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PROOF ANALYSIS IN MODAL LOGIC

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ABSTRACT. A general method for generating contraction- and cut-free sequent calculi for a large family of normal modal logics is presented. The method covers all modal logics characterized by Kripke frames determined by universal or geometric properties and it can be extended to treat also Gödel–Löb provability logic. The calculi provide direct decision methods through terminating proof search. Syntactic proofs of modal undefinability results are obtained in the form of conservativity theorems.

1. INTRODUCTION

The possibility of a systematic development of a proof theory of modal logic in terms of Gentzen sequent calculus has been looked at with overall skepticism. The question has been stepped over very often in the literature by choosing Hilbert type proof systems.

The difficulties in finding cut-free sequent systems are encountered already for quite simple modal systems such as S5. Partial solutions have been proposed by relaxing on the usual requirements on sequent formulations, such as the subformula property (in Mints, 1968) or locality of the rules (in Braüner, 2000). An indirect solution, characterizing S5 derivability of A as derivability of $\Box A$ in a system without cut, was given in Shvarts (1989).

Standard sequent systems for modal logic typically fail to be modular and do not satisfy most of the properties usually demanded on sequent calculus (as observed in Wansing, 2002). These shortcomings have led to look for generalizations of standard sequent systems, such as higher-level sequents, hypersequents, and display logic (cf. the above-cited survey).

In recent years an approach based on the internalization of the Kripke semantics of modal logic into its inference systems has gained prominence. This idea, with precursors as far as in Kanger (1957), has taken several forms. The accessibility relation of Kripke frames has, either implicitly or explicitly, become part of the syntax in tableau systems (Fitting, 1983; Catach, 1991; Nerode, 1991; Goré, 1998), in natural deduction (Simpson, 1994; Basin, Matthews, and Viganò, 1998), in sequent calculus (Mints, 1997; Viganò, 2000; Kushida and Okada, 2003), in first-order encodings of modal logic (Ohlbach, 1993), and in hybrid logic (Blackburn, 2000).

In this work, we present a method for generating sequent calculi with excellent structural properties, with the accessibility relation as part of the syntax, for a large family of modal logics, including Gödel–Löb provability logic.

The modal systems we present are obtained by extending a basic modal sequent system with rules for the accessibility relation. The method is founded on a general result (Negri and von Plato, 1998) by which it is possible to extend contraction-free sequent calculi with rules for mathematical axioms in such a way that the resulting system maintains the good structural properties of the basic, purely logical, system. The method encompasses all universal axioms and is extended to cover also what are known as geometric implications (Negri, 2003).

Our calculi carefully avoid any use of context-dependent rules that usually make cut elimination problematic and impair modularity. The rule of necessitation need not be imposed but is shown admissible.

Validity of the structural properties, invertibility of all the rules, admissibility of substitution, contraction, and cut, are proved following the standard methods as for the **G3** sequent calculi (cf. Troelstra and Schwichtenberg, 2000; Negri and von Plato, 2001). We establish cut elimination in a direct way through an algorithm of proof transformation, whereas other approaches obtain only the weaker property of closure with respect to cut as a consequence of semantical completeness.

The modal logic of provability, also called Gödel–Löb provability logic (GL), is characterized by the frame condition that the accessibility relation is transitive and has no infinite chains. The latter is not a first-order property, and therefore the method of extensions with rules for the accessibility relation cannot be applied. We can, nevertheless, extend to GL all our results on sequent calculi with internalized Kripke semantics by a simple, semantically justified, modification of the rules for the necessity operator in the basic modal calculus. In this way we solve the long-standing open problem of finding a contraction- and cut-free sequent calculus for GL.

Our calculi, although not satisfying a full subformula property, enjoy a *subterm* property: all terms in minimal derivations are terms found in the endsequent. This property, together with height-preserving admissibility of contraction, makes our calculi suitable for proof search. In particular, decidability properties get established in the strong form of an effective bound on proof search.

In the final section, we show that a relation of equality can be added to the frame properties along with suitable replacement rules (following the

TABLE I

The sequent calculus **G3c**.

Initial sequents:

$P, \Gamma \Rightarrow \Delta, P$

Logical rules:

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A\&B, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\&} \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A\&B}\,{\scriptstyle R\&}$$

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\vee} \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\,{\scriptstyle R\vee}$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\supset} \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \supset B}\,{\scriptstyle R\supset}$$

$$\frac{}{\bot, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\bot}$$

$$\frac{A(t/x), \forall x A, \Gamma \Rightarrow \Delta}{\forall x A, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\forall} \qquad \frac{\Gamma \Rightarrow \Delta, A(y/x)}{\Gamma \Rightarrow \Delta, \forall x A}\,{\scriptstyle R\forall}$$

$$\frac{A(y/x), \Gamma \Rightarrow \Delta}{\exists x A, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\exists} \qquad \frac{\Gamma \Rightarrow \Delta, \exists x A, A(t/x)}{\Gamma \Rightarrow \Delta, \exists x A}\,{\scriptstyle R\exists}$$

treatment of equality in Negri and von Plato, 2001) while maintaining the structural properties of the resulting systems.

We conclude with an alternative approach to proofs of negative results in correspondence theory (Van Benthem, 1984; Blackburn, Rijke, and Venema, 2001). The lack of modal correspondents to certain frame properties, usually proved by not so straightforward model extension methods, is here an immediate consequence of a conservativity theorem.

2. PRELIMINARIES

We recall here the basic results on the method for generating extensions of sequent calculi with rules for axiomatic theories while preserving all the structural properties of the logical calculus. For historical background, proofs, and further details, the reader is referred to Negri and von Plato (1998, 2001) and Negri (2003).

The sequent calculus we shall be using here is the contraction- and cut-free sequent calculus **G3c**.

The structural rules of weakening, contraction, and cut are formulated as follows:

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} RW$$

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} LC \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A} RC$$

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} Cut$$

The system **G3c** has the remarkable structural properties that all its rules are invertible and all the structural rules are admissible. We recall that a rule is *admissible* in a given system if, whenever its premisses are derivable, then also its conclusion is derivable. Weakening and contraction are *height-preserving* admissible, that is, whenever their premiss is derivable with height bounded by n, also the conclusion is derivable, with the same bound on the derivation height (recall that the *height* of a derivation is its height as a tree, that is, the length of its longest branch). Moreover, the calculus enjoys height-preserving admissibility of substitution. Also invertibility of the rules of **G3c** is height-preserving. For a proof of these properties cf. Chapters 3 and 4 of Negri and von Plato (2001)

It was proved in Negri and von Plato (1998) that these remarkable structural properties of **G3c** are maintained in extensions of the logical calculus with suitably formulated *mathematical rules* representing axioms for specific theories. Universal axioms are first transformed, through the rules of **G3c**, into conjunctive normal form, that is conjunctions of formulas of the form $P_1 \& \cdots \& P_m \supset Q_1 \vee \cdots \vee Q_n$, where the consequent is $\bot$ if $n = 0$. Each conjunct is then converted into a rule, called *regular rule scheme*, of the form

$$\frac{Q_1, \Gamma \Rightarrow \Delta \quad \ldots \quad Q_n, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_m, \Gamma \Rightarrow \Delta} Reg$$

Two more details need to be added: (1) The formulas $P_1, \ldots, P_m$ in the antecedent of the conclusion of the scheme above have to be repeated in the antecedent of each of the premisses. (2) It can happen that instantiation of free parameters in atoms produces a duplication (two identical atoms in the conclusion of a rule instance), say $P_1, \ldots, P, P, \ldots, P_m, \Gamma \Rightarrow \Delta$. Then, by (1) each premiss has the duplication. We now require that the rule with the duplication P, P *contracted* into a single P is added to the system of rules (closure condition). For each axiom system, there is only a bounded number of possible cases of contracted rules to be added, very often none at all.

We have:

THEOREM 2.1. *The structural rules are admissible in extensions of* **G3c** *following the rule-scheme and satisfying the closure condition. Weakening and contraction are height-preserving admissible.*

By this method, all universal theories can be formulated as contraction and cut-free systems of sequent calculi. Examples and applications can be found in Chapter 6 of Negri and von Plato (2001).

Recently (Negri, 2003), the method has been extended to cover also geometric theories, that is, theories axiomatized by geometric implications.

We recall that a *geometric formula* is a formula not containing $\supset$ or $\forall$ and a *geometric implication* is a sentence of the form

$$\forall \overline{z}(A \supset B)$$

where A and B are geometric formulas. Geometric implications can be reduced to a normal form consisting of conjunctions of formulas, called *geometric axioms*, of the form

$$\forall \overline{z}(P_1 \& \cdots \& P_m \supset \exists \overline{x}(M_1 \vee \cdots \vee M_n))$$

where M_j is the conjunction of the atomic formulas $Q_{j_1}, \ldots, Q_{j_{k_j}}$. For simplicity we only deal with the case where the sequence $\overline{x}$ of bound variables has length 1, and distribute the existential quantifier over the disjunctions, as in $\exists x_1 M_1 \vee \cdots \vee \exists x_n M_n$.

The left rule scheme for geometric axioms takes the form

$$\frac{\overline{Q}_1(y_1/x_1), \overline{P}, \Gamma \Rightarrow \Delta \quad \ldots \quad \overline{Q}_n(y_n/x_n), \overline{P}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta}\ GRS$$

where $\overline{Q}_j$ and $\overline{P}$ indicate the multisets of atomic formulas $Q_{j_1}, \ldots, Q_{j_{k_j}}$ and $P_1, \ldots, P_m$, respectively, and the eigenvariables $y_1, \ldots, y_n$ of the premisses satisfy the condition of not having free occurrences in the conclusion of the scheme.

As in the rule scheme for universal axioms, the repetition of the principal atom in the premisses and satisfaction of the closure condition are required in order to satisfy height-preserving admissibility of contraction.

We recall from Negri (2003, Theorems 2, 4, 5):

TABLE II

The system **G3K**.

Initial sequents:

$x : P, \Gamma \Rightarrow \Delta, x : P \qquad xRy, \Gamma \Rightarrow \Delta, xRy$

Propositional rules:

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A\&B, \Gamma \Rightarrow \Delta}\,L\& \qquad \frac{\Gamma \Rightarrow \Delta, x : A \quad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A\&B}\,R\&$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \quad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta}\,L\vee \qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}\,R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad x : B, \Gamma \Rightarrow \Delta}{x : A \supset B, \Gamma \Rightarrow \Delta}\,L\supset \qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \supset B}\,R\supset$$

$$\frac{}{x : \bot, \Gamma \Rightarrow \Delta}\,L\bot$$

Modal rules:

$$\frac{y : A, x : \Box A, xRy, \Gamma \Rightarrow \Delta}{x : \Box A, xRy, \Gamma \Rightarrow \Delta}\,L\Box \qquad \frac{xRy, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}\,R\Box$$

$$\frac{xRy, y : A, \Gamma \Rightarrow \Delta}{x : \Diamond A, \Gamma \Rightarrow \Delta}\,L\Diamond \qquad \frac{xRy, \Gamma \Rightarrow \Delta, x : \Diamond A, y : A}{xRy, \Gamma \Rightarrow \Delta, x : \Diamond A}\,R\Diamond$$

Rules $R\Box$ and $L\Diamond$ have the condition that y is not in the conclusion.

THEOREM 2.2. *The structural rules are admissible in extensions of* **G3c** *following the geometric rule-scheme and satisfying the closure condition. Weakening and contraction are height-preserving admissible.*

3. SEQUENT CALCULI FOR MODAL LOGIC

In this section we shall present a G3-style system for the basic modal logic K and a general method for extending the system to cover a wide range of modal logics: These are all the modal logics characterized by frame properties expressible by means of universal axioms, and more generally by means of geometric implications. Admissibility of the structural rules will be proved in a uniform way for all such systems.

The method for generating the basic sequent calculus and its extensions is based on the internalization of the Kripke semantics. The rules for the modal operators of necessity and possibility are obtained directly from their semantical explanation in terms of Kripke frames.

Instead of the forcing relation $x \Vdash A$ of Kripke models, we have an internal relation written $x : A$. The accessibility relation is denoted as usual by R. In our calculus, sequents are expressions of the form $\Gamma \Rightarrow \Delta$ where Γ and Δ are multisets of relational atoms xRy and labelled formulas $x : A$, with x, y ranging in a set W and with A any formula in the language of propositional logic extended with the modal operators of necessity and possibility, $\Box$ and $\Diamond$. Relational atoms will sometimes be simply called atoms, and labels will also be called worlds, variables, or prefixes.

The systems of sequent calculi for the modal logics that will be considered are all obtained as extensions of the basic sequent calculus for the modal logic K.

In the first initial sequent, P is an arbitrary atomic formula. Observe that no rule removes an atom of the form xRy from the right-hand side of sequents, and such atoms are never active in the logical rules. Moreover, the modal axioms corresponding to the properties of the accessibility relation are derived from their rule presentations alone. As a consequence, initial sequents of the form $xRy, \Gamma \Rightarrow \Delta, xRy$ are needed only for deriving properties of the accessibility relation, namely, the axioms corresponding to the rules for R given below. Thus such initial sequents can as well be left out from the calculus without impairing completeness of the system.

All the rules of the calculus are obtained from the inductive definition of validity in a Kripke frame. For the propositional part this is straightforward. From the semantical explanation of the modality $\Box$,

$$x \Vdash \Box A \text{ iff for all } y,\ xRy \text{ implies } y \Vdash A$$

we arrive at the rule: If $y : A$ can be derived for an arbitrary y accessible from x, then $x : \Box A$ can be derived. In the presence of contexts, this gives rule $R\Box$ above, in which the arbitrariness of y is expressed by the variable condition that y is not (free) in Γ, Δ. This restriction is identical to the variable restriction in rule $R\forall$ of first-order sequent calculi. Rule $L\Box$ expresses the other side of the equivalence, namely that if $x : \Box A$ and y is accessible from x, then $y : A$. The principal formula $x : \Box A$ is repeated in the premiss of the rule in order to make the rule invertible. This is analogous to the repetition of $\forall x A$ in the premiss of rule $L\forall$ in **G3c**.

The rules for $\Diamond$ are obtained similarly from the semantical explanation

$$x \Vdash \Diamond A \text{ iff for some } y,\ xRy \text{ and } y \Vdash A$$

The left to right direction gives the rule $L\Diamond$. The converse direction gives the two-premiss rule

$$\frac{\Gamma \Rightarrow \Delta, xRy \qquad \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Diamond A}\, R\Diamond$$

TABLE III

Modal logics, axioms, and frame properties.

Name	Axiom	Frame property
T	$\Box A \supset A$	$\forall x xRx$ reflexivity
4	$\Box A \supset \Box\Box A$	$\forall xyz(xRy \& yRz \supset xRz)$ transitivity
5	$\Diamond A \supset \Box\Diamond A$	$\forall xyz(xRy \& xRz \supset yRz)$ euclideanness
B	$A \supset \Box\Diamond A$	$\forall xy(xRy \supset yRx)$ symmetry
3	$\Box(\Box A \supset B) \vee \Box(\Box B \supset A)$	$\forall xyz(xRy \& xRz \supset yRz \vee zRy)$ connectedness
D	$\Box A \supset \Diamond A$	$\forall x \exists y xRy$ seriality
2	$\Diamond\Box A \supset \Box\Diamond A$	$\forall xyz(xRy \& xRz \supset \exists w(yRw \& zRw))$ directedness
W	$\Box(\Box A \supset A) \supset \Box A$	no infinite R-chains + transitivity

which is turned into the equivalent one-premiss rule

$$\frac{xRy, \Gamma \Rightarrow \Delta, x : \Diamond A, y : A}{xRy, \Gamma \Rightarrow \Delta, x : \Diamond A} {\scriptstyle R\Diamond}$$

Dually to the rules for $\Box$, the left rule for $\Diamond$ has the variable restriction that y must not occur in the conclusion, and the right rule has the repetition of the principal formula in the premiss, as in rule $R\exists$ of **G3c**.

The modal logic K is characterized by arbitrary frames. Correspondingly, there are no rules for the accessibility relation. The sequent calculi for the modal logics T, K4, KB, S4, B, S5 are obtained by adding to **G3K** the rules expressing the properties of the accessibility relation characterizing their frames. The rules for the accessibility relation, given below, are in the form of "mathematical rules" and follow the regular rule scheme described in Negri and von Plato (1998, 2001) and recalled in Section 2.

Before presenting the rules, we give a table of some well-known modal logics with their characteristic axioms and frame properties.

The standard logics that can be obtained by the addition of any of the above axioms to K are usually denoted by K followed by the axioms' identifiers, but alternative nomenclatures are found in the literature, for instance T for KT, S4 for KT4, S5 for KT4B or equivalently KT45, Deontic T for KD, Deontic S5 for KD45, Brouwer system for KTB, ... Axiom 3 is known as Geach's axiom.

Observe that the frame properties in the first group (T, 4, 5, B, 3) are universal axioms, and those in the second group geometric implications, whereas the last one is not expressible as a first-order property. Our method for obtaining cut- and contraction-free sequent calculi will directly cover any modal logic with a Kripke frame characterized by universal axioms or geometric implications, but we shall see in Section 5 how the method can be extended to cover also the case of provability logic.

In order to illustrate the method, we focus first on the modal logics of the first group. By transforming axioms into rules, we obtain the following sequent calculus rules for R:

$$\frac{xRx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}{\scriptstyle Ref} \qquad \frac{xRz, xRy, yRz, \Gamma \Rightarrow \Delta}{xRy, yRz, \Gamma \Rightarrow \Delta}{\scriptstyle Trans}$$

$$\frac{yRx, xRy, \Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}{\scriptstyle Sym} \qquad \frac{yRz, xRy, xRz, \Gamma \Rightarrow \Delta}{xRy, xRz, \Gamma \Rightarrow \Delta}{\scriptstyle Eucl}$$

The corresponding Gentzen systems are obtained by adding combinations of the above rules to the basic modal sequent system **G3K**. Among the systems obtained are

G3T = **G3K** + *Ref*
G3K4 = **G3K** + *Trans*
G3KB = **G3K** + *Sym*
G3S4 = **G3K** + *Ref* + *Trans*
G3TB = **G3K** + *Ref* + *Sym*
G3S5 = **G3K** + *Ref* + *Trans* + *Sym*

Seriality is converted into a rule following the general pattern of the geometric rule scheme

$$\frac{xRy, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}{\scriptstyle Ser}$$

with the variable condition $y \notin \Gamma, \Delta$.

Directedness can be converted into the rule

$$\frac{yRu, zRu, xRy, xRz, \Gamma \Rightarrow \Delta}{xRy, xRz, \Gamma \Rightarrow \Delta}{\scriptstyle Dir}$$

with the condition that u is not in the conclusion.

Thus, a Gentzen system for Deontic logic is obtained by adding rule *Ser* to **G3K**. Similarly, a Gentzen system for the extension of K with axiom 2 is obtained with the addition of the rule *Dir*.

In analogy with the notation used in Negri von Plato (1998, 2001) for extensions of **G3c** with rules, we shall denote with **G3K*** any extension of

G3K with rules following the regular rule scheme or, more generally, the geometric rule scheme.

4. Admissibility of the Structural Rules

In this section we shall prove that all the structural rules – weakening, contraction, and cut – are admissible in the system **G3K** and all its extensions with rules for the accessibility relation R. We shall also prove that the characteristic axioms are derivable in each of the systems considered and that the necessitation rule is admissible, so that our systems are complete.

Some features of the calculi will be highlighted that will be crucial for proving decidability in Section 6. These features are the invertibility of all the rules and the height-preserving admissibility of contraction.

LEMMA 4.1. *Sequents of the form* $x : A, \Gamma \Rightarrow \Delta, x : A$, *with* A *an arbitrary modal formula, are derivable in* **G3K***.

Proof. By induction on A. □

LEMMA 4.2. *For arbitrary* A *and* B*, the sequent*

$$\Rightarrow x : \Box(A \supset B) \supset (\Box A \supset \Box B)$$

is derivable in **G3K***.

Proof. Apply root-first the rules of **G3K** and Lemma 4.1. □

In order to prove that the necessitation rule of the basic modal system K is admissible, we first need a substitution lemma. Although we are considering a propositional system, the use of possible worlds as explicit elements of the syntax creates a strong analogy to first-order logic. The substitution lemma is similar, both in the statement and in the proof, to the substitution lemma of the classical predicate calculus (Lemma 4.1.2 in Negri and von Plato, 2001).

We define substitution in the obvious way as follows:

$xRy(z/w) \equiv xRy$ if $w \neq x$ and $w \neq y$
$xRy(z/x) \equiv zRy$ if $x \neq y$
$xRy(z/y) \equiv xRz$ if $x \neq y$
$xRx(z/x) \equiv zRz$
$x : A(z/y) \equiv x : A$ if $y \neq x$
$x : A(z/x) \equiv z : A$

and extend the definition to multisets componentwise. We then have:

LEMMA 4.3. *If* $\Gamma \Rightarrow \Delta$ *is derivable in* **G3K***, *then* $\Gamma(y/x) \Rightarrow \Delta(y/x)$ *is also derivable, with the same derivation height.*

Proof. By induction on the height n of the derivation of $\Gamma \Rightarrow \Delta$.

If $n = 0$, and (y/x) is not a vacuous substitution, the sequent can either be an initial sequent of the form $x : P, \Gamma' \Rightarrow \Delta', x : P$ or of the form $xRy, \Gamma' \Rightarrow \Delta', xRy$, or a conclusion of $L\bot$, $\bot, \Gamma' \Rightarrow \Delta$. In each case $\Gamma(y/x) \Rightarrow \Delta(y/x)$ is either an initial sequent of the same form or a conclusion of $L\bot$.

Suppose $n > 0$, and consider the last rule applied in the derivation. If it is a propositional rule, apply the inductive hypothesis to the premiss(es) of the rule, and then the rule. Proceed similarly if the last rule is a modal rule without variable condition, i.e., $L\Box$ or $R\Diamond$. If the last rule is a modal rule with variable condition, observe that either the substitution is vacuous or x is not an eigenvariable of the rule. If the first case, the result of the substitution is identical to $\Gamma \Rightarrow \Delta$ and there is nothing to prove. In the second case, assume that neither y is an eigenvariable. We have, in case the last rule is $R\Box$ and $x : \Box A$ appears as principal, a derivation ending with

$$\dfrac{\begin{matrix}\vdots\\ xRz, \Gamma \Rightarrow \Delta', z : A\end{matrix}}{\Gamma \Rightarrow \Delta', x : \Box A}\,{\scriptstyle R\Box}$$

where $z \neq x$ and z is not in Γ, Δ. By applying the inductive hypothesis to the shorter derivation of the premiss, and $R\Box$, we obtain the derivation in n steps

$$\dfrac{\begin{matrix}\vdots\\ yRz, \Gamma(y/x) \Rightarrow \Delta'(y/x), z : A\end{matrix}}{\Gamma(y/x) \Rightarrow \Delta'(y/x), y : \Box A}\,{\scriptstyle R\Box}$$

If y is the eigenvariable, the derivation ends with

$$\dfrac{\begin{matrix}\vdots\\ xRy, \Gamma \Rightarrow \Delta', y : A\end{matrix}}{\Gamma \Rightarrow \Delta', x : \Box A}\,{\scriptstyle R\Box}$$

We first apply the inductive hypothesis in order to replace the eigenvariable y with a fresh variable w. By the variable condition the substitution does not affect Γ or Δ', and we obtain a derivation of height $n - 1$ of

$$xRw, \Gamma \Rightarrow \Delta', w : A$$

then we apply the inductive hypothesis to substitute x with y and conclude with $R\Box$ in n steps

$$\frac{\begin{array}{c}\vdots\\ yRw, \Gamma(y/x) \Rightarrow \Delta'(y/x), w : A\end{array}}{\Gamma(y/x) \Rightarrow \Delta'(y/x), y : \Box A}\ R\Box$$

If x is not the label of a formula principal in the rule, the proof does not present any significant difference, and the case of $L\Diamond$ is detailed similarly.

For extensions of **G3K** with regular rules for R, observe that the rules are schematic, thus closed under substitution. In other words, the induction proceeds as for the propositional rules.

For geometric extensions some care is needed in order to avoid a clash with the eigenvariables of the geometric rule scheme. Suppose the last rule in the derivation is one of the form

$$\frac{\overline{Q}_1(y_1/x_1), \overline{P}, \Gamma \Rightarrow \Delta \quad \dots \quad \overline{Q}_n(y_n/x_n), \overline{P}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta}\ GRS$$

If $y \neq y_i$ for all $i = 1, \dots, n$, apply the inductive hypothesis to each of the premisses to obtain derivations of

$$\overline{Q}_i(y_i/x_i)(y/x), \overline{P}(y/x), \Gamma(y/x) \Rightarrow \Delta(y/x)$$

and by application of the geometric rule scheme obtain

$$\overline{P}(y/x), \Gamma(y/x) \Rightarrow \Delta(y/x)$$

If $y = y_i$ for some i, we first replace the eigenvariable y_i with a fresh variable y_i' by inductive hypothesis applied to the i-th premiss of the rule. Then by inductive hypothesis applied to each of the new premisses, we perform the substitution y/x and obtain the conclusion by applying rule *GRS*. $\Box$

PROPOSITION 4.4. *The rules of weakening*

$$\frac{\Gamma \Rightarrow \Delta}{x : A, \Gamma \Rightarrow \Delta}\ LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x : A}\ RW$$

$$\frac{\Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}\ LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, xRy}\ RW$$

are height-preserving admissible in **G3K***.

Proof. Straightforward induction on the height of the derivation of the premiss, for the propositional rules and the modal and nonlogical rules without variable condition. In case the last step is a modal rule with variable condition, the substitution lemma is applied to the premisses of the rule in order to have a fresh eigenvariable not clashing with those in $x : A$ or xRy. The conclusion is then obtained by applying the inductive hypothesis and the modal rule. An identical procedure is applied if the last step is a geometric rule and $x : A$ or xRy contain some of its eigenvariables. □

We are now ready to prove admissibility of the necessitation rule:

PROPOSITION 4.5. *The necessitation rule*

$$\frac{\Rightarrow x : A}{\Rightarrow x : \Box A}$$

is admissible in **G3K***.

Proof. Suppose we have a derivation of $\Rightarrow x : A$. By the substitution lemma we obtain a derivation of $\Rightarrow y : A$ and, by admissibility of weakening, of $xRy \Rightarrow y : A$. By $R\Box$ we have $\Rightarrow x : \Box A$. □

Observe that having the necessitation rule admissible rather than as an explicit rule of the calculus is crucial for our development. A proper addition of a rule like the necessitation rule above (a context-dependent rule) usually destroys any possibility of attaining a satisfactory system of sequent calculus.

The standard properties of distribution of the modalities over the connectives and of the interdefinability of the modal operators are easily derivable in **G3K**:

PROPOSITION 4.6. *The following sequents are derivable in* **G3K**:

(1) $\Rightarrow x : \Box(A \supset B) \supset (\Diamond A \supset \Diamond B)$
(2) $\Rightarrow x : \sim\Diamond\bot$
(3) $\Rightarrow x : \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$
(4) $\Rightarrow x : (\Diamond A \supset \Box B) \supset \Box(A \supset B)$
(5) $\Rightarrow x : \Box A \supset \sim\Diamond\sim A$ *and* $\Rightarrow x : \sim\Diamond\sim A \supset \Box A$
(6) $\Rightarrow x : \Diamond A \supset \sim\Box\sim A$ *and* $\Rightarrow x : \sim\Box\sim A \supset \Diamond A$

Proof. By root-first proof search from the sequent to be derived. □

Next, we show that the characteristic axioms of the modal logics T, K4, B, S5 are derivable in the respective sequent calculi **G3T**, **G3K4**, **G3TB**, **G3S5**. The simple proofs of Lemmas 4.7–4.9 are left to the reader.

PROPOSITION 4.7. *The following sequents are derivable in* **G3K** + Ref (**G3T**):

(1) $\Rightarrow x : \Box A \supset A$
(2) $\Rightarrow x : A \supset \Diamond A$.

PROPOSITION 4.8. *The following sequents are derivable in* **G3K**+Trans (**G3K4**):

(1) $\Rightarrow x : \Box A \supset \Box\Box A$
(2) $\Rightarrow x : \Diamond\Diamond A \supset \Diamond A$.

PROPOSITION 4.9. *The following is derivable in* **G3K** + Sym (**G3KB**): $\Rightarrow x : A \supset \Box\Diamond A$.

PROPOSITION 4.10. *The sequent* $\Rightarrow x : \Box A \supset \Diamond A$ *is derivable in* **G3K** + Ser. *The sequent* $\Rightarrow x : \Diamond\Box A \supset \Box\Diamond A$ *is derivable in* **G3K** + Dir.

Proof. We have the inferences

$$\dfrac{\dfrac{\dfrac{\dfrac{y : A, xRy, x : \Box A \Rightarrow x : \Diamond A, y : A}{xRy, x : \Box A \Rightarrow x : \Diamond A, y : A}\,L\Box}{xRy, x : \Box A \Rightarrow x : \Diamond A}\,R\Diamond}{x : \Box A \Rightarrow x : \Diamond A}\,Ser}{\Rightarrow x : \Box A \supset \Diamond A}\,R\supset$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{xRy, xRz, yRu, zRu, u : A, z : \Box A \Rightarrow y : \Diamond A, u : A}{xRy, xRz, yRu, zRu, u : A, z : \Box A \Rightarrow y : \Diamond A}\,R\Diamond}{xRy, xRz, yRu, zRu, z : \Box A \Rightarrow y : \Diamond A}\,L\Box}{xRy, xRz, z : \Box A \Rightarrow y : \Diamond A}\,Dir}{xRy, x : \Diamond\Box A \Rightarrow y : \Diamond A}\,L\Diamond}{x : \Diamond\Box A \Rightarrow x : \Box\Diamond A}\,R\Box}{\Rightarrow x : \Diamond\Box A \supset \Box\Diamond A}\,R\supset$$

in which the topsequents are derivable by Lemma 4.1. □

In order to prove height-preserving admissibility of contraction we need to show height-preserving invertibility of the rules of the modal sequent calculi.

PROPOSITION 4.11. *All the rules of* **G3K*** *are height-preserving invertible.*

Proof. The proof of height-preserving invertibility for the propositional rules is done exactly as for **G3c** (Theorem 3.1.1 in Negri and von Plato, 2001). Rules $L\Box$ and $R\Diamond$ are trivially height-preserving invertible, since their premisses are obtained by weakening from the conclusion, and weakening is height-preserving invertible. The same holds for the rules for R. As usual, some care is needed for the rules with variable condition.

We show height-preserving invertibility of $R\Box$ by induction on the height n of the derivation of $\Gamma \Rightarrow \Delta, x : \Box A$. If $n = 0$, it is an initial sequent or conclusion of $L\bot$, but then also $xRy, \Gamma \Rightarrow \Delta, y : A$ is an initial sequent or conclusion of $L\bot$ (observe that it is essential here that the initial sequents are restricted to atomic formulas). If $n > 0$ and $\Gamma \Rightarrow \Delta, x : \Box A$ is concluded by any rule $\mathcal{R}$ other than $R\Box$ or $L\Diamond$, we apply the inductive hypothesis to the premiss(es) $\Gamma' \Rightarrow \Delta', x : \Box A$ ($\Gamma'' \Rightarrow \Delta'', x : \Box A$) and obtain derivation(s) of height $n - 1$ of $xRy, \Gamma' \Rightarrow \Delta', y : A$ ($xRy, \Gamma'' \Rightarrow \Delta'', y : A$). By applying rule $\mathcal{R}$ we obtain a derivation of height n of $xRy, \Gamma \Rightarrow \Delta, y : A$. If $\Gamma \Rightarrow \Delta, x : \Box A$ is concluded by $L\Diamond$, we have, in the first case, a derivation ending with

$$\frac{zRw, w : B, \Gamma \Rightarrow \Delta, x : \Box A}{z : \Diamond B, \Gamma \Rightarrow \Delta, x : \Box A}\, L\Diamond$$

where without loss of generality we can assume that the eigenvariable of $L\Diamond$ is not y (else apply the substitution lemma). By inductive hypothesis applied to the premiss we obtain a derivation with the same derivation height ending with

$$\frac{xRy, zRw, w : B, \Gamma \Rightarrow \Delta, y : A}{xRy, z : \Diamond B, \Gamma \Rightarrow \Delta, y : A}\, L\Diamond$$

If $\Gamma \Rightarrow \Delta, x : \Box A$ is conclusion of $R\Box$ with principal formula in Δ, we proceed in a similar way. If instead the principal formula is $\Box A$, the premiss of the last step gives the conclusion (possibly with a different eigenvariable, but the desired one can be obtained by height-preserving substitution). The proof of height-preserving invertibility of $L\Diamond$ is similar. □

We are now in a position to prove the most important structural property of our calculi besides cut-admissibility, namely height-preserving admissibility of contraction. First observe that there are, a priori, four contraction rules, left and right contraction for expressions of the form $x : A$ and of the form xRy. Explicitly stated, the rules of left and right contraction are:

$$\frac{x : A, x : A, \Gamma \Rightarrow \Delta}{x : A, \Gamma \Rightarrow \Delta}\, L\text{-}Ctr \qquad \frac{xRy, xRy, \Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}\, L\text{-}Ctr_R$$

$$\frac{\Gamma \Rightarrow \Delta, x : A, x : A}{\Gamma \Rightarrow \Delta, x : A}\, R\text{-}Ctr \qquad \frac{\Gamma \Rightarrow \Delta, xRy, xRy}{\Gamma \Rightarrow \Delta, xRy}\, R\text{-}Ctr_R$$

Observe that rule $R\text{-}Ctr_R$ is not needed in case we are using the calculus without the initial sequent $xRy, \Gamma \Rightarrow \Delta, xRy$.

THEOREM 4.12. *The rules of contraction are height-preserving admissible in* **G3K***.

Proof. By simultaneous induction on the height of derivation for left and right contraction.

If $n = 0$ the premiss is either an initial sequent or conclusion of $L\bot$. In each case the contracted sequence is also an initial sequent or conclusion of $L\bot$.

If $n > 0$, consider the last rule $\mathcal{R}$ used to derive the premiss of contraction. If the contraction formula is not principal in it, both occurrences are found in the premiss(es) of the rule, which have smaller derivation height. By the induction hypothesis, they can be contracted and the conclusion is obtained by applying rule $\mathcal{R}$ to the contracted premiss(es). If the contraction formula is principal in it, we distinguish three cases: Either $\mathcal{R}$ is a rule in which the principal formulas appear also in the premiss (such as $L\Box$ or $R\Diamond$ or the rules for R), or it is a rule where active formulas are proper subformulas[1] of the principal formula (such as the rules for &, $\vee$, $\supset$), or it is a rule where active formulas are atoms xRy and proper subformulas of the principal formula (like the rules $R\Box$ or $L\Diamond$).

In the first case we have, for instance,

$$\frac{x : \Box A, x : \Box A, xRy, y : A, \Gamma \Rightarrow \Delta}{x : \Box A, x : \Box A, xRy, \Gamma \Rightarrow \Delta}\, L\Box$$

By induction hypothesis applied to the premiss we obtain

$$\frac{x : \Box A, xRy, y : A, \Gamma \Rightarrow \Delta}{x : \Box A, xRy, \Gamma \Rightarrow \Delta}\, L\Box$$

Observe that the case in which both contraction formulas are principal in a rule for R is taken care of by the closure condition.

In the second case, contraction is reduced to contraction on smaller formulas as in the standard proof for **G3c**.

In the third case, a subformula of the contraction formula and an atom xRy are found in the premiss, for instance

$$\frac{x : \Diamond A, xRy, y : A, \Gamma \Rightarrow \Delta}{x : \Diamond A, x : \Diamond A, \Gamma \Rightarrow \Delta}\, L\Diamond$$

By height-preserving invertibility applied to the premiss, we obtain a derivation of height $n - 1$ of

$$xRy, y : A, xRy, y : A, \Gamma \Rightarrow \Delta$$

that yields, by induction hypothesis for both forms of contraction, a derivation of height $n - 1$ of

$$xRy, y : A, \Gamma \Rightarrow \Delta$$

and the conclusion $x : \Diamond A, \Gamma \Rightarrow \Delta$ follows in one more step by $L\Diamond$. □

Also cut can take two forms, namely

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad x : A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \textit{Cut}$$

and

$$\frac{\Gamma \Rightarrow \Delta, xRy \quad xRy, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \textit{Cut}_R$$

However, Cut_R is not needed if the variant of **G3K** without the initial sequent $xRy, \Gamma \Rightarrow \Delta, xRy$ is used.

We have:

THEOREM 4.13. *The cut rule is admissible in* **G3K***.

Proof. The proof has the same structure as the proof of admissibility of cut for sequent calculus extended with the left rule-scheme (Theorem 6.2.3 in Negri and von Plato, 2001). In case the geometric rule-scheme is considered, the proof follows the pattern of Negri (2003). We observe that in all the cases of permutation of cuts that may give a clash with the variable conditions in the modal rules (and in the rules for R in case of geometric extensions), an appropriate substitution (Lemma 4.3) prior to the permutation will be used.

We recall that the proof is by induction on the length of the cut formula with subinduction on the sum of the heights of the derivations of the premisses of cut. We consider in detail only the case of a cut with cut formula principal in modal rules in both premisses of cuts.

If the cut formula is $x : \Box A$, we transform the derivation

$$\frac{\dfrac{xRy, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}\ R\Box \qquad \dfrac{xRz, x : \Box A, z : A, \Gamma' \Rightarrow \Delta'}{xRz, x : \Box A, \Gamma' \Rightarrow \Delta'}\ L\Box}{xRz, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \textit{Cut}_1$$

into

$$\frac{\dfrac{xRz, \Gamma \Rightarrow \Delta, z : A \qquad \dfrac{\Gamma \Rightarrow \Delta, x : \Box A \quad xRz, x : \Box A, z : A, \Gamma' \Rightarrow \Delta'}{xRz, z : A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \textit{Cut}_1}{xRz, xRz, \Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta'}\ \textit{Cut}_1}{xRz, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \textit{Ctr}^*$$

where the upper cut is of smaller derivation height and the lower on a smaller cut formula, *Ctr** denotes repeated applications of contraction rules, and the leftmost premiss is obtained by the substitution (z/y) from $xRy, \Gamma \Rightarrow \Delta, y : A$.

If the cut formula is $x : \Diamond A$, we transform the derivation

$$\dfrac{\dfrac{xRy, \Gamma \Rightarrow \Delta, x : \Diamond A, y : A}{xRy, \Gamma \Rightarrow \Delta, x : \Diamond A}{\scriptstyle R\Diamond} \quad \dfrac{xRz, z : A, \Gamma' \Rightarrow \Delta'}{x : \Diamond A, \Gamma' \Rightarrow \Delta'}{\scriptstyle L\Diamond}}{xRy, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{\scriptstyle Cut_1}$$

into

$$\dfrac{\dfrac{\dfrac{xRy, \Gamma \Rightarrow \Delta, x : \Diamond A, y : A \quad x : \Diamond A, \Gamma' \Rightarrow \Delta'}{xRy, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', y : A}{\scriptstyle Cut_1} \quad xRy, y : A, \Gamma' \Rightarrow \Delta'}{xRy, xRy, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta', \Delta'}{\scriptstyle Cut_1}}{xRy, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{\scriptstyle Ctr^*}$$

where the upper cut is of smaller derivation height and the lower on a smaller cut formula, and the rightmost premiss is obtained by the substitution (y/z) from $xRz, z : A, \Gamma' \Rightarrow \Delta'$. □

5. Gödel–Löb Provability Logic

Gödel-Löb provability logic, nowadays commonly called GL, is characterized by the frame condition that the accessibility relation is irreflexive, transitive and Noetherian (every chain eventually becomes stationary), or, equivalently, transitive and with no infinite R-chains. The Noetherian condition is also called converse well-foundedness. The method detailed in the previous sections covers all the modal logics with frame conditions characterized by universal axioms or geometric implications. The Noetherian property is not even first-order, so a cut-free sequent calculus for GL cannot be obtained as a direct application.

We can, nevertheless, extend to GL all our results on sequent calculi with internalized Kripke semantics by a simple modification the $\Box$ rules of **G3K**. The rules are formulated in a way that reflects a characterization of the standard forcing relation in irreflexive, transitive, and Noetherian Kripke frames.

LEMMA 5.1. *For every interpretation in irreflexive, transitive, and Noetherian Kripke frames we have, for all x and for all A,*

$$x \Vdash \Box A \textit{ iff for all } y,\ xRy \textit{ and } y \Vdash \Box A \textit{ implies } y \Vdash A$$

Proof. Suppose $x \Vdash \Box A$ and let y be such that xRy. Then $y \Vdash A$ holds, and *a fortiori* it follows with the extra assumption $y \Vdash \Box A$.

For the converse, assume the right-hand side and suppose that $x \nVdash \Box A$. Then there exists x_1 such that xRx_1 and $x_1 \nVdash A$. From the assumption it follows that $x_1 \nVdash \Box A$, hence there exists x_2 such that x_1Rx_2 and $x_2 \nVdash A$. By transitivity we have xRx_2 and so from the assumption $x_2 \nVdash \Box A$ follows. In this way, we build a chain $xRx_1, x_1Rx_2, \dots$ which never becomes stationary because of irreflexivity, so we have a contradiction. □

The above characterization of the forcing relation justifies the following rules for □: the right rule

$$\frac{xRy, y : \Box A, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}\ R\Box\text{-}L$$

with the same variable condition as $R\Box$, namely, that y must not appear in the conclusion, and the left rule

$$\frac{x : \Box A, xRy, \Gamma \Rightarrow \Delta, y : \Box A \quad y : A, x : \Box A, xRy, \Gamma \Rightarrow \Delta}{x : \Box A, xRy, \Gamma \Rightarrow \Delta}\ L\Box\text{-}L$$

Let **G3GL** be the system obtained from **G3K** by allowing also initial sequents of the form

$$x : \Box A, \Gamma \Rightarrow \Delta, x : \Box A$$

by replacing the rules $R\Box$ and $L\Box$ with $R\Box$-L and $L\Box$-L, respectively, and by adding rule *Trans* that corresponds to the axiom $\Box A \supset \Box\Box A$, and rule *Irref* in the form of the zero-premiss rule $xRx, \Gamma \Rightarrow \Delta$.

It is an easy task to verify that all the preliminary results (from Lemma 4.1 to Proposition 4.5) proved for **G3K*** continue to hold for **G3GL**. In particular, we have height-preserving admissibility of the necessitation rule and height-preserving admissibility of weakening.

In addition, we have invertibility (not height-preserving) of all the rules of **G3GL** and of contraction:

PROPOSITION 5.2. *All the rules of the system* **G3GL** *are invertible.*

Proof. For invertibility of the rules for &, ∨, ⊃ see Proposition 4.11.

Rule $L\Box$-L is (height-preserving) invertible by (height-preserving) invertibility of weakening. We show invertibility of $R\Box$-L by induction on the height n of the derivation of $\Gamma \Rightarrow \Delta, x : \Box A$. If $n = 0$ and $x : \Box A$ is not principal, then also $xRy, y : \Box A, \Gamma \Rightarrow \Delta, y : A$ is an initial sequent, or an instance of *Irref*. If it is principal, we have $\Gamma \equiv x : \Box A, \Gamma'$, and we need to prove that $xRy, y : \Box A, x : \Box A, \Gamma' \Rightarrow \Delta, y : A$ is derivable. This follows by $L\Box$-L from the initial sequent $xRy, y : \Box A, x : \Box A, \Gamma' \Rightarrow \Delta, y : A, y : \Box A$ and the derivable sequent $y : A, xRy, y : \Box A, x : \Box A, \Gamma' \Rightarrow \Delta, y : A$. □

Without loss of generality we shall assume that derivations are *pure*, i.e., the eigenvariables used at steps of $R\Box$-L appear only in the subtree above the rule introducing them. Clearly, by height-preserving substitution, such a condition can always be met.

Before proving admissibility of contraction, we introduce the notion of range of a world in a derivation, to be used as one component of the inductive parameter in the proof of cut elimination:

DEFINITION 5.3. The *range* of x in a derivation $\mathcal{D}$ is the (finite) set of worlds y such that either xRy or for some $n \geqslant 1$ and for some $x_1, \ldots, x_n$, the atoms xRx_1, x_1Rx_2, ..., x_nRy appear in the antecedent of sequents of $\mathcal{D}$. Ranges of variables are ordered by set inclusion.

We shall say that a rule is *range-preserving* admissible if the elimination of the rule does not increase the ranges of variables in the derivation. We then have:

THEOREM 5.4. *The rules of contraction are range-preserving admissible in* **G3GL**.

Proof. By simultaneous induction for left and right contraction, with induction on the size of the contraction formula and subinduction on derivation height. We detail the proof in one case specific to **G3GL**: assume we have proved admissibility of contraction for formulas of size up to n on the left and up to $n-1$ on the right and assume the contraction formula is $x : \Box A$ on the right, of size n. If the last rule in the derivation is not $R\Box$-L on the contraction formula, we apply the inductive hypothesis to the premiss of the rule (of smaller height) and then apply the rule. If the last step is $R\Box$-L, the premiss is $xRy, y : \Box A, \Gamma \Rightarrow \Delta, x : \Box A, y : A$. By using invertibility of $R\Box$-L, we derive a sequent of the form $xRy, xRy, y : \Box A, y : \Box A, \Gamma \Rightarrow \Delta, y : A, y : A$, and by using the inductive hypotheses we obtain a derivation of $xRy, y : \Box A, \Gamma \Rightarrow \Delta, y : A$ hence the conclusion of contraction by $R\Box$-L.

Although invertibility of $R\Box$-L is not, in general, range-preserving, because it introduces a new world, the special instance of invertibility used here does not, as the world needed in the inversion is already a label used in the derivation. It follows that contraction is range-preserving admissible. □

THEOREM 5.5. *The cut rule is admissible in* **G3GL**.

Proof. The proof follows the structure of the proof of Theorem 4.13, but with a modified induction parameter: an uppermost cut is shown admissible by induction on the weight of the cut. The *weight* of a cut on $x : A$ is defined as the triple consisting of

(1) The size of the cut formula A;
(2) The range of x;
(3) The sum of the heights of the derivations of the two premisses of the cut.

Observe that we can exclude the possibility of a cut formula labelled by a variable occurring in the range of itself: if a loop $xRx_1, x_1Rx_2, \ldots, x_nRx$ occurs in the conclusion of cut, then the conclusion can be obtained without cut from *Irref* and steps of *Trans*. Else, if there is no loop in the conclusion of cut, there is no loop in the premisses either. The only way then to produce a loop (possibly spread among different antecedents) would be by introduction of eigenvariables at steps of $R\Box$-L that violate either the variable or the pureness condition.

Clearly, if y is in the range of x, then the range of y is included in the range of x, and by the above argument, we can always assume that the inclusion is proper.

The triples are ordered lexicographically in the usual way.

The proof of cut elimination for **G3GL** is structured as the proof for **G3K***. The cases in which the cut formula is not principal in both premisses of cut are dealt with as usual, with the additional observation that permutations do not increase the range since they change neither the cut formula nor its label. The only case specific to **G3GL** is the one in which the cut formula is principal in both premisses of *Cut*:

$$\dfrac{\dfrac{xRy, y:\Box A, \Gamma \Rightarrow \Delta, y:A}{\Gamma \Rightarrow \Delta, x:\Box A}\,{\scriptstyle R\Box\text{-}L} \qquad \dfrac{xRz, x:\Box A, \Gamma' \Rightarrow \Delta', z:\Box A \quad z:A, xRz, x:\Box A, \Gamma' \Rightarrow \Delta'}{xRz, x:\Box A, \Gamma' \Rightarrow \Delta'}\,{\scriptstyle L\Box\text{-}L}}{xRz, \Gamma', \Gamma \Rightarrow \Delta, \Delta'}\,{\scriptstyle Cut}$$

The derivation is transformed into one containing four cuts, each of lower weight.

(1) First, we construct the derivation

$$\dfrac{\Gamma \Rightarrow \Delta, x:\Box A \quad xRz, x:\Box A, \Gamma', \Rightarrow \Delta', z:\Box A}{xRz, \Gamma, \Gamma' \Rightarrow \Delta, \Delta', z:\Box A}\,{\scriptstyle Cut}$$

using a cut of smaller weight, on the same labelled formula $x : \Box A$ (and thus the same range) but with lower sum of heights of derivations.

(2) Second, we construct the derivation

$$\dfrac{\Gamma \Rightarrow \Delta, x:\Box A \quad xRz, x:\Box A, z:A, \Gamma' \Rightarrow \Delta'}{xRz, z:A, \Gamma', \Gamma \Rightarrow \Delta, \Delta'}\,{\scriptstyle Cut}$$

reduced in weight in the same way.

(3) Third, we use derivation (1) and height-preserving substitution (z/y) on the premiss of $R\Box$-L to obtain

$$\dfrac{\begin{array}{c}\vdots\\ xRz, \Gamma', \Gamma \Rightarrow \Delta, \Delta', z : \Box A\end{array} \quad xRz, z : \Box A, \Gamma \Rightarrow \Delta, z : A}{xRz, xRz, \Gamma, \Gamma', \Gamma \Rightarrow \Delta, \Delta', \Delta, z : A}\ \textit{Cut}$$

using a cut on the labelled formula $z : \Box A$, of smaller range, given by the union of the range of y and the range of z. This is strictly included in the range of x because by the pureness condition z cannot be in the range of y.

(4) Fourth, we combine (3) and (2) by a cut on the labelled formula $z : A$ of smaller size, followed by several contractions:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ xRz, xRz, \Gamma', \Gamma, \Gamma \Rightarrow \Delta, \Delta, \Delta', z : A\end{array} \quad \begin{array}{c}\vdots\\ xRz, z : A, \Gamma', \Gamma \Rightarrow \Delta, \Delta'\end{array}}{xRz, xRz, xRz, \Gamma', \Gamma', \Gamma, \Gamma, \Gamma \Rightarrow \Delta, \Delta, \Delta, \Delta', \Delta'}\ \textit{Cut}}{xRz, \Gamma', \Gamma \Rightarrow \Delta, \Delta'}\ \textit{Ctr*}$$

Since no transformation in the process of cut elimination increases the range and contraction is range-preserving admissible, we can conclude that cut is range-preserving admissible as well. $\Box$

We observe that Theorem 5.5 gives a solution to the problem of finding a syntactic proof of cut-elimination for Gödel–Löb provability logic. A proof of cut elimination was proposed by Leivant (1981), but a gap was found by Valentini (cf. Sambin and Valentini, 1982 and Valentini, 1983). In the former, the proposed sequent calculus for provability logic was proved to be semantically complete, and therefore the weaker property of closure with respect to cut was established. In the latter, a syntactic proof of cut elimination was given. However, the proofs use a treatment of contexts as sets that cannot be made formally complete without the addition of a rule of contraction. When an explicit rule of contraction is added to the calculus, the reduction used as starting point in the procedure of cut elimination needs further justification. For a discussion of the difficulties, see Moen (2001). Our approach avoids the problem altogether by using a calculus in which contraction is not part of the system but an admissible rule.

As an application of the cut-free calculus we have:

COROLLARY 5.6 (Second incompleteness theorem). *The sequent* $\Rightarrow x : \sim\Box\bot$ *is not derivable in* **G3GL**.

Proof. Proceeding root-first, if a derivation exists, it ends with

$$\frac{x : \Box\bot \Rightarrow x : \bot}{\Rightarrow x : \Box\bot\supset\bot}\ R\supset$$

but no rule of **G3GL** is applicable to the premiss. □

Finally, in order to obtain the derivability of the characteristic axiom of GL we show:

LEMMA 5.7. *All sequents of the form* $xRy, x : \Box A, \Gamma \Rightarrow \Delta, y : \Box A$ *are derivable in* **G3GL**.

Proof. Root-first, by steps of $R\Box$-L, *Trans*, and $L\Box$-L. □

COROLLARY 5.8. *The standard rule* $L\Box$ *is derivable in* **G3GL**.

Proof. By Lemma 5.7, the left premiss of $L\Box$-L is derivable in **G3GL**. □

Although the two left □ rules are interderivable, the use of $L\Box$-L seems essential in the proof of cut elimination. If the standard $L\Box$ were used instead, a cut with a sequent of the form $xRy, x : \Box A, \Gamma \Rightarrow \Delta, y : \Box A$ would be needed. However, its derivation introduces new worlds, thus breaking the property of range admissibility of all cut reductions.

COROLLARY 5.9. *The Löb axiom is derivable in* **G3GL**.

Proof. Using Corollary 5.8, we have the derivation:

$$\dfrac{\dfrac{\dfrac{y : \Box A \supset A, xRy, x : \Box(\Box A \supset A), y : \Box A \Rightarrow y : A}{xRy, x : \Box(\Box A \supset A), y : \Box A \Rightarrow y : A}\ L\Box}{x : \Box(\Box A \supset A) \Rightarrow x : \Box A}\ R\Box\text{-}L}{\Rightarrow x : \Box(\Box A \supset A) \supset \Box A}\ R\supset$$

□

By Corollary 5.8, the system **G3GL** (with rules $R\Box$-L and $L\Box$-L) and the system with rules $R\Box$-L and $L\Box$ (that we shall call **G3KGL**) are equivalent. In the latter system, initial sequents can be restricted to atomic formulas, as in **G3K**, and therefore stronger structural properties such as height-preserving admissibility of contraction, hold with no limitations. Cut elimination for **G3KGL** can be established through translation to **G3GL**, cut elimination in this system, and translation back to **G3KGL**. The structural properties of **G3KGL** will be exploited in the following section.

6. DECIDABILITY

In general, cut-elimination alone does not ensure terminating proof search in a given system of sequent calculus. Cut-elimination often has the subformula property as one of its immediate consequences. Sometimes the subformula property does not require full cut-elimination, as in systems with *analytic cut*, i.e., with cut restricted to subformulas of the conclusion. Even the subformula property is not always sufficient to delimit the space of proof search, either because the notion of subformula is extended (as in first-order logic) to include all substitution instances of a given formula, or because of the presence in the calculus of structural rules like contraction.

In our systems, a suitable version of the subformula property, adequate for proving syntactic decidability, will emerge as a consequence of the structural properties of the calculi.

Before proceeding with the analysis of the subformula properties of our systems we state precisely what we mean by "subformula" and "subformula property" of derivations in the context of prefixed formulas $x : A$:

SUBFORMULA. *For every propositional connective* $\circ$, *the subformulas of* $x : A \circ B$ *are* $x : A \circ B$ *and all the subformulas of* $x : A$ *and of* $x : B$. *The subformulas of* $x : \Box A$ *and* $x : \Diamond A$, *resp., are* $x : \Box A$ *and* $x : \Diamond A$, *resp., and all the subformulas of* $y : A$ *for arbitrary* y.

SUBFORMULA PROPERTY. *All formulas in a derivation are subformulas of formulas in the endsequent.*

WEAK SUBFORMULA PROPERTY. *All formulas in a derivation are either subformulas of formulas in the endsequent or atomic formulas of the form* xRy.

A priori, such properties do not ensure decidability, unless a bound is found on the number of eigenvariables and "new worlds" in a derivation of a given sequent.

For obtaining a bound on the number of atomic formulas that can appear in a derivation it is useful to look at *minimal* derivations, that is, derivations where shortenings are not possible. A derivation where a rule, read root first, produces a duplication of an atom xRy can be shortened by applying height-preserving admissibility of contraction in place of the rule that introduces that atom. Similarly, a derivation that contains a sequent that matches the conclusion of a zero-premiss rule can be shortened by removing the subtree concluding that sequent.

More precisely, we have:

PROPOSITION 6.1. *All variables* (*worlds*) *in a minimal derivation of a sequent* $\Gamma \Rightarrow \Delta$ *in* **G3K** *and in its extensions with rules for* R, **G3T**, **G3K4**, **G3KB**, **G3S4**, **G3TB**, **G3S5**, *and in* **G3KGL**, *are either eigenvariables or else variables in* Γ, Δ.

Proof. Immediate for **G3K** and its extensions with *Trans* and/or *Sym* (**G3K4**, **G3KB**) and for **G3KGL**. For extension with *Ref*, the proof follows from the lemma below. □

Before stating the lemma, we observe that the hypothesis of minimality is redundant in the absence of *Ref*. Nevertheless, it is useful in any case since it precludes the possibility of applying rules that produce duplications.

LEMMA 6.2. *All variables in atoms of the form* xRx *removed by* Ref *in a minimal derivation of a sequent* $\Gamma \Rightarrow \Delta$ *in* **G3T**, **G3S4**, **G3TB**, **G3S5**, *are variables in* Γ, Δ.

Proof. Consider a minimal derivation of a sequent $\Gamma \Rightarrow \Delta$ and suppose there is a variable x in an atom xRx removed by *Ref*. Consider a last occurrence of x and the step of *Ref* removing it

$$\frac{xRx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Ref$$

Trace the atom xRx up in the derivation (observe that nothing, in particular no atom xRy, is removed going up in the derivation).

If xRx is never principal in a rule, we trace it up to the leaves (initial sequents) of the derivation tree. If it is principal in the initial sequent, it has the form

$$xRx, \Gamma \Rightarrow \Delta, xRx$$

and we find x in the succedent. No atom of the form xRy is removed from the right-hand side of sequents in a derivation, so x is found in the conclusion. If xRx is not principal in any of the leaves, it can be removed *tout court* from the derivation, together with the step of *Ref*, so the derivation is shortened, contrary to the assumption.

If xRx is principal in a rule, this can happen in a step of $L\Box$, or $R\Diamond$, or *Trans*, or *Sym*. We analyze each of these possibilities.

If xRx is principal in $L\Box$, we have the derivation steps

$$\begin{array}{c}\dfrac{x : A, xRx, x : \Box A, \Gamma' \Rightarrow \Delta'}{xRx, x : \Box A, \Gamma' \Rightarrow \Delta'}\ L\Box \\ \vdots\ \mathcal{D} \\ \dfrac{xRx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Ref\end{array}$$

By tracing the variable x, another occurrence of the variable has been found in a modal expression $x : \Box A$. Since by hypothesis the premiss of *Ref* contains the last occurrence of x, the occurrence in $x : \Box A$ has been removed from the derivation before the step of *Ref*. The expression $x : \Box A$ can be active in propositional rules that either maintain x on the left-hand side of the sequent, or that move it to the right-hand side. Eventually we find in $\mathcal{D}$

(1) $xRx, x : B, \Gamma'' \Rightarrow \Delta''$ or

(2) $xRx, \Gamma'' \Rightarrow \Delta'', x : B$

Observe that, because of the variable condition, x cannot disappear from (1) by $L\Diamond$, nor from (2) by $R\Box$. If $x : B$ is active in $L\Box$ (1) or $R\Diamond$ (2), then we find another occurrence of x in an atom zRx in the conclusion of the rule. The atom zRx can be removed only by *Ref*, so we must have $z \equiv x$ and therefore, for each of the two alternatives

$$\frac{x : B, xRx, x : \Box B, \Gamma \Rightarrow \Delta}{xRx, x : \Box B, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\Box} \qquad \frac{xRx, \Gamma \Rightarrow \Delta, x : \Diamond B, x : B}{xRx, \Gamma \Rightarrow \Delta, x : \Diamond B}\,{\scriptstyle R\Diamond}$$

so we still have x in the conclusion.

If xRx is principal in $R\Diamond$, the analysis is similar.

If xRx is principal in *Trans*, we have the derivation

$$\cfrac{\cfrac{xRx, xRz, xRz, \Gamma' \Rightarrow \Delta'}{xRx, xRz, \Gamma' \Rightarrow \Delta'}\,{\scriptstyle Trans}}{\cfrac{\vdots}{\cfrac{xRx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,{\scriptstyle Ref}}}$$

By applying height-preserving admissibility of contraction to the premiss of *Trans*, we obtain a shorter derivation of the same endsequent, contrary to the assumption:

$$\begin{array}{c} xRx, xRz, \Gamma' \Rightarrow \Delta' \\ \vdots \\ \cfrac{xRx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,{\scriptstyle Ref} \end{array}$$

If xRx is principal in *Sym*, we have

$$\begin{array}{c} \cfrac{xRx, xRx, \Gamma' \Rightarrow \Delta'}{xRx, \Gamma' \Rightarrow \Delta'}\,{\scriptstyle Sym} \\ \vdots \\ \cfrac{xRx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,{\scriptstyle Ref} \end{array}$$

Again, by applying height-preserving admissibility of contraction as above, we obtain a shorter derivation of the same endsequent, with the step of *Sym* made superfluous, contrary to the assumption. □

The property stated by the above proposition will be referred to in brief as *subterm property* of a derivation:

SUBTERM PROPERTY. *All terms* (*variables, worlds*) *in a derivation are either eigenvariables or terms* (*variables, worlds*) *in the conclusion.*

Proofs of the subterm property for systems of linear order and lattice theory have been obtained by similar methods in Negri and von Plato (2004) and in Negri, von Plato and Coquand (2004).

Another source of potentially non-terminating proof search is the repetition of the principal formulas in the premisses of $L\Box$ and $R\Diamond$. By the following lemmas and their corollary, it is enough to apply them only once on any given pair of principal formulas $xRy,\ x : \Box A$ or $xRy,\ x : \Diamond A$. First we prove that if there are two applications of $L\Box$ or $R\Diamond$ on the same pair of principal formulas, such applications can be made consecutive by the permutation of rules:

LEMMA 6.3. *Rule $L\Box$ permutes down with respect to rules $L\&$, $R\&$, $L\vee$, $R\vee$, $L\supset$, $R\supset$, $L\Box$, $R\Diamond$. It also permutes with instances of $R\Box$, $R\Box$-L, $L\Diamond$, and with mathematical rules in case the principal atom of $L\Box$ is not active in them.*

Proof. The permutation is straightforward in the case of a one-premiss propositional rule. For instance, for $L\&$ we have

$$\dfrac{\dfrac{y : A, x : \Box A, xRy, z : C, z : D, \Gamma \Rightarrow \Delta}{x : \Box A, xRy, z : C, z : D, \Gamma \Rightarrow \Delta}{\scriptstyle L\Box}}{x : \Box A, xRy, z : C\&D, \Gamma \Rightarrow \Delta}{\scriptstyle L\&}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{y : A, x : \Box A, xRy, z : C, z : D, \Gamma \Rightarrow \Delta}{y : A, x : \Box A, xRy, z : C\&D, \Gamma \Rightarrow \Delta}{\scriptstyle L\&}}{x : \Box A, xRy, z : C\&D, \Gamma \Rightarrow \Delta}{\scriptstyle L\Box}$$

In the case of a two-premiss rule, use of height-preserving admissibility of weakening is needed; for instance, the derivation

$$\dfrac{\dfrac{y : A, x : \Box A, xRy, \Gamma \Rightarrow \Delta, z : C}{x : \Box A, xRy, \Gamma \Rightarrow \Delta, z : C}{\scriptstyle L\Box} \quad x : \Box A, xRy, \Gamma \Rightarrow \Delta, z : D}{x : \Box A, xRy, \Gamma \Rightarrow \Delta, z : C\&D}{\scriptstyle R\&}$$

is transformed into

$$\dfrac{\dfrac{y : A, x : \Box A, xRy, \Gamma \Rightarrow \Delta, z : C \quad \dfrac{x : \Box A, xRy, \Gamma \Rightarrow \Delta, z : D}{y : A, x : \Box A, xRy, \Gamma \Rightarrow \Delta, z : D}}{y : A, x : \Box A, xRy, \Gamma \Rightarrow \Delta, z : C\&D}{\scriptstyle R\&}}{x : \Box A, xRy, \Gamma \Rightarrow \Delta, z : C\&D}{\scriptstyle L\Box}$$

in which the right premiss of $R\&$ is obtained by weakening from the right premiss of the given derivation.

The permutation is performed similarly for the other propositional rules.

For the modal rules and the mathematical rules, by the additional hypothesis of no clash of active/principal formulas, the permutation is straightforward. For instance, the permutation of $L\Box$ over $R\Box$ is as follows:

$$\dfrac{\dfrac{y : A, x : \Box A, xRy, zRw, \Gamma \Rightarrow \Delta, w : B}{x : \Box A, xRy, zRw, \Gamma \Rightarrow \Delta, w : B}{\scriptstyle L\Box}}{x : \Box A, xRy, \Gamma \Rightarrow \Delta, z : \Box B}{\scriptstyle R\Box}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{y : A, x : \Box A, xRy, zRw, \Gamma \Rightarrow \Delta, w : B}{y : A, x : \Box A, xRy, \Gamma \Rightarrow \Delta, z : \Box B}{\scriptstyle R\Box}}{x : \Box A, xRy, \Gamma \Rightarrow \Delta, z : \Box B}{\scriptstyle L\Box}$$

In the system **G3KGL**, rule $L\Box$ permutes down as follows with respect to $R\Box$-L in case the principal atoms of $L\Box$ are not principal in it:

$$\dfrac{\dfrac{y : A, x : \Box A, xRy, zRw, w : \Box B, \Gamma \Rightarrow \Delta, w : B}{x : \Box A, xRy, zRw, w : \Box B, \Gamma \Rightarrow \Delta, w : B}{\scriptstyle L\Box}}{x : \Box A, xRy, \Gamma \Rightarrow \Delta, z : \Box B}{\scriptstyle R\Box\text{-}L}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{y : A, x : \Box A, xRy, zRw, w : \Box B, \Gamma \Rightarrow \Delta, w : B}{y : A, x : \Box A, xRy, \Gamma \Rightarrow \Delta, z : \Box B}{\scriptstyle R\Box\text{-}L}}{x : \Box A, xRy, \Gamma \Rightarrow \Delta, z : \Box B}{\scriptstyle L\Box}$$

□

A similar lemma holds, *mutatis mutandis*, for the dual case of rule $R\Diamond$:

LEMMA 6.4. *Rule $R\Diamond$ permutes down with respect to rules $L\&$, $R\&$, $L\vee$, $R\vee$, $L\supset$, $R\supset$, $L\Box$, $R\Diamond$. It also permutes with instances of $R\Box$, $L\Diamond$, and with mathematical rules in case the principal atom of $R\Diamond$ is not active in them.*

COROLLARY 6.5. *In a minimal derivation in* **G3K** *and in any of its extensions with rules for R, rules $L\Box$ and $R\Diamond$ cannot be applied more than once on the same pair of principal formulas on any branch. In* **G3KGL**, *rule $L\Box$ cannot be applied more than once on the same pair of principal formulas on any branch.*

Proof. Suppose we have, say, $L\Box$ applied twice on $x : \Box A, xRy$. Then the derivation contains the steps

$$\dfrac{y : A, x : \Box A, xRy, \Gamma' \Rightarrow \Delta'}{x : \Box A, xRy, \Gamma' \Rightarrow \Delta'}{\scriptstyle L\Box}$$

$$\vdots$$

$$\dfrac{y : A, x : \Box A, xRy, \Gamma \Rightarrow \Delta}{x : \Box A, xRy, \Gamma \Rightarrow \Delta}{\scriptstyle L\Box}$$

By permuting down $L\Box$ with respect to the steps in the dotted part of the derivation, we obtain a derivation of the same height ending with

$$\cfrac{\cfrac{y : A, y : A, x : \Box A, xRy, \Gamma \Rightarrow \Delta}{y : A, x : \Box A, xRy, \Gamma \Rightarrow \Delta}\, L\Box}{x : \Box A, xRy, \Gamma \Rightarrow \Delta}\, L\Box$$

By applying height-preserving contraction on $y : A$ in place of the upper $L\Box$, a shorter derivation is obtained, contrary to the assumption of minimality. □

Decidability for the basic modal logic **K** is obtained in the strongest form of an effective bound on proof search in the system **G3K**:

THEOREM 6.6. *The system* **G3K** *allows terminating proof search.*

Proof. Consider any given sequent to be shown derivable. Apply root-first the (applicable) propositional rules and the modal rules. All the propositional rules reduce the complexity of the sequents. Rules $R\Box$ and $L\Diamond$ remove one modal operator and add an atomic relation, rules $L\Box$ and $R\Diamond$ increase the complexity. However, by the corollary above, rules $L\Box$ and $R\Diamond$, once applied on a given pair of formulas, need not be so applied again. Thus the number of applications of $L\Box$ with principal formula $x : \Box A$ is bounded by the number of atoms of the form xRy that may appear on the left-hand side of sequents in the derivation. This, in turn, is bounded by the number of existing atoms of that form and atoms that can be introduced by applications of $R\Box$ with principal formula $x : \Box B$ or applications of $L\Diamond$ with principal formula $x : \Diamond B$. A similar bound holds for the number of applications of $R\Diamond$ on a given principal formula. □

Explicit bounds are computed as follows: First define *negative* and *positive* parts of a sequent $\Gamma \Rightarrow \Delta$ as the negative and positive parts of the formula $\&\Gamma \supset \vee\Delta$. For any given sequent, let $n(\Box)$ be the number of $\Box$ in the negative part of the sequent, $p(\Box)$ the number of $\Box$ in the positive part of the sequent, $n(\Diamond)$ the number of $\Diamond$ in the negative part of the sequent, $p(\Diamond)$ the number of $\Diamond$ in the positive part of the sequent.

In case the endsequent does not contain any atom xRy, the number of applications of $L\Box$ in a minimal derivation is bounded by

$$n(\Box)(p(\Box) + n(\Diamond))$$

and in case there are r atoms in the antecedent of the endsequent, it is bounded by

$$n(\Box)(p(\Box) + n(\Diamond) + r)$$

The number of applications of $R\Diamond$ is bounded by

$$p(\Diamond)(p(\Box) + n(\Diamond))$$

in case there are no atoms xRy in the endsequent, and by

$$p(\Diamond)(p(\Box) + n(\Diamond) + r)$$

if there are r such atoms.

By a similar argument we have:

THEOREM 6.7. *The system* **G3T** *allows terminating proof search.*

Proof. First, observe that by the subterm property, reflexivity can be restricted to atoms xRx where x is a world in the conclusion or an eigenvariable introduced by $R\Box$ or $L\Diamond$. Therefore, if w denotes the number of worlds in the endsequent, the bound to the number of applications of $L\Box$ and $R\Diamond$ is as above, with the parameter r replaced by $r + w + p(\Box) + n(\Diamond)$. □

The addition of rule *Sym* to **G3K** or **G3T** has the following effect on proof search (of minimal derivations): Whenever an atom xRy appears on the left-hand side of sequents, the symmetric atom yRx has to be added. In case $x \equiv y$, no addition is needed, because such addition would cause a duplication and use of height-preserving admissibility of contraction would shorten the derivation. With the notation introduced above, in **G3KB**, the bound to the number of applications of $L\Box$ is $n(\Box)(2p(\Box) + 2n(\Diamond) + 2r)$ and for $R\Diamond$, $p(\Diamond)(2p(\Box) + 2n(\Diamond) + 2r)$. For **G3TB**, the bounds are given by $n(\Box)(3p(\Box) + 3n(\Diamond) + 2r + w)$ and $p(\Diamond)(3p(\Box) + 3n(\Diamond) + 2r + w)$, respectively. We have thus proved:

THEOREM 6.8. *The systems* **G3KB** *and* **G3TB** *allow terminating proof search.*

In **G3S4**, the situation is more complicated: by the rule of transitivity and its interaction with $R\Box$ that brings in new accessible worlds, we can build chains of accessible worlds on which $L\Box$ can be applied *ad infinitum*. However, by our results on height-preserving admissibility of substitution and height-preserving admissibility of contraction, we can truncate an attempted proof search after a finite number of steps. Before giving precise bounds, we illustrate the method with an example (based upon a similar example discussed in Section 11.2 of Viganò, 2000). In what follows, we shall for simplicity restrict the language to the $\Box$ modality alone. The results can be generalized by symmetry to the full language that includes $\Diamond$.

We attempt to find a proof for the sequent $\Rightarrow x : \Box{\sim}\Box A \supset \Box B$. Proceeding root-first, we build the following inference tree (in which we have omitted the derivable right premisses of $L\supset$)

$$
\begin{array}{c}
\vdots \\
\underline{zRw, xRz, xRy, yRz, x : \Box{\sim}\Box A \Rightarrow y : B, z : A, w : A}\ {\scriptstyle R\Box} \\
\underline{xRz, xRy, yRz, x : \Box{\sim}\Box A \Rightarrow y : B, z : A, z : \Box A}\ {\scriptstyle L\supset} \\
\underline{z : {\sim}\Box A, xRz, xRy, yRz, x : \Box{\sim}\Box A \Rightarrow y : B, z : \Box A}\ {\scriptstyle L\Box} \\
\underline{xRz, xRy, yRz, x : \Box{\sim}\Box A \Rightarrow y : B, z : \Box A}\ {\scriptstyle Trans} \\
\underline{xRy, yRz, x : \Box{\sim}\Box A \Rightarrow y : B, z : A}\ {\scriptstyle R\Box} \\
\underline{xRy, x : \Box{\sim}\Box A \Rightarrow y : B, y : \Box A}\ {\scriptstyle L\supset} \\
\underline{y : {\sim}\Box A, xRy, x : \Box{\sim}\Box A \Rightarrow y : B}\ {\scriptstyle L\Box} \\
\underline{xRy, x : \Box{\sim}\Box A \Rightarrow y : B}\ {\scriptstyle R\Box} \\
\underline{x : \Box{\sim}\Box A \Rightarrow x : \Box B}\ {\scriptstyle R\supset} \\
\Rightarrow x : \Box{\sim}\Box A \supset \Box B
\end{array}
$$

Consider now the topsequent. By applying the substitution z/w we obtain a derivation of the same height of

$$zRz, xRz, xRy, yRz, x : \Box{\sim}\Box A \Rightarrow y : B, z : A, z : A$$

hence, by height-preserving contraction, of

$$zRz, xRz, xRy, yRz, x : \Box{\sim}\Box A \Rightarrow y : B, z : A$$

By a step of reflexivity we obtain a derivation of

$$xRz, xRy, yRz, x : \Box{\sim}\Box A \Rightarrow y : B, z : A$$

with a resulting shortening by two steps of the original derivation. Since we can assume that the attempted proof search is for a minimal derivation, we have a contradiction, thus the sequent is not derivable.

This argument can be formalized through providing a bound to the number of successive applications of $R\Box$ with principal formula $x : \Box A$ on successive worlds accessible from x. Intuitively, only those applications that contribute to unfold all the boxed negative subformulas of the endsequent through steps of $L\Box$ are needed. Additional steps are superfluous as they give rise to duplications (modulo substitution) as soon as the innermost boxed formula in the negative part has been reached, as shown in the above example.

PROPOSITION 6.9. *In a minimal derivation of a sequent in* **G3S4***, for each formula $x : \Box A$ in its positive part there are at most $n(\Box)$ applications of $R\Box$ iterated on a chain of accessible worlds xRx_1, $x_1Rx_2, \ldots,$ with principal formula $x_i : \Box A$.*

Proof. Let m be $n(\Box)$, and suppose that the antecedent of the derivable sequent contains a formula of the form $\Box^m Q$, where $\Box^m$ denotes a block of m boxes. This assumption can be done without loss of generality: The modalities in the negative part of the sequent do not necessarily occur in a block, but may be interleaved with propositional connectives. However, these connectives can be unfolded by the application, root-first, of propositional rules without changing the number of applications of $R\Box$ that are necessary to reach the innermost non-modal formula. Suppose that we iterate $R\Box$ on a chain of accessible worlds $x_0Rx_1, \ldots,$ etc., with $x_0 \equiv x$. After the first application of $R\Box$, we have the accessibility x_0Rx_1 and application of $L\Box$ produces an antecedent containing $x_0 : \Box^m Q$, $x_1 : \Box^{m-1}Q$. After the second application we have the new accessibility x_1Rx_2, and, by transitivity, x_0Rx_2, and applications of $L\Box$ add to the antecedent the formulas $x_2 : \Box^{m-2}Q$, $x_2 : \Box^{m-1}Q$. After m applications, the antecedent contains in addition $x_m : Q, \ldots, x_m : \Box^{m-1}Q$ and the succedent $x_{m-1} : A$. If we apply $R\Box$ one more time, by the newly available steps of $L\Box$ licensed by the accessibility x_mRx_{m+1}, we add to the antecedent also the formulas $x_{m+1} : Q, \ldots, x_{m+1} : \Box^{m-1}Q$. These latter steps are superfluous. By Lemma 4.3, we can perform the height-preserving substitution x_{m+1}/x_m, and by Theorem 4.12 eliminate all the duplications that arise, while maintaining the derivation height. By the single steps of reflexivity that eliminates the atom x_mRx_m, we obtain a shorter derivation of the sequent reached after m steps of $R\Box$. □

We therefore have:

COROLLARY 6.10. *The systems* **G3S4** *allows terminating proof search.*

By the remarks before Theorem 6.8, the result above directly extends to **G3S5**.

7. Extensions and Applications

7.1. *Frame Properties with Equality*

If we allow also the relation of equality in frame properties, then we need to extend **G3K*** with appropriate rules. The rules are similar to the rules added to the sequent systems **G3c** in Section 6.5 of Negri and von Plato (2001) for obtaining predicate logic with equality. A contraction- and cut-free system **G3K-Eq** of modal logic with equality is obtained by adding to **G3K** the rules of reflexivity and Euclidean transitivity of equality, and rules of substitution of equals:

$$\frac{x = x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \textit{Eq-Ref} \qquad \frac{y = z, x = y, x = z, \Gamma \Rightarrow \Delta}{x = y, x = z, \Gamma \Rightarrow \Delta}\ \textit{Eq-Trans}$$

$$\frac{yRz, x = y, xRz, \Gamma \Rightarrow \Delta}{x = y, xRz, \Gamma \Rightarrow \Delta}\ \textit{Repl}_{R_1} \qquad \frac{xRz, y = z, xRy, \Gamma \Rightarrow \Delta}{y = z, xRy, \Gamma \Rightarrow \Delta}\ \textit{Repl}_{R_2}$$

$$\frac{y : A, x = y, x : A, \Gamma \Rightarrow \Delta}{x = y, x : A, \Gamma \Rightarrow \Delta}\ \textit{Repl}$$

As in Section 6.5 of Negri and von Plato (2001), it can be shown that rule *Repl* can be restricted to atomic formulas

$$\frac{y : P, x = y, x : P, \Gamma \Rightarrow \Delta}{x = y, x : P, \Gamma \Rightarrow \Delta}\ \textit{Repl}_{At}$$

because its general form *Repl* becomes admissible.

For instance, the modal axiom

$$\Diamond(A \& \Box B) \supset \Box(A \vee \Diamond A \vee B)$$

corresponds to the frame property

$$\forall xyz(xRy \& xRz \supset z = y \vee zRy \vee yRz)$$

that can be represented as the following rule

$$\frac{z = y, xRy, xRz, \Gamma \Rightarrow \Delta \quad zRy, xRy, xRz, \Gamma \Rightarrow \Delta \quad yRz, xRy, xRz, \Gamma \Rightarrow \Delta}{xRy, xRz, \Gamma \Rightarrow \Delta}$$

added to the sequent calculus **G3K-Eq**.

All the extensions with frame properties involving equality are obtained in a modular way by adding the corresponding rules to the system **G3K-Eq**.

7.2. *Negative Results in Correspondence Theory*

The range of modal logics for which the method is available goes beyond the commonly known and studied modal logics: There are in fact simple first-order frame properties that do not correspond to any modally expressible axiom (cf. Example 2.4.3 in Van Benthem, 1984, and Exercise 4.12 in Popkorn, 1994) but that can nevertheless receive a complete syntactic treatment through a cut-free sequent calculus. By the same method we can give a proof-theoretical proof of a class of negative results in what is known as correspondence theory. The lack of a modal correspondent to a given frame property is obtained as a straightforward corollary to a conservativity theorem.

We start with the frame property of irreflexivity

$$\forall x \sim xRx$$

that corresponds to the zero-premiss rule

$$\frac{}{xRx, \Gamma \Rightarrow \Delta}\ \textit{Irref}$$

We have:

THEOREM 7.1. *The system* **G3K** + Irref *is conservative over* **G3K**.

Proof. Suppose that the sequent $\Gamma \Rightarrow \Delta$ (not containing relational atoms) is derivable in **G3K** + *Irref*. The atoms of the form xRy that appear on the left-hand side of sequents in the derivation originate from applications of rule $R\Box$. By the variable condition, $x \neq y$, so the derivation contains no atom of the form xRx, hence no application of *Irref*. Therefore the sequent is derivable in **G3K**. □

Intransitivity is the axiom

$$\forall x \forall y \forall z (xRy \& yRz \supset \sim xRz)$$

that correspond to the zero-premiss rule

$$\frac{}{xRy, yRz, xRz\Gamma \Rightarrow \Delta}\ \textit{Intrans}$$

A similar result obtains:

THEOREM 7.2. *The system* **G3K** + Intrans *is conservative over* **G3K**.

Proof. As above, observe that relational atoms on the left in derivations of a sequent $\Gamma \Rightarrow \Delta$ originate from applications of $R\Box$. In order to have both xRz and yRz, two applications of $R\Box$ with the same eigenvariable are needed, but this is ruled out by the variable condition. □

We can generalize the above two results to the following results for a generalization of intransitivity:

THEOREM 7.3. *Let* $P_1, \ldots, P_n\Gamma \Rightarrow \Delta$ *be a rule, called* G-Intrans, *that corresponds to the axiom* $\sim(P_1 \& \cdots \& P_n)$ *with* $P_i \equiv x_i R y_i$, *and assume that for some* i, j, $y_i = y_j$. *Then* **G3K** + G-Intrans *is conservative over* **G3K**.

Proof. Straightforward. □

By similar arguments, we can prove conservativity for extensions with rules for geometric axioms, such as the property that there exists a reflexive world, $\exists x xRx$, or compositions thereof, such as the property stating that every world has access to a reflexive one, $\forall x \exists y(xRy \,\&\, yRy)$.

Let *Eref* be the geometric rule stating the existence of a reflexive world

$$\frac{xRx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

with the variable condition that x is not in Γ, Δ. We have

THEOREM 7.4. *The system* **G3K** + Eref *is conservative over* **G3K**.

Proof. Assume that $\Gamma \Rightarrow \Delta$ is derivable in **G3K** + *Eref* and consider a step of *Eref* in the derivation. Trace its active atom xRx until it is principal in a rule. The rule can be $L\Box$ of $R\Diamond$. In the former case, the derivation above the step of *Eref* contains a sequent of the form $xRx, x : \Box A, \Gamma' \Rightarrow \Delta'$. By the variable condition on *Eref*, the label x in $x : \Box A$ has to disappear before the application of *Eref*. However, by the presence of x in xRx in the context, such a step would not be correct. The other possibility (xRx principal in $R\Diamond$) is excluded in a similar way. If xRx is principal in an initial sequent, then xRx is found in the succedent Δ since no relational atom disappear from the right-hand side of sequents. But this violates the variable condition of *Eref*. The only possibility left is that the atom xRx is nowhere principal. Then we can remove it everywhere from the derivation, together with the step of *Eref*.

This procedure, combined with an induction on the number of occurrences of *Eref* in the derivation, produces a derivation in **G3K**. □

Therefore we have:

COROLLARY 7.5. *The frame properties of irreflexivity, intransitivity and its generalization, and existence of a reflexive world do not have any modal correspondent.*

Proof. By the conservativity theorems, there is no modal formula that can be proved in the systems extended with the above frame properties that could not be proved in the ground system. □

8. Related Work

When this work was essentially completed we were informed of the related development of Viganò (2000), where sequent systems with internalized

Kripke semantics are presented in the style of labelled deductive systems. There are some major points where our approach differs from Viganò's: first, the use of a contraction-free sequent calculus gives us straightforward decision procedures; in contrast, a partial elimination of contraction is a major issue in Viganò's approach, and it is limited to only some of the systems considered; second, our method for converting axioms into rules permits to treat frame properties beyond the limitation to the Harrop class (Horn clauses) of Viganò (2000).

Further recent related work on labelled systems for modal logic, by Castellini and Smaill (Castellini and Smaill, 2002; Castellini, 2005) was pointed out to us after the present article was submitted. In this work, arbitrary first-order frame properties are reduced to universal axioms using Skolemization. Then universal axioms A are turned into rules by a procedure called "strengthening" that consists in decomposing the sequent $A, \Gamma \Rightarrow \Delta$ by the logical rules and taking as premisses of the rule the leaves of the tree, and as conclusion the sequent $\Gamma \Rightarrow \Delta$. This is equivalent to a cut with the sequent $\Rightarrow A$.

A translation of modal axioms into rules in an unlabelled style can be found in Kracht (1996), where extensions of a modal display calculus are investigated.

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In the summer of 2004 Grisha Mints suggested to me to find some characterization of the accessibility relation for Noetherian frames as a direct justification of the modality rules for provability logic. By the characterization given in Section 5, not only the right rule for $\Box$ of **G3GL**, but also the left rule, depart from those of **G3K**. This turned out particularly useful for correcting a gap in an earlier proof of cut elimination for **G3GL** that used the standard $L\Box$ rule. I thank Silvio Valentini for a discussion that helped me see the gap, and Roy Dyckhoff for his constant interest in this work, pointers to recent literature, and for substantial help with some of the proofs in Section 5. The definition of range was suggested by Jan von Plato.

NOTES

[1] A formal definition of subformula of a labelled formula is given in Section 6.

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