Diamond$ are obtained similarly from the semantic explanation

$$x : \Diamond A \ \ \textit{iff} \ \ \textit{for some} \ \ y, \ xRy \ \textit{and} \ \ y : A$$

The semantic explanation of the classical propositional connectives is flat, so the result of the above procedure is just a labelling with the same variable of the active formulas in the premisses and conclusion of each rule of the calculus **G3c**.

Our sequent calculus **G3K** for basic modal logic is thus obtained:

Initial sequents:

$$x : P, \Gamma \Rightarrow \Delta, x : P \qquad\qquad xRy, \Gamma \Rightarrow \Delta, xRy$$

Propositional rules:

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A\&B, \Gamma \Rightarrow \Delta}\,L\& \qquad \frac{\Gamma \Rightarrow \Delta, x : A \quad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A\&B}\,R\&$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \quad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta}\,L\vee \qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}\,R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad x : B, \Gamma \Rightarrow \Delta}{x : A \supset B, \Gamma \Rightarrow \Delta}\,L\supset \qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \supset B}\,R\supset$$

$$\frac{}{x : \perp, \Gamma \Rightarrow \Delta}\,L\perp$$

Modal rules:

$$\frac{y : A, x : \Box A, xRy, \Gamma \Rightarrow \Delta}{x : \Box A, xRy, \Gamma \Rightarrow \Delta}\,L\Box \qquad \frac{xRy, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}\,R\Box$$

$$\frac{xRy, y : A, \Gamma \Rightarrow \Delta}{x : \Diamond A, \Gamma \Rightarrow \Delta}\,L\Diamond \qquad \frac{xRy, \Gamma \Rightarrow \Delta, x : \Diamond A, y : A}{xRy, \Gamma \Rightarrow \Delta, x : \Diamond A}\,R\Diamond$$

Table 1. The sequent calculus G3K

2.2 Extensions

We present, by way of an example, a table of modal systems with their characterizing Hilbert-style axioms and corresponding frame properties.

	Axiom	Frame property
T	$\Box A \supset A$	$\forall x\, xRx$ reflexivity
4	$\Box A \supset \Box\Box A$	$\forall xyz(xRy \,\&\, yRz \supset xRz)$ transitivity
E	$\Diamond A \supset \Box\Diamond A$	$\forall xyz(xRy \,\&\, xRz \supset yRz)$ euclideanness
B	$A \supset \Box\Diamond A$	$\forall xy(xRy \supset yRx)$ symmetry
3	$\Box(\Box A \supset B) \vee \Box(\Box B \supset A)$	$\forall xyz(xRy \,\&\, xRz \supset yRz \vee zRy)$ connectedness
D	$\Box A \supset \Diamond A$	$\forall x \exists y\, xRy$ seriality
2	$\Diamond\Box A \supset \Box\Diamond A$	$\forall xyz(xRy \,\&\, xRz \supset \exists w(yRw \,\&\, zRw))$ directedness
W	$\Box(\Box A \supset A) \supset \Box A$	no infinite R-chains + trans.

The frame properties in the first group (T, 4, E, B, 3) are universal axioms, those in the second group are geometric implications, as defined in Section 1, whereas the last one is not expressible as a first-order property.

The systems T, K4, KB, S4, B, S5, ... are obtained by adding one or more axioms to the system K. Sequent calculi are obtained by adding to the system **G3K** the rule(s) corresponding to the properties of the accessibility relation characterizing their frames. For instance, a sequent calculus for S4 is obtained by adding to **G3K** the rules corresponding to the axiom of reflexivity and transitivity of the accessibility relation

$$\frac{xRx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,{\scriptstyle Ref} \qquad \frac{xRz, xRy, yRz, \Gamma \Rightarrow \Delta}{xRy, yRz, \Gamma \Rightarrow \Delta}\,{\scriptstyle Trans}$$

and a system for S5 by adding also the rule corresponding to symmetry

$$\frac{yRx, xRy, \Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}\,{\scriptstyle Sym}$$

Extensions are obtained in a modular way for all possible combinations of properties:

G3T = **G3K** + *Ref*
G3K4 = **G3K** + *Trans*
G3KB = **G3K** + *Sym*
G3S4 = **G3K** + *Ref* + *Trans*
G3TB = **G3K** + *Ref* + *Sym*
G3S5 = **G3K** + *Ref* + *Trans* + *Sym*

A system for Deontic logic is obtained by adding the geometric rule

$$\frac{xRy, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,{\scriptstyle Ser}$$

with the variable condition $y \notin \Gamma, \Delta$.

Directedness is another property that follows the pattern of a geometric implication, and it is converted into the rule

$$\frac{yRu, zRu, xRy, xRz, \Gamma \Rightarrow \Delta}{xRy, xRz, \Gamma \Rightarrow \Delta}\,{\scriptstyle Dir}$$

with the variable condition $u \notin xRy, xRz, \Gamma, \Delta$.

The treatment of a modal logic with a frame property not expressible as a first-order sentence, namely provability logic, is postponed to the following section.

2.3 Structural properties

Let **G3K*** be any extension of **G3K** with rules for the accessibility relation following the regular rule scheme or the more general geometric rule scheme. The following properties of any system belonging to the class **G3K*** can all be established uniformly. We refer to Negri (2005) for the details.

Lemma 2.1 *Sequents of the form*

$$x : A, \Gamma \Rightarrow \Delta, x : A$$

with A an arbitrary modal formula (not just atomic), are derivable in **G3K***.

In order to prove the correspondence between our systems and their Hilbert-style presentations it is necessary to show that the characteristic axioms are derivable and the systems closed under the rules of necessitation and modus ponens.

Lemma 2.2 *For arbitrary A and B, the sequent*

$$\Rightarrow x : \Box(A \supset B) \supset (\Box A \supset \Box B)$$

is derivable in **G3K***.

The rule of necessitation

$$\frac{\Rightarrow x : A}{\Rightarrow x : \Box A}$$

is a context-dependent rule, as it requires both the antecedent and succedent contexts to be empty. As an explicit rule it would destroy the flexibility of the systems in the permutations needed to prove cut elimination; However, we do not need to add any such rule because we can show that it is admissible. In order to prove this we exploit the first-order features of the system in proving a lemma about substitution.

Substitution of labels is defined in the obvious way as follows for relational atoms and labelled formulas:

$$\begin{array}{rcl} xRy(z/w) & \equiv & xRy \text{ if } w \neq x \text{ and } w \neq y \\ xRy(z/x) & \equiv & zRy \text{ if } x \neq y \\ xRy(z/y) & \equiv & xRz \text{ if } x \neq y \\ xRx(z/x) & \equiv & zRz \\ x : A(z/y) & \equiv & x : A \text{ if } y \neq x \\ x : A(z/x) & \equiv & z : A \end{array}$$

and is extended to multisets componentwise. We have

Lemma 2.3 *If $\Gamma \Rightarrow \Delta$ is derivable in* **G3K***, *then $\Gamma(y/x) \Rightarrow \Delta(y/x)$ is also derivable, with the same derivation height.*

An immediate consequence is

Corollary 2.4 *The necessitation rule is admissible in* **G3K***.

We also obtain a desirable property of a sequent calculus, namely:

Proposition 2.5 *All the rules of* **G3K*** *are height-preserving invertible.*

Finally, we have:

Theorem 2.6 *All the structural rules–weakening, contraction, and cut–are admissible in the system* **G3K***.

2.4 Equality and undefinability

The syntax for system **G3K*** can be extended with equality. The treatment of equality as a left rule system, following Negri and von Plato (2001, Section 6.5), is easily implemented in the context of labelled calculi. We shall not give here the details, that can be found in Negri (2005, Section 7), but just observe by way of an example that the modal axiom

$$\Diamond(A\&\Box B) \supset \Box(A \vee \Diamond A \vee B)$$

corresponding to the frame property

$$\forall xyz(xRy \,\&\, xRz \supset z = y \vee zRy \vee yRz)$$

converts to the rule

$$\frac{z = y, xRy, xRz, \Gamma \Rightarrow \Delta \quad zRy, xRy, xRz, \Gamma \Rightarrow \Delta \quad yRz, xRy, xRz, \Gamma \Rightarrow \Delta}{xRy, xRz, \Gamma \Rightarrow \Delta}$$

The corresponding sequent system is obtained by adding the above rule to the system **G3K** augmented with the rules for equality. All the structural properties of the resulting system hold as a consequence of the general results.

The use of proof systems that unify the syntax and semantics of modal logic permits to obtain very simple proofs of negative results in correspondence theory. These results state that certain frame properties (such as irreflexivity and intransitivity) do not have any modal correspondent. The usual proofs are based on model extension methods: in order to prove that a frame property is not modally definable it is shown that the corresponding class of frames is not closed under the constructions of disjoint union, generated subframes, bounded morphic images, and ultrafilter extensions (cf. Blackburn, de Rijke, and Venema 2001, section 3.3; see also Van Benthem 1984). In our systems, the lack of a modal correspondent is an immediate consequence of a conservativity theorem. Consider, for instance, the frame property of irreflexivity $\forall x \sim xRx$ that corresponds to the rule

$$\frac{}{xRx, \Gamma \Rightarrow \Delta}\ \textit{Irref}$$

By a straightforward proof analysis (see Theorem 7.1 of Negri 2005 for the complete, five-line proof) we observe

Theorem 2.7 *The system* **G3K**+Irref *is conservative over* **G3K**.

It follows that the property of irreflexivity does not have any modal correspondent, because, if it had, there would be some formula that is provable in the extension **G3K**+*Irref* but not in **G3K**.

The result is easily generalized to any property, generalizing intransitivity, of the form $\sim (P_1\& \ldots \&P_n)$ where P_i is x_iRy_i and for some i, j, $y_i = y_j$. A similar result holds for $\exists x\ xRx$ and $\forall x \exists y(xRy \,\&\, yRy)$.

3 Provability logic

After Solovay's landmark paper (1976) that presented axiomatically GL as the logic of arithmetic provability and characterized its Kripke models as the transitive and Noetherian frames, a lot of interest has been directed to the search of an adequate, cut-free sequent system for GL.

Semantic proofs of closure of a certain system with respect to cut, based on completeness arguments, were presented in Sambin and Valentini (1982) and in Avron (1984). Syntactic proofs, aimed at providing explicit proof transformations that would describe a procedure of cut elimination, were proposed by Leivant (1981), Valentini (1983), and Borga (1983). Valentini (1983) gave a counterexample to the proof presented by Levaint. More recently Moen (2003) observed that the proof by Valentini assumes as a starting point a reduction of a cut on $\Box A$ to a detour cut, which is not fully justified in a calculus with explicit contraction. However, in all the proofs given in the 1980s (and also in more recent proposals, see Sasaki 2001) calculi with *contexts-as-sets* have been used. There are good reasons for objecting to such an approach to sequent calculus that would deserve a more thorough discussion, but we shall not go into this issue here.

Another problematic aspect of the proposed calculi for provability logic is a so-called lack of harmony[3]: In fact, there is only one rule (both left and right) for $\Box$

$$\frac{\Box\Gamma, \Gamma, \Box A \Rightarrow A}{\Box\Gamma, \Gamma' \Rightarrow \Delta, \Box A}$$

that does not respect any of the design requirements of separation, symmetry, uniqueness recalled in the Introduction.

Here we shall show how a calculus with admissible contraction for sequents labelled by possible worlds, with harmonic, semantically originated left and right rules for $\Box$, permits a transparent proof of cut elimination.

In the Kripke frames for provability logic the accessibility relation R is irreflexive, transitive, and Noetherian (every R-chain eventually becomes stationary). Equivalently, we can say that R is transitive and all R-chains are finite. Clearly, this characterizing frame condition is not first order, so the method of universal/geometric extensions exploited in Section 2 cannot be applied directly. However, the condition can be internalized in the explanation of the meaning of the modality as follows:

Lemma 3.1 *In irreflexive, transitive, and Noetherian Kripke frames*

$$x \Vdash \Box A \ \ \textit{iff for all } y,\ xRy \textit{ and } y \Vdash \Box A \textit{ implies } y \Vdash A$$

Proof: See Negri (2005).

The right-to-left direction of the implication stated above gives the right rule for $\Box$

$$\frac{xRy, y : \Box A, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}\ {\scriptstyle R\Box\text{-}L}$$

[3]Cf. Read 2005 for a discussion of this notion in the context of modal logic and von Plato 2005 for an application of (harmonic) general elimination rules to the solution of the problem of normal form for S4.

with the variable condition that y is not in the conclusion. The left-to-right direction gives the left rule

$$\frac{x : \Box A, xRy, \Gamma \Rightarrow \Delta, y : \Box A \quad y : A, x : \Box A, xRy, \Gamma \Rightarrow \Delta}{x : \Box A, xRy, \Gamma \Rightarrow \Delta} {\scriptstyle L\Box\text{-}L}$$

The systems **G3GL** is thus determined as follows:

Initial sequents:

$x : P, \Gamma \Rightarrow \Delta, x : P \qquad x : \Box A, \Gamma \Rightarrow \Delta, x : \Box A$

Logical rules:

As in **G3K** for &, $\vee$, $\supset$, $\bot$; $L\Box$-L, $R\Box$-L

Mathematical rules: *Ref, Trans*

Table 2. The sequent calculus G3GL

All the properties that have been established for **G3K*** hold for **G3GL**, namely:

Theorem 3.2 *1. The axioms of the Hilbert-type system for GL are derivable in* **G3GL**.

2. The rules of substitution, weakening, and necessitation are height-preserving admissible in **G3GL**.

3. The rules of contraction are admissible in **G3GL**. *Elimination of contraction does not introduce new worlds in the derivation.*

The last item in the theorem introduces a new notion that was not needed before, namely the notion of *range* of a label in a derivation. Roughly, the range of a label x in a derivation $\mathcal{D}$ is the set of labels belonging to the transitive closure of all the relations xRy occurring in the left hand side of sequents of $\mathcal{D}$. The need for this new notion becomes clear from the proof of cut elimination for **G3GL**. We shall not give all the details here (for which we refer to Negri 2005), but just focus on the main ideas.

A typical procedure of cut elimination for **G3**-like systems considers topmost cuts and performs reductions that either decrease the *height* of one of the two premisses of cut (for permutation cuts, that is, cuts in which the cut formula is not principal in at least one of the premisses) or the *size* of the cut formula (for detour, or principal, cuts, that is cuts in which the formula principal in both premisses). The reductions are repeated until cuts reach initial sequents and disappear. This procedure does not work for **G3GL** in the case of detour cuts on $x : \Box A$. Consider a principal cut on $x : \Box A$

$$\frac{\dfrac{xRy, y : \Box A, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}{\scriptstyle R\Box\text{-}L} \quad \dfrac{xRz, x : \Box A, \Gamma' \Rightarrow \Delta', z : \Box A \quad z : A, xRz, x : \Box A, \Gamma' \Rightarrow \Delta'}{xRz, x : \Box A, \Gamma' \Rightarrow \Delta'}{\scriptstyle L\Box\text{-}L}}{xRz, \Gamma', \Gamma \Rightarrow \Delta, \Delta'}{\scriptstyle Cut}$$

this is transformed into four cuts as follows

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ \vdots \\ xRz, xRz, \Gamma', \Gamma, \Gamma \Rightarrow \Delta, \Delta, \Delta', z : A\end{matrix} \quad \begin{matrix}\mathcal{D}_2 \\ \vdots \\ xRz, z : A, \Gamma', \Gamma \Rightarrow \Delta, \Delta'\end{matrix}}{xRz, xRz, xRz, \Gamma', \Gamma', \Gamma, \Gamma, \Gamma \Rightarrow \Delta, \Delta, \Delta, \Delta', \Delta'}\ \textit{Cut}}{xRz, \Gamma', \Gamma \Rightarrow \Delta, \Delta'}\ \textit{Ctr*}$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are the following two derivations

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x : \Box A \quad xRz, x : \Box A, \Gamma', \Rightarrow \Delta', z : \Box A}{xRz, \Gamma', \Gamma \Rightarrow \Delta, \Delta', z : \Box A}\ \textit{Cut} \qquad xRz, z : \Box A, \Gamma \Rightarrow \Delta, z : A}{xRz, xRz, \Gamma, \Gamma', \Gamma \Rightarrow \Delta, \Delta', \Delta, z : A}\ \textit{Cut}$$

$$\dfrac{\Gamma \Rightarrow \Delta, x : \Box A \quad xRz, x : \Box A, z : A, \Gamma' \Rightarrow \Delta'}{xRz, z : A, \Gamma', \Gamma \Rightarrow \Delta, \Delta'}\ \textit{Cut}$$

Observe that the cuts on $x : \Box A$ and on $z : A$ are all reduced according to the standard procedure, whereas the cut on $z : \Box A$ is not, because neither the complexity of the cut formula nor the height of the cut is reduced.

However, if the range of z in the new derivation is strictly smaller than the range of x in the original derivation, then we have for all the cuts in the transformed derivation a reduced inductive parameter given by the triple consisting of the complexity of the cut formula, the range of its label, and the height of the cut, ordered lexicographically.

In order to prove the reduction in range, two extra assumptions are needed, namely, that there be no cuts with xRx or $xRx_1, \ldots, x_nRx$ in the antecedents of their conclusions and that eigenvariables be *pure*, i.e., appear only in the subtree above the step introducing them. The first condition is met by observing that if there are cuts of that form, they are eliminated using *Irref* and *Trans*, the second by a fresh renaming of eigenvariables. It then follows that no x can be in the range of itself, that if y is in the range of x then the range of y is properly included in the range of x, and that if y, z are in the range of x and y is an eigenvariable, then the union of the range of y and the range of z is properly included in the range of x.

We conclude with observing that the Löb axiom can be derived in this system, and Gödel's second incompleteness theorem follows as an immediate consequence of cut elimination. See Negri (2005).

4. Intermediate logics

It is well known that intuitionistic logic can be embedded into the classical modal logic S4, and actually all the intermediate logics between intuitionistic and classical logic can be embedded into the intermediate modal logics between S4 and S5. The analogy between these two families of logics is best seen at the level of their Kripke semantics. The explanation of the meaning of implication in intuitionistic logic reflects the explanation of the modality in K. As for normal modal logics, we can internalize the inductive definition of validity in a Kripke frame for obtaining uniform **G3**-style sequent calculi for intermediate logics. The accessibility relation for intuitionistic logic is a *partial order.* By requiring additional properties, logics above intuitionistic logic are obtained. We observe that all the properties of the accessibility relation characterizing the *interpolable* propositional logics fall under the geometric rule scheme. By applying the results on geometric extensions we can therefore obtain complete calculi with good structural properties. In addition, the uniformity in the syntax allows immediate proofs of the faithfulness of the embeddings. The details of the proofs can be found in Dyckhoff and Negri (2005).

From the inductive definition of validity of implication in a Kripke frame,

$$x \Vdash A \supset B \;\; \textit{iff} \;\; \textit{for all} \;\; y, \; x \leqslant y \; \textit{and} \; y \Vdash A \; \textit{implies} \;\; y \Vdash A$$

we obtain the left and right rules for intuitionistic implication. Arbitrariness in y in the right rule is again expressed by a variable condition.

The rules for the other connectives are exactly as the rules in **G3K**. The initial sequents of **G3K** are instead modified in order to guarantee the property of monotonicity of forcing. In compliance with the features of the **G3**-style calculi, it is enough to have monotonicity with respect to atomic formulas to have full monotonicity admissible. The mathematical rules for the accessibility relation $\leqslant$ are the rules *Ref* and *Trans*, expressing that $\leqslant$ is a partial order. We have thus determined the following system **G3I** for intuitionistic propositional logic:

Initial sequents:

$x \leqslant y, x : P, \Gamma \Rightarrow \Delta, y : P$

Logical rules:

As in **G3K** for &, $\vee$, $\bot$;

$$\frac{x \leqslant y, x : A \supset B, \Gamma \Rightarrow y : A, \Delta, \quad x \leqslant y, x : A \supset B, y : B, \Gamma \Rightarrow \Delta}{x \leqslant y, x : A \supset B, \Gamma \Rightarrow \Delta} \, L\supset$$

$$\frac{x \leqslant y, y : A, \Gamma \Rightarrow \Delta, y : B}{\Gamma \Rightarrow \Delta, x : A \supset B} \, R\supset$$

Order rules:

$$\frac{x \leqslant x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \, \textit{Ref} \qquad\qquad \frac{x \leqslant z, x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}{x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta} \, \textit{Trans}$$

Table 3. The sequent calculus G3I

Let **G3I*** be any extension of **G3I** with rules following the geometric rule scheme. Following the method presented in Section 3, the structural properties of **G3I*** are proved uniformly for any extension. We summarize the results in the following

Theorem 3.3 *1.* **G3I*** $\vdash x \leqslant y, x : A, \Gamma \Rightarrow \Delta, y : A$ *(monotonicity).*

2. *If* **G3I*** $\vdash_n \Gamma \Rightarrow \Delta$, *then* **G3I*** $\vdash_n \Gamma(y/x) \Rightarrow \Delta(y/x)$ *(height-preserving substitution).*

3. *Weakening and contraction are height-preserving admissible.*

4. *All the rules of* **G3I*** *are height-preserving invertible.*

5. *Cut is admissible.*

We obtain at once that each of the seven interpolable intermediate logics (cf. Maksimova 1979, Chagrov and Zakharyaschev 1997) belong to the class **G3I***: The point is simply that all these have frame conditions expressible geometrically.

1. **Int** *Intuitionistic Logic*: as already built in above, the accessibility relation $\leqslant$ is reflexive and transitive, i.e.

$$\forall x(x \leqslant x) \text{ and } \forall xyz(x \leqslant y \,\&\, y \leqslant z \;\supset\; x \leqslant z).$$

2. **Jan** *Jankov-De Morgan Logic* (cf. Jankov 1968): The relation $\leqslant$ is *directed* or *convergent*, i.e.,

$$\forall xyz(x \leqslant y \,\&\, x \leqslant z \;\supset\; \exists w(y \leqslant w \,\&\, z \leqslant w)).$$

This logic, also known as **KC** (cf. Chagrov and Zakharyaschev 1997) and as the "logic of weak excluded middle," is axiomatised by either $\sim A \vee \sim\sim A$ or $\sim(A \& B) \supset \sim A \vee \sim B$.

3. **GD** *Gödel-Dummett Logic*: The accessibility relation is *linear*, i.e.,

$$\forall xy(x \leqslant y \vee y \leqslant x).$$

This logic (also known as **LC**, for "linear chains") has as characteristic axiom scheme either $(A \supset B) \vee (B \supset A)$ or $(A \supset B) \supset C) \supset (((B \supset A) \supset C) \supset C)$.

4. $\mathbf{Bd}_2$: The accessibility relation has depth at most 2, i.e., it satisfies

$$\forall xyz(x \leqslant y \,\&\, y \leqslant z \;\supset\; z \leqslant y \vee y \leqslant x).$$

This logic is axiomatised by for example $A \vee (A \supset (B \vee \sim B))$.

5. **Sm**: Smetanich logic, also known as $\mathbf{LC}_2$ (cf. Chagrov and Zakharyaschev 1997) or the "logic of here and there." The accessibility relation is linear and has depth at most 2, i.e., the conditions for **GD** and $\mathbf{Bd}_2$. It is axiomatised by the **GD** axiom plus the $\mathbf{Bd}_2$ axiom, or, equivalently, $(\sim B \supset A) \supset (((A \supset B) \supset A) \supset A)$.

6. **GSc**: The accessibility relation has depth at most 2 and at most 2 final elements, i.e., the following holds in addition to the frame condition for $\mathbf{Bd}_2$:

$$\forall xyzw(x \leqslant y \,\&\, x \leqslant z \,\&\, x \leqslant w \supset w \leqslant y \vee w \leqslant z)$$

The logic is axiomatized by $(A \supset B) \vee (B \supset A) \vee ((A \supset \sim B) \& (\sim B \supset A))$ and $A \vee (A \supset B \vee \sim B)$.

7. **Cl** *Classical logic*: The accessibility relation is *symmetric*, i.e.,

$$\forall xy(x \leqslant y \supset y \leqslant x).$$

The logic is axiomatised by $A \vee \sim A$ or by $\sim\sim A \supset A$.

There are the following containments between these logics: **Int** $\subset$ **Jan** $\subset$ **GD** $\subset$ **Sm**, **Int** $\subset$ $\mathbf{BD}_2$ $\subset$ **GSc** $\subset$ **Sm** and **Sm** $\subset$ **Cl**.

We recall the standard translation $^\Box$ of **Int** into **S4**, a variant (cf. Troelstra and Schwichtenberg 2000) of the translation given in Gödel (1933):

$$\begin{array}{rcl} P^\Box & \equiv & \Box P \\ \bot^\Box & \equiv & \bot \\ (A \supset B)^\Box & \equiv & \Box(A^\Box \supset B^\Box) \\ (A\&B)^\Box & \equiv & A^\Box \& B^\Box \\ (A \vee B)^\Box & \equiv & A^\Box \vee B^\Box \\ (A_1, \ldots, A_n)^\Box & \equiv & A_1^\Box, \ldots, A_n^\Box \end{array}$$

We obtain a uniform proof of the faithful embeddings of intermediate logics between **Int** and **Cl** and intermediate modal logics between **S4** and **S5**.

Theorem 3.4 *Given an extension* **G3I*** *of* **G3I** *with rules for* $\leqslant$ *, let* **G3S4*** *be the corresponding extension of* **G3S4**. *We then have* **G3I*** $\vdash \Gamma \Rightarrow \Delta$ *iff* **G3S4*** $\vdash \Gamma^\Box \Rightarrow \Delta^\Box$.

4 Substructural logics

Among the logics that can be characterized in terms of a relational semantics is the family of relevant, and, in more generality, substructural logics. Here we shall show how our method can be successfully applied for obtaining sequent calculi for these logics. For a general background, history, motivations, applications, and references to the vast literature on the field we refer to the survey by Dunn and Restall (2002) and to the two recent monographs Restall (2000) and Mares (2004).

Our starting point for the development of uniform calculi for substructural logics is given by the *Routley-Meyer relational semantics*. This semantics is a generalization of the standard relational semantics for intuitionistic and modal logic: Instead of a binary accessibility relation, we have a ternary relation R on a set of worlds W. A distinguished element 0 of W defines a projection of R, namely $a \leqslant b \equiv R0ab$ that turns out to be a partial order.

For basic relevant logic, R satifies the properties:

$$\begin{array}{ll} Ref & R0xx \\ Mon_1 & R0x'x \,\&\, Rxyz \supset Rx'yz \\ Mon_2 & R0y'y \,\&\, Rxyz \supset Rxy'z \\ Mon_3 & R0z'z \,\&\, Rxyz' \supset Rxyz \end{array}$$

Following the method recalled in Section 2, all the above properties can be given as rules for the accessibility relation to be added to an appropriate labelled calculus.

As for intuitionistic logic, the only connective with a non-trivial semantics is implication, with validity defined inductively by

$$x \Vdash A \supset B \equiv \textit{for all } y, z,\ Rxyz \textit{ and } y \Vdash A \textit{ implies } z \Vdash A$$

This semantic explanation justifies the rules

$$\frac{Rxyz, x : A \supset B, \Gamma \Rightarrow \Delta, y : A \quad Rxyz, x : A \supset B, z : B, \Gamma \Rightarrow \Delta}{Rxyz, x : A \supset B, \Gamma \Rightarrow \Delta}\ L\supset$$

$$\frac{Rxyz, y : A, \Gamma \Rightarrow \Delta, z : B}{\Gamma \Rightarrow \Delta, x : A \supset B}\ R\supset$$

where the latter has the variable condition $y, z \notin \Gamma, \Delta, x : A \supset B$.

A cut-free complete sequent calculus for basic relevance logic is obtained, with initial sequents given by

$$R0xy, x : P, \Gamma \Rightarrow \Delta, y : P$$

The logical rules for implication are as above, the rules for & and $\vee$ as in **G3K** and **G3I**, and the mathematical rules are given by the monotonicity properties of R.

Besides cut, also the other structural rules (weakening and contraction) are admissible. We observe that this does not contradict the substructural nature of these logics. These admissible rules are what could be called (borrowing terminology from hypersequents) *external* structural rules. In fact, we can easily verify that the axiom $A \supset (B \supset A)$ that corresponds to weakening is not derivable in the above system despite the admissibility of weakening.

Logics extending the basic relevant logic can be obtained by assuming additional properties for the accessibility relation. We recall some correspondences between axioms and frame properties for a variety of relevant logics. First, define $R^2abcd \equiv R^2(ab)cd \equiv \exists x(Rabx \& Rxcd)$ and $R^2a(bc)d \equiv \exists x(Raxd \& Rbcx)$

Axiom	Frame property
$A \& (A \supset B) \supset B$	$Raaa$ or $R0ab \supset Raab$ idempotence
$(A \supset B) \& (B \supset C) \supset (A \supset C)$	$Rabc \supset R^2a(ab)c$ transitivity
$(A \supset B) \supset ((B \supset C) \supset (A \supset C))$	$R^2abcd \supset R^2b(ac)d$ suffixing
$(A \supset B) \supset ((C \supset A) \supset (C \supset B))$	$R^2abcd \supset R^2a(bc)d$ associativity
$(A \supset (A \supset B)) \supset (A \supset B)$	$Rabc \supset R^2abbc$ contraction
$((A \supset A) \supset B) \supset B$	$Ra0a$ specialized assertion
$A \supset ((A \supset B) \supset B)$	$Rabc \supset Rbac$ commutativity
$A \supset (A \supset A)$	$Rabc \supset (R0ac \vee R0bc)$ mingle

Observe that all the properties of R are geometric. As a consequence, the basic calculus can be extended by rules representing the frame properties and its structural properties follow from the general result on extensions with the geometric rule-scheme.

A similar approach to substructural logics is presented in Viganó (2000). The main difference with respect to our method consists in the use of a basic sequent calculus with explicit structural rules and in a presentation of mathematical rules for the accessibility relation in the form of rules with a single conclusion (Horn clauses) that cannot be extended beyond Harrop theories (theories that do not have disjunctions in positive parts of axioms). This excludes, for instance, the treatment of the last frame property in the above table.

5 Concluding remarks

We have presented a uniform way of generating sequent calculi with good structural properties for a variety of non-classical logics, including most standard normal modal logics, provability logic, intermediate logics, and substructural logics. The calculi are all in the form of Gentzen sequent calculi, with an extra syntactic element given by the labels and rules governing them that formally encode the Kripke semantics into the sequent systems. We can now relate the proposed solution to the requirements on good sequent systems that we quoted in the Introduction. The first property, separation, is satisfied as each connective/modality has rules given through its meaning explanation, independent of any other connective. Symmetry also clearly holds, and, in particular for the case of provability logic, it simplifies previous proof of cut elimination that were based on a non-symmetric rule for the modality. As for the third property, we observe that there are rules in which the connective/modality appears also in the premisses. This is an unavoidable feature of certain calculi, such as **G3i**, and it is needed for obtaining admissibility of contraction. Some extra care is therefore needed in proofs of termination of proof search. The fourth and fifth requirements are clearly satisfied. As for the sixth, we observe that our calculi satisfy a similar requirement: We have a core basic logical calculus and different systems are obtained by modifying only the mathematical rules added to the ground calculus, that is, the rules for the accessibility relation. The structural rules, instead, are absent from all our calculi, because they are admissible. In particular, cut is admissible, so also property 7 is satisfied. As for property 8, the subformula property, we do not have a priori a full subformula property. In rules for frame properties there are relational atoms that disappear from the premisses to the conclusion. However, we can prove in most cases a suitable version of the subformula property, adequate for proving syntactic decidability, as consequence of the structural properties of the calculi. Our calculi all clearly satisfy a *weak subformula property*, that is, all formulas in a derivation are either subformulas of (formulas in) the endsequent or atomic formulas of the form xRy. By considering minimal derivations, that is, derivations in which shortenings are not possible, the weak subformula property can be strengthened by restricting the labels that can appear in the relational atoms to those in the conclusion. The *subterm property* states that all terms (variables, worlds) in a derivation are either eigenvariables or terms (variables, worlds) in the conclusion. This property, together with height-preserving admissibility of contraction, ensures the consequences of the full subformula property and it has been used for establishing decidability through terminating proof search for logics extending basic modal logic in section 6 of Negri (2005). The same approach to proof-theoretic decidability can be extended to the other non-classical logics treated in this article.

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