PROOF ANALYSIS FOR LEWIS COUNTERFACTUALS

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Abstract. A deductive system for Lewis counterfactuals is presented, based directly on the influential generalisation of relational semantics through ternary similarity relations introduced by Lewis. This deductive system builds on a method of enriching the syntax of sequent calculus by labels for possible worlds. The resulting labelled sequent calculus is shown to be equivalent to the axiomatic system **VC** of Lewis. It is further shown to have the structural properties that are needed for an analytic proof system that supports root-first proof search. Completeness of the calculus is proved in a direct way, such that for any given sequent either a formal derivation or a countermodel is provided; it is also shown how finite countermodels for unprovable sequents can be extracted from failed proof search, by which the completeness proof turns into a proof of decidability.

§1. Introduction. Counterfactual conditionals have been of interest in Philosophy ever since the classic accounts of Chisholm (1946) and Goodman (1947), which were motivated by the large role that counterfactual reasoning appears to play in ordinary life and scientific inquiry. Counterfactuals still play a crucial role in current theories of causation, natural laws, in epistemology, and in metaphysics. In English, counterfactuals are usually expressed by subjunctive conditionals; to use the famous example by Lewis (1973), followed by his proposed two-place connective for the counterfactual conditional:

If kangaroos had no tails, they would topple over.
Kangaroos have no tail $\Box\!\!\rightarrow$ *Kangaroos topple over*

Counterfactuals share some properties with other kinds of conditionals, most notably the material conditional and the strict conditional introduced by C. I. Lewis. For instance, they all validate modus ponens and modus tollens. Yet the logic of counterfactuals can't be simply analyzed in terms of either. Transitivity (Hypothetical Syllogism) is validated by material and strict conditionals, but fails for counterfactuals:

If Hoover had been a Communist, he would have been a traitor.
If Hoover had been born in Russia, he would have been a Communist.

∴ If Hoover had been born in Russia, he would have been a traitor.

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The first sentence is intuitively true: had Hoover been a (secret) Communist, then since he was head of the FBI, he would have been a traitor. The second sentence is also true: had Hoover been a Russian, involved in politics as he was, it can be credibly assumed that he would have been a member of the Party. It doesn't seem, however, that being born on Russian soil would have made Hoover a traitor. This Cold-War era example is taken from Stalnaker's (1968) pioneering work in the semantics of counterfactuals. Stalnaker (1968) and Lewis (1971, 1973, 1973a) developed independently what is now considered - aside from some important distinctions - the classic analysis of counterfactuals as "variable strict conditionals" (the expression is from Lewis). Informally, a counterfactual $A \mathrel{\Box\!\!\rightarrow} B$ is true under this analysis if and only if B is true in all possible worlds in which A is true that are most similar ("closest") to the world of evaluation. As for the Hoover case, Transitivity fails because of the variability of the similarity relation: intuitively, in all worlds closest to our world in which Hoover is a Russian, he is a Communist, and in all worlds closest to our world in which he is a Communist, he is a traitor, but it is not the case that all worlds closest to our world in which he is a Russian must be worlds in which he is a traitor. For instance, he could have been born in Russia and lived as a good Communist.

The difference between Stalnaker's and Lewis's system is that Stalnaker makes, but Lewis does not, two crucial stipulations: (i) the Uniqueness Assumption that there is at most one closest world to the world of evaluation; (ii) the Limit Assumption that there is at least one closest world to the world of evaluation.[1] This makes for some subtle logical distinctions, to which we will turn below, by way of concluding remarks. In this paper, we concern ourselves only with Lewis's account.

The classic analysis of counterfactuals is made possible by a Kripke-style relational semantics for modal logic,[2] one that Lewis augmented with a "similarity" metrics - in effect a topological condition on a frame. The advent of Kripke frames marked a decisive turning point for philosophical logic: earlier axiomatic studies of modal concepts were replaced by a solid semantic method that displayed the connections between modal axioms and conditions on the accessibility relation between possible worlds. However, the success of the semantic methods has not been followed by equally powerful syntactic theories of modal and conditional concepts and reasoning, and this includes the case of counterfactuals: Concerning the former, the situation was depicted by Melvin Fitting in his article (2007) in the Handbook of Modal Logic as: "No proof procedure suffices for every normal modal logic determined by a class of frames"; In the chapter on tableau systems for conditional logics, Graham Priest states that there are presently no known tableau systems for Lewis's logic for counterfactuals (2008, p. 93). The present paper is a contribution to fill in this gap by providing a proof-theoretic investigation of Lewis counterfactuals.[3]

In her article *Proof analysis in modal logic*, the first author showed in 2005 how Kripke semantics can be exploited to enrich the syntax of systems of proof to answer to the challenge of developing analytic proof systems for standard systems of modal logic.

1 Formally, the limit assumption states that for every world w and every proposition A there is a world that satisfies A (in short, A-world) and which is closest to w. Lewis showed through an example (Lewis 1973, p. 20) that for some propositions A a closest A-world does not exist. If l is a given segment of length arbitrarily close to, say, one inch, and A is the proposition stating that the length of l is greater than one inch, there are worlds w_n where the length of l is $1 + 1/n$ inches, but there is no world closest to the actual one where A is satisfied.

2 Cf. Kripke (1963).

3 From now on, we adopt the convention of writing "Lewis counterfactual" for "Lewis's counterfactual" much in the same style as "Kripke semantics."

The approach has been extended to wider frame classes in later work (Negri 2014), and in Dyckhoff & Negri (2015) it was shown how the method can capture any nonclassical logic characterized by arbitrary first-order frame conditions in their relational semantics.[4] Notably, in these calculi, all the rules are invertible and a strong form of completeness holds for them, with a simultaneous construction of formal proofs, for derivable sequents, or countermodels, for underivable ones (Negri 2014a).

In particular, it has turned out that a more expressive language, with a formal notation of labels to represent possible worlds, is the crucial component in answering to the challenge, set by the successful semantic methods, for a general proof theory of philosophical logic. The way the rules of a labelled sequent calculus with good structural properties are obtained from the semantic explanation of the logical constants is detailed for intuitionistic and modal logic in Negri & von Plato (2015).[5]

The whole methodology is here extended to obtain analogous results for Lewis counterfactuals. As mentioned above, Lewis's truth conditions for counterfactuals are formulated in terms of a notion of similarity that is defined by a ternary relation among possible worlds. Formally, Lewis's account is based on sphere semantics, a special form of neighbourhood semantics for conditionals. In this semantics, there is an $\exists\forall$ nesting of quantifiers in the truth conditions for the counterfactual conditional, which makes the determination of the rules of the calculus an interesting and challenging task. Our solution uses indexed modalities, which allow the splitting of the semantic clause in two separate parts, and correspondingly, the dependence of the rules for the counterfactual conditional on rules for the indexed modality, which are standard modal labelled rules. The result is a system, called **G3LC** below, which is a sound and complete Gentzen-style sequent calculus for the original Lewis counterfactual. The system has all the structural rules—weakening, contraction, and cut—admissible, and all its rules are invertible.

Further comments on the characteristic properties of our approach in the context of a broader discussion of previous proposals for the proof theory of counterfactuals are postponed to the conclusion.

We introduce the system **G3LC** in Section 2 and then briefly discuss its extension to a system provided with alethic modalities. In Section 3, we present some interesting structural properties of **G3LC** and in particular a cut elimination theorem. Then we show (Section 4) that Lewis's axioms and rules are, respectively, admissible and derivable, which allows us to show that the calculus is complete (by soundness and by Lewis's own proof of completeness). Finally, we prove directly completeness and decidability results in Section 5. Related literature and further work are discussed in the concluding section.

§2. A sequent calculus for Lewis conditional. In this section, we shall present a labelled sequent calculus for Lewis conditional starting from Lewis's own semantics based on a ternary relation (Lewis 1973, 1973a).

On the one hand the formulation of an analytic calculus for the Lewis counterfactual is not an ad hoc process, but can be regarded as descending from an established methodology, that we shall briefly summarize below. On the other hand, we are here considering a much larger phenomenon that requires an extension of the methodology to account

[4] Frame conditions beyond first-order formulations, such as Noetherianity, are covered in Negri (2005) and Dyckhoff and Negri (2013).

[5] See also the discussion in Read (2015), who shows—against charges of impurity—how the rules of a labelled system obey the principles of inferentialism in no lesser way than the rules of traditional Gentzen systems.

for modalities—beyond the standard alethic ones—the explanation of which, in terms of possible words, contains a nesting of quantifiers. Before diving into the subtleties of the extension, we shall present the main ideas of the method.

The methodology for obtaining analytic sequent calculi for logics endowed with a relational semantics (first presented in Negri 2005) is based on an internalization of the semantics into the syntax of the calculus: to start with, the language is extended by labelled formulas, of the form $x : A$, and by expressions of the form xRy. Labelled formulas $x : A$ correspond to the statement that A is true at node/possible world x; expressions of the form xRy correspond to relations between nodes/possible worlds in a frame. Once the language has been extended, the way from meaning to rules goes along the following stages:[6] first, the compositional semantic clauses that define the truth of a formula at a world on the basis of the truth of its components are translated into natural deduction inference rules for labelled expressions. The logical structure of the semantic condition is replaced by the structure of the rules, and quantification over worlds is replaced by the condition that certain variables in the rules (eigenvariables) should be fresh. Secondly, through the use of inversion principles one finds the corresponding elimination rules. As a third stage, such rules are appropriately converted into sequent calculus rules; the translation from natural deduction to sequent calculus will produce almost automatically a sequent calculus with independent contexts, with right rules corresponding to the introduction rules of natural deduction and the left rules corresponding to elimination rules. As a fourth stage the calculus thus obtained is refined into a G3-style sequent calculus with invertible rules and good structural properties. The characteristic frame properties that distinguish the modal system at hand are converted into rules for the relational part of the calculus following the method of translation of axioms into sequent calculus rules introduced in Negri & von Plato (1998) for universal axioms and further developed in Negri (2003) and in Negri & von Plato (2011) for geometric ones. In this way, the frame properties are carried over to the calculus by the addition of rules for binary accessibility relations regarded as binary atomic predicates with the labels as arguments. The same generality of relational semantics is then achieved by proof systems. In the case of normal modal logics the method allows one to obtain analytic sequent calculi for any system characterised by arbitrary first-order frame conditions (Dyckhoff & Negri 2015).

The truth conditions for Lewis's conditional are spelled out in terms of a three-place similarity relation $\preceq$ among worlds, with the intuitive meaning of "$x \preceq_w y$" being "x is at least as similar to w as y is".

Lewis's truth conditions for the counterfactual conditional are as follows:

$w \Vdash A \Box\!\!\rightarrow B$ iff either

1. There is no x such that $x \Vdash A$, or
2. there is an x such that $x \Vdash A$ and for all y, if $y \preceq_w x$ then $y \Vdash A \supset B$.

As previously anticipated, the truth condition for $A \Box\!\!\rightarrow B$ has a universal quantification in the scope of an existential, and thus it is not of a form that can be directly translated into rules following the method of generation of labelled sequent rules for intensional operators (as expounded in Negri 2005); a more complex formalism in the line of the method of *systems of rules* (cf. Negri 2014) would have to be invoked to maintain the primitive language.

[6] All these stages from meaning to rules are detailed in the case of intuitionistic and standard modal logics in Negri & von Plato (2015).

We observe however that the rules for the labelled calculus for Lewis's conditional can be presented following a general method for embedding the neighbourhood semantics for non-normal modal logics into the standard relational semantics for normal modal systems through the use of *indexed modalities*. The method is formulated in general terms in Gasquet & Herzig (1996) for classical modal logics and used in Giordano *et al.* (2008) for obtaining a tableau calculus for preference-based conditional logics.[7] Specifically, the relation of similarity is used to define a ternary accessibility relation

$$xR_w y \equiv y \preceq_w x\,.$$

In turn, this relation defines an indexed necessity modality with the truth condition

$$x \Vdash \Box_w A \text{ iff for all } y,\ \text{if } xR_w y \text{ then } y \Vdash A\,.$$

Then the truth condition for the conditional gets replaced by the following

$w \Vdash A \Box\!\!\rightarrow B$ iff either

1. There is no x such that $x \Vdash A$, or
2. there is x such that $x \Vdash A$ and $x \Vdash \Box_w(A \supset B)$.

Observe that the presentation of a calculus formulated in terms of indexed modalities is directly faithful to Lewis's original idea of conditional implication as a variably strict conditional.

The sequent system is obtained as an extension of the propositional part of the contraction and cut-free sequent calculus G3K for basic modal logic introduced in Negri (2005). In addition there are rules for the similarity and the equality relation. For the latter, we have just two rules, reflexivity and the scheme of *replacement*, $Repl_{At}$, where $At(x)$ stands for an atomic labelled formula $x : P$ or a relation of the form $y = z$, $yR_w z$, with x one of y, w, z. Symmetry of equality follows as a special case of $Repl_{At}$ as well as *Euclidean transitivity* which, together with symmetry, gives the usual *transitivity*.[8]

The following properties are generally assumed for the ternary relation[9]:

1. *Transitivity*: If $x \preceq_w y$ and $y \preceq_w z$ then $x \preceq_w z$,
2. *Strong connectedness*: Either $x \preceq_w y$ or $y \preceq_w x$,
3. $\preceq_w$-*Minimality*: If $x \preceq_w w$ then $x = w$.

These correspond to the following sequent calculus rules:

$$\frac{x \preceq_w z, x \preceq_w y, y \preceq_w z, \Gamma \Rightarrow \Delta}{x \preceq_w y, y \preceq_w z, \Gamma \Rightarrow \Delta}\,\textit{Trans} \qquad \frac{x \preceq_w y, \Gamma \Rightarrow \Delta \quad y \preceq_w x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\textit{SConn}$$

$$\frac{x = w, \Gamma \Rightarrow \Delta}{x \preceq_w w, \Gamma \Rightarrow \Delta}\,\textit{LMin}$$

7 Further discussion on indexed modalities is postponed to the conclusion.

8 The general reasons for the architecture behind the rules of equality are discussed in section 6.5 of Negri & von Plato (2001) for extensions of first-order systems and the equality rules for labelled systems in Negri (2005) and Negri & von Plato (2011).

9 Lewis defined a strict relation $\prec_w$ as $x \prec_w y \equiv \neg y \preceq_w x$ and the condition of minimality was formulated as: World w is $\prec_w$-minimal, that is, for every x distinct from w we have $w \prec_w x$. We have preferred to give all the conditions in terms of just the primitive relation and use *LMin* as a name of the corresponding rule-scheme.

The intuitive meaning of the rules for the similarity relation is obtained directly by rephrasing them in terms of the meaning of the ternary similarity relation. For example, $\preceq_w$-Minimality asserts that every world is as close as possible to itself.

Strongly connected similarity systems are immediately seen to be reflexive. It follows that the rule corresponding to reflexivity of the relation R_w

$$\frac{x R_w x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, \mathit{RefR}\ (x,\, w\ in\ \Gamma, \Delta)$$

need not be taken as a primitive rule of the calculus because it is derivable, simply by writing its premiss twice and applying *SConn*. This admissible rule will be useful to simplify the derivation of the third of the Lewis axioms below.

With this preparation, the rules for Lewis conditional can be obtained from its truth condition following the general method for turning the truth conditions of alethic modalities into rules of a labelled sequent calculus. Since the truth condition for the Lewis conditional is a disjunction of two conditions, there are two right rules and one left rule with two premisses.

$$\frac{z : A, \Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B}\, R\Box\!\!\rightarrow_1\ (z\ \mathit{fresh})$$

$$\frac{\Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B, x : A \qquad \Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B, x : \Box_w(A \supset B)}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B}\, R\Box\!\!\rightarrow_2$$

$$\frac{w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta, z : A \qquad x : A, x : \Box_w(A \supset B), \Gamma \Rightarrow \Delta}{w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta}\, L\Box\!\!\rightarrow\ (x\ \mathit{fresh})$$

The rules for the propositional part are standard and the rules for indexed modalities are very much like the rules for the alethic modality. The system is presented in Table 1.

We can now state a couple of important results that will be needed below. First we need a definition of weight of formulas:

DEFINITION 2.1. *The weight* $\mathtt{w}(A)$ *of a formula* A *is defined inductively by the following:*
$\mathtt{w}(\gamma) = 1$ *for* γ *the constant* $\bot$*, an atomic formula, or a relational atom,*
$\mathtt{w}(A \circ B) = \mathtt{w}(A) + \mathtt{w}(B) + 1$ *for* $\circ$ *conjunction, disjunction, or implication,*
$\mathtt{w}(\Box_x A) = \mathtt{w}(A) + 1$,
$\mathtt{w}(A \Box\!\!\rightarrow B) = \mathtt{w}(A) + \mathtt{w}(B) + 3$.

Observe that since we have taken negation $\neg A$ as defined by $A \supset \bot$, we have $\mathtt{w}(\neg A) \equiv \mathtt{w}(A) + 2$. Observe also that $\mathtt{w}(\Box_x(A \supset B)) < \mathtt{w}(A \Box\!\!\rightarrow B)$.

We can further prove the following lemma:

LEMMA 2.2. *All the sequents of the form* $x : A, \Gamma \Rightarrow \Delta, x : A$ *are derivable in* **G3LC**.

Proof. By induction on the weight of A. The base cases, with weight 1, hold because $x : A, \Gamma \Rightarrow \Delta, x : A$ is either an initial sequent or conclusion of $L\bot$. The inductive steps, the propositional cases as well as the case of $A \equiv \Box_w A$ are obtained by root-first application of the right and left rules of the outermost connective of A and the inductive hypothesis. If A is $B \Box\!\!\rightarrow C$ we have the following derivation, with the leaves derivable by inductive hypothesis:

$$\dfrac{\dfrac{z : B, x : B \Box\!\!\rightarrow C, \Gamma \Rightarrow \Delta, x : B \Box\!\!\rightarrow C, z : B \qquad \dfrac{y : B, y : \Box_x(B \supset C), z : B, \Gamma \Rightarrow \Delta, x : B \Box\!\!\rightarrow C, y : B \quad y : B, y : \Box_x(B \supset C), z : B, \Gamma \Rightarrow \Delta, x : B \Box\!\!\rightarrow C, y : \Box_x(B \supset C)}{y : B, y : \Box_x(B \supset C), z : B, \Gamma \Rightarrow \Delta, x : B \Box\!\!\rightarrow C}\, R\Box\!\!\rightarrow_2}{z : B, x : B \Box\!\!\rightarrow C, \Gamma \Rightarrow \Delta, x : B \Box\!\!\rightarrow C}\, L\Box\!\!\rightarrow}{x : B \Box\!\!\rightarrow C, \Gamma \Rightarrow \Delta, x : B \Box\!\!\rightarrow C}\, R\Box\!\!\rightarrow_1$$

□

Table 1. *Lewis counterfactual conditional sequent system* (**G3LC**) *based on ternary similarity*

Initial sequents:

$x : P, \Gamma \Rightarrow \Delta, x : P$

Propositional rules:

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A\&B, \Gamma \Rightarrow \Delta}\, L\& \qquad \frac{\Gamma \Rightarrow \Delta, x : A \quad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A\&B}\, R\&$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \quad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta}\, L\vee \qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}\, R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad x : B, \Gamma \Rightarrow \Delta}{x : A \supset B, \Gamma \Rightarrow \Delta}\, L\supset \qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \supset B}\, R\supset$$

$$\frac{}{x : \bot, \Gamma \Rightarrow \Delta}\, L\bot$$

Similarity rules:

$$\frac{xR_wz, xR_wy, yR_wz, \Gamma \Rightarrow \Delta}{xR_wy, yR_wz, \Gamma \Rightarrow \Delta}\, \mathit{Trans} \qquad \frac{xR_wy, \Gamma \Rightarrow \Delta \quad yR_wx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, \mathit{SConn}\ (x, y, w\ in\ \Gamma, \Delta)$$

$$\frac{x = x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, \mathit{Ref}\ (x\ in\ \Gamma, \Delta) \qquad \frac{x = y, At(x), At(y), \Gamma \Rightarrow \Delta}{x = y, At(x), \Gamma \Rightarrow \Delta}\, \mathit{Repl}_{At}$$

$$\frac{x = w, wR_wx, \Gamma \Rightarrow \Delta}{wR_wx, \Gamma \Rightarrow \Delta}\, \mathit{LMin}$$

Conditional rules:

$$\frac{xR_wy, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box_w A}\, R\Box_w\ (y\ \mathit{fresh}) \qquad \frac{xR_wy, x : \Box_w A, y : A, \Gamma \Rightarrow \Delta}{xR_wy, x : \Box_w A, \Gamma \Rightarrow \Delta}\, L\Box_w$$

$$\frac{z : A, \Gamma \Rightarrow \Delta, w : A \,\Box\!\!\rightarrow B}{\Gamma \Rightarrow \Delta, w : A \,\Box\!\!\rightarrow B}\, R\Box\!\!\rightarrow_1\ (z\ \mathit{fresh})$$

$$\frac{\Gamma \Rightarrow \Delta, w : A \,\Box\!\!\rightarrow B, x : A \quad \Gamma \Rightarrow \Delta, w : A \,\Box\!\!\rightarrow B, x : \Box_w(A \supset B)}{\Gamma \Rightarrow \Delta, w : A \,\Box\!\!\rightarrow B}\, R\Box\!\!\rightarrow_2$$

$$\frac{w : A \,\Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta, z : A \quad x : A, x : \Box_w(A \supset B), \Gamma \Rightarrow \Delta}{w : A \,\Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta}\, L\Box\!\!\rightarrow\ (x\ \mathit{fresh})$$

2.1. Adding the alethic modality. We can consider also alethic modalities defined in terms of a suitable binary accessibility relation R defined in terms of the given ternary relation. Building on Lewis's idea[10] that worlds accessible from x are worlds in the union of the set of spheres around x, and using the relation between spheres and ternary relations,

[10] Cf. *Counterfactuals*, pp. 22–23, where such modalities—called *outer modalities*—are defined through the conditional and characterized semantically through systems of spheres.

we establish the following equivalence:

$$xRy \equiv zR_x y \textit{ for some } z$$

By standard prenex conversions we thus have that the alethic modality defined by R is expressed, in terms of the ternary accessibility relations, as follows:

$$x \Vdash \Box A \text{ iff } \textit{for all } z \textit{ and } y, \textit{ if } zR_x y \textit{ then } y \Vdash A$$

The forcing condition gives the following rules for $\Box$ in terms of the ternary accessibility relation

$$\dfrac{y : A, x : \Box A, zR_x y, \Gamma \Rightarrow \Delta}{x : \Box A, zR_x y, \Gamma \Rightarrow \Delta}\, L\Box \qquad\qquad \dfrac{zR_x y, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}\, R\Box \ (y, z \textit{ fresh})$$

As usual for systems with a classical base, $\Diamond$ does not have to be treated separately since it can defined in terms of $\Box$ and negation.

The definition of necessity in terms of conditional

$$\Box A \equiv \neg A \mathrel{\Box\!\!\rightarrow} A$$

can now be formally derived as an equivalence in the calculus **G3LC** extended with the above two rules as follows (for brevity, we omit to copy the main formula in the premisses). In one direction we have

$$\dfrac{\dfrac{z : \neg A \Rightarrow z : \neg A \qquad \dfrac{\dfrac{\dfrac{zR_x y, y : \neg A, x : \Box A, y : A \Rightarrow y : A}{zR_x y, y : \neg A, x : \Box A \Rightarrow y : A}\, L\Box}{zR_x y, x : \Box A \Rightarrow y : \neg A \supset A}\, R\supset}{x : \Box A \Rightarrow z : \Box_x(\neg A \supset A)}\, R\Box_x}{z : \neg A, x : \Box A \Rightarrow x : \neg A \mathrel{\Box\!\!\rightarrow} A}\, R\Box\!\!\rightarrow_2}{x : \Box A \Rightarrow x : \neg A \mathrel{\Box\!\!\rightarrow} A}\, R\Box\!\!\rightarrow_1$$

where the topmost sequents are derivable by Lemma 2.2. For the converse direction we have

$$\dfrac{\dfrac{\dfrac{zR_x y, y : A, \Rightarrow y : A}{zR_x y \Rightarrow y : A, y : \neg A}\, R\neg \qquad \dfrac{\dfrac{zR_x y, wR_x w, w : \neg A, w : \neg A \supset A \Rightarrow y : A}{zR_x y, wR_x w, w : \neg A, w : \Box_x(\neg A \supset A) \Rightarrow y : A}\, L\Box_x}{zR_x y, w : \neg A, w : \Box_x(\neg A \supset A) \Rightarrow y : A}\, \textit{RefR}}{zR_x y, x : \neg A \mathrel{\Box\!\!\rightarrow} A \Rightarrow y : A}\, L\Box\!\!\rightarrow}{x : \neg A \mathrel{\Box\!\!\rightarrow} A \Rightarrow x : \Box A}\, R\Box$$

where the topmost sequents are derivable, the left one by Lemma 2.2 and the right one by a step of $L\supset$ and Lemma 2.2. The similar proof of the equivalence

$$\Diamond A \equiv \neg(A \mathrel{\Box\!\!\rightarrow} \neg A)$$

is left to the reader.

The above rules for $\Box$ are found in full generality. In the case we are considering the rules can be simplified because R_x is a reflexive relation and therefore R then turns out to be total; the necessity modality then becomes the universal modality and the rules are simply obtained by removing the accessibility relation.

$$\dfrac{y : A, x : \Box A, \Gamma \Rightarrow \Delta}{x : \Box A, \Gamma \Rightarrow \Delta}\, L\Box \qquad\qquad \dfrac{\Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \Box A}\, R\Box \ (y \textit{ fresh})$$

If the analysis of counterfactuals is embedded in a Kripke frame for modalities, then the truth conditions are the following[11]:

$w \Vdash A \,\Box\!\!\rightarrow B$ iff either

1. There is no z such that wRz and $z \Vdash A$, or
2. there is x such that wRx and $x \Vdash A$ and for all y, if $y \preceq_w x$ then $y \Vdash A \supset B$.

Lewis also considers another possibility for the definition of an alethic modality in terms of counterfactuals[12], namely what he calls *inner modality* and denotes by $\boxdot A$, with the condition of being true at a world w if there is a non-empty sphere around w where A is true in every world. In our setting, in terms of similarity systems, the condition becomes

$$x \Vdash \boxdot A \textit{ iff there is } y \textit{ such that for all } z\textit{, if } yR_x z \textit{ then } z \Vdash A$$

and the corresponding rules are then easily formulated in terms of indexed modalities

$$\frac{y : \Box_x A, \Gamma \Rightarrow \Delta}{x : \boxdot A, \Gamma \Rightarrow \Delta}\, L\boxdot \; (y \textit{ fresh}) \qquad \frac{\Gamma \Rightarrow \Delta, x : \boxdot A, y : \Box_x A}{\Gamma \Rightarrow \Delta, x : \boxdot A}\, R\boxdot$$

The equivalence of $\boxdot A$ and $\top \,\Box\!\!\rightarrow A$ is then shown by the following derivations

$$\cfrac{\ldots \Rightarrow z : \top \qquad \cfrac{y : \Box_x(\top \supset A) \Rightarrow x : \boxdot A, y : \Box_x A}{y : \Box_x(\top \supset A) \Rightarrow x : \boxdot A}\, L\boxdot}{x : \top \,\Box\!\!\rightarrow A \Rightarrow x : \boxdot A}\, L\Box\!\!\rightarrow$$

and

$$\cfrac{\cfrac{\ldots \Rightarrow z : \top \qquad y : \Box_x A \Rightarrow y : \Box_x(\top \supset A)}{y : \Box_x A \Rightarrow x : \top \,\Box\!\!\rightarrow A}\, R\Box\!\!\rightarrow}{x : \boxdot A \Rightarrow x : \top \,\Box\!\!\rightarrow A}\, L\boxdot$$

where the topsequents are easily derivable.

For the sake of simplicity, we shall refer below only to the version of the system without an accessibility relation, but the same results can be obtained, *mutatis mutandis* for the full systems. To consider the system without an accessibility relation is equivalent to assuming that the accessibility relation is total, i.e. that all the worlds are accessible as in the case here considered. The analysis thus becomes relevant when one considers weaker conditions on the similarity system.

§3. Structural properties. The proof of admissibility of the structural rules in **G3LC** follows the pattern presented in Negri & von Plato (2011), section 11.4. Likewise, some preliminary results are needed, namely height-preserving admissibility of substitution (in short, hp-substitution) and height-preserving invertibility (in short, hp-invertibility) of the rules. We recall that the *height* of a derivation is its height as a tree, i.e. the length of its longest branch, and that $\vdash_n$ denotes derivability with derivation height bounded by n in a given system. In what follows, the results are all referred to system **G3LC**. The definition of substitution of labels is obtained by extending in the obvious way the clauses given in the definition of substitution given in the aforementioned reference to the ternary relations and the indexed modalities of **G3LC**.

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[11] Cf. *Counterfactuals*, p. 49.

[12] Cf. *Counterfactuals*, p. 30.

PROPOSITION 3.1. *If* $\vdash_n \Gamma \Rightarrow \Delta$, *then* $\vdash_n \Gamma(y/x) \Rightarrow \Delta(y/x)$.

Proof. By induction on the height of the derivation. If it is 0, then $\Gamma \Rightarrow \Delta$ is an initial sequent or a conclusion of $L\bot$. Then the same is true of $\Gamma(y/x) \Rightarrow \Delta(y/x)$. If the derivation height is $n > 0$, we consider the last rule in the derivation. If $\Gamma \Rightarrow \Delta$ has been derived by a propositional or a relational rule, or by $L\Box_w$ or $R \Box\!\!\rightarrow_2$, we apply the induction hypothesis and then the rule. Rules with variable conditions require that we avoid a clash of the substituted variable with the fresh variable in the premiss. This is the case of $R\Box_w$, $R \Box\!\!\rightarrow_1$, and $L \Box\!\!\rightarrow$. So, if $\Gamma \Rightarrow \Delta$ has been derived by any of these rules, we apply the inductive hypothesis twice to the premiss, first to replace the fresh variable with another fresh variable different, if necessary, from the one we want to substitute, then to make the substitution, and then apply the rule. □

PROPOSITION 3.2. *The rules of left and right weakening are hp-admissible in* **G3LC**.

Proof. Straightforward induction, with a similar proviso as in the above proof for rules with variable conditions. □

Next, we prove *hp-invertibility* of the rules of **G3LC**, i.e. for every rule of the form $\frac{\Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta}$, if $\vdash_n \Gamma \Rightarrow \Delta$ then $\vdash_n \Gamma' \Rightarrow \Delta'$, and for every rule of the form $\frac{\Gamma' \Rightarrow \Delta' \quad \Gamma'' \Rightarrow \Delta''}{\Gamma \Rightarrow \Delta}$ if $\vdash_n \Gamma \Rightarrow \Delta$ then $\vdash_n \Gamma' \Rightarrow \Delta'$ and $\vdash_n \Gamma'' \Rightarrow \Delta''$.

LEMMA 3.3. *All the propositional rules of* **G3LC** *are hp-invertible.*

Proof. Similar to the proof of Lemma 11.7 in Negri & von Plato (2011). □

LEMMA 3.4. *The following hold:*

(i) *If* $\vdash_n \Gamma \Rightarrow \Delta, x : \Box_w A$, *then* $\vdash_n xR_w y, \Gamma \Rightarrow \Delta, y : A$,

(ii) *If* $\vdash_n w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta$, *then* $\vdash_n x : A, x : \Box_w(A \supset B), \Gamma \Rightarrow \Delta$.

Proof. Both proofs are by induction on n.

(i) *Base case*: suppose that $\Gamma \Rightarrow \Delta, x : \Box_w A$ is an initial sequent or conclusion of $L\bot$. Then, in the former case, $x : \Box_w A$ not being atomic, $xR_w y, \Gamma \Rightarrow \Delta, y : A$ is an initial sequent, in the latter it is a conclusion of $L\bot$. *Inductive step*: assume hp-invertibility up to n, and assume $\vdash_{n+1} \Gamma \Rightarrow \Delta, x : \Box_w A$. If $x : \Box_w A$ is principal, then the premiss $xR_w y, \Gamma \Rightarrow \Delta, y : A$ (possibly obtained through hp-substitution) has a derivation of height n. Otherwise, it has one or two premisses of the form $\Gamma' \Rightarrow \Delta', x : \Box_w A$ of derivation height $\leq n$. By induction hypothesis we have $xR_w y, \Gamma' \Rightarrow \Delta', y : A$ for each premiss, with derivation height at most n. Thus, $\vdash_{n+1} xR_w y, y : A, \Gamma \Rightarrow \Delta, y : A$.

(ii) *Base case*: Similar to the above. *Inductive step*: assume hp-invertibility up to n, and assume $\vdash_{n+1} w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta$. If $w : A \Box\!\!\rightarrow B$ is principal in the last rule, then the left premiss is, or gives by hp-substitution on the eigenvariable, a derivation of height at most n of $x : A, x : \Box_w(A \supset B), \Gamma \Rightarrow \Delta$. If the last rule is a relational rule, or a logical rule in which $w : A \Box\!\!\rightarrow B$ is not principal, consider the one or two premisses of the form $w : A \Box\!\!\rightarrow B, \Gamma' \Rightarrow \Delta'$ of derivation height $\leq n$. Then, by induction hypotesis, we have at most $\vdash_n x : A, x : \Box_w(A \supset B), \Gamma' \Rightarrow \Delta'$ for each premiss. By application of the last rule, the conclusion $\vdash_{n+1} x : A, x : \Box_w(A \supset B), \Gamma \Rightarrow \Delta$ follows. □

Observe that Lemma 3.4(ii) states hp-invertibility of $L \Box\!\!\rightarrow$ with respect to the second premiss; its hp-invertibility with respect to the first premiss is a special case of Proposition 3.2. Therefore, as a general result, we have:

COROLLARY 3.5. *All the rules of* **G3LC** *are hp-invertible.*

Proof. By Lemmas 3.3 and 3.4, and Proposition 3.2 for all the other cases. □

The rules of contraction of **G3LC** have the following form, where ϕ is either a relational atom of the form $x R_w y$ or a labelled formula $x : A$:

$$\frac{\phi, \phi, \Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta}\, LC \qquad \frac{\Gamma \Rightarrow \Delta, \phi, \phi}{\Gamma \Rightarrow \Delta, \phi}\, RC$$

Since relational atoms never appear on the right, there are just three contraction rules to be considered. We do not need to give different names for these rules since we can prove that all of them are hp-admissible:

THEOREM 3.6. *The rules of left and right contraction are hp-admissible in* **G3LC**.

Proof. By simultaneous induction on the height of derivation for left and right contraction.

If $n = 0$ the premiss is either an initial sequent or a conclusion of a zero-premiss rule. In each case, the contracted sequent is also an initial sequent or a conclusion of the same zero-premiss rule.

If $n > 0$, consider the last rule used to derive the premiss of contraction. If the contraction formula is not principal in it, both occurrences are found in the premisses of the rule and they have a smaller derivation height. By the induction hypothesis, they can be contracted and the conclusion is obtained by applying the rule to the contracted premisses. If the contraction formula is principal in it, we distinguish three cases: 1. A rule in which the principal formulas appear also in the premiss (such as *Trans*, *SConn*, *LMin*, the left rule for $\Box_w$ or the rules for $\Box\!\!\rightarrow$). 2. A rule in which the active formulas are proper subformulas of the principal formula (such as the rules for &, $\vee$, $\supset$). 3. A rule in which active formulas are relational atoms and proper subformulas of the principal formula (like rule $R\Box_w$). 4. Rule $L\Box\!\!\rightarrow$.

In the first case we have, for instance,

$$\frac{x R_w y, x : \Box_w A, x : \Box_w A, y : A, \Gamma \Rightarrow \Delta}{x R_w y, x : \Box_w A, x : \Box_w A, \Gamma \Rightarrow \Delta}\, L\Box_w$$

By the induction hypothesis applied to the premiss we obtain

$$x R_w y, x : \Box_w A, y : A, \Gamma \Rightarrow \Delta$$

with height n, and by a step of $L\Box_w$ we have

$$x R_w y, x : \Box_w A, \Gamma \Rightarrow \Delta$$

with height $n + 1$.

In the second case, contraction is reduced to contraction on shorter derivations (and smaller formulas).

In the third case, a subformula of the contraction formula and a relational atom are found in the premiss, for instance

$$\frac{x R_w y, \Gamma \Rightarrow \Delta, x : \Box_w A, y : A}{\Gamma \Rightarrow \Delta, x : \Box_w A, x : \Box_w A}\, R\Box_w$$

By Lemma 3.4 applied to the premiss, we obtain a derivation of height $n - 1$ of

$$x R_w y, x R_w y, \Gamma \Rightarrow \Delta, y : A, y : A$$

that yields, by the induction hypothesis for left and right contraction, a derivation of height $n-1$ of

$$xR_w y, \Gamma \Rightarrow \Delta, y : A$$

and the conclusion $\Gamma \Rightarrow \Delta, x : \Box_w A$ follows in one more step by $R\Box_w$.

Finally, if the premiss of contraction is derived by $L\,\Box\!\!\rightarrow$ we have

$$\dfrac{w : A \Box\!\!\rightarrow B, w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta, z : A \quad x : A, x : \Box_w(A \supset B), w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta}{w : A \Box\!\!\rightarrow B, w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta}\; L\Box\!\!\rightarrow$$

By inductive hypothesis applied to the left premiss we have a derivation of height $n-1$ of $w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta$; by Lemma 3.4 and inductive hypothesis applied to the second premiss we have a derivation of height $n-1$ of $x : A, x : \Box_w(A \supset B), \Gamma \Rightarrow \Delta$. Thus, by a step of $L\,\Box\!\!\rightarrow$ we obtain a derivation of height n of $w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta$. □

THEOREM 3.7. *Cut is admissible in* **G3LC**.

Proof. The proof is by induction on the weight of the cut formula and subinduction on the sum of the heights of derivations of the premisses (cut-height). The cases pertaining initial sequents and the propositional rules of the calculus are dealt with as in Theorem 11.9 of Negri & von Plato (2011) and therefore omitted here. Also the cases with cut formula not principal in both premisses of cut are dealt with in the usual way by permutation of cut, with possibly an application of hp-substitution to avoid a clash with the fresh variable in rules with variable condition. So, the only cases to focus on are those with cut formula of the form $\Box_w A$ or $A \Box\!\!\rightarrow B$ which is principal in both premisses of cut. The former case presents, apart from the indexing on the accessibility relation, no difference with respect to the case of a plain modality, so we proceed to analyse the latter. This case splits into two cases, depending on whether the left premiss is derived by $R\,\Box\!\!\rightarrow_1$ or $R\,\Box\!\!\rightarrow_2$.

In the first case we have a derivation of the form

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{y : A, \Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B}}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B}\; R\Box\!\!\rightarrow_1 \quad \dfrac{\overset{\mathcal{D}_2}{w : A \Box\!\!\rightarrow B, \Gamma' \Rightarrow \Delta', z : A} \quad \overset{\mathcal{D}_3}{y : A, y : \Box_w(A \supset B), \Gamma' \Rightarrow \Delta'}}{w : A \Box\!\!\rightarrow B, \Gamma' \Rightarrow \Delta'}\; L\Box\!\!\rightarrow}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\; Cut$$

This is converted into a derivation with three cuts of reduced height as follows (we have to split the result of the conversion to fit it in the page): First, we have a derivation $\mathcal{D}_4$

$$\dfrac{\overset{\mathcal{D}_1}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B} \quad \overset{\mathcal{D}_2}{w : A \Box\!\!\rightarrow B, \Gamma' \Rightarrow \Delta', z : A}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', z : A}\; Cut$$

Further, by application of hp-substitution, we have another derivation $\mathcal{D}_5$

$$\dfrac{\overset{\mathcal{D}_1(z/y)}{z : A, \Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B} \quad \dfrac{\overset{\mathcal{D}_2}{w : A \Box\!\!\rightarrow B, \Gamma' \Rightarrow \Delta', z : A} \quad \overset{\mathcal{D}_3}{y : A, y : \Box_w(A \supset B), \Gamma' \Rightarrow \Delta'}}{w : A \Box\!\!\rightarrow B, \Gamma' \Rightarrow \Delta'}\; L\Box\!\!\rightarrow}{z : A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\; Cut$$

The two derivations are then used as premisses of a third cut of reduced weight as follows

$$\dfrac{\dfrac{\overset{\mathcal{D}_4}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', z : A} \quad \overset{\mathcal{D}_5}{z : A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}}{\Gamma^2, \Gamma'^2 \Rightarrow \Delta^2, \Delta'^2}\; Cut}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\; Ctr^*$$

In the second case we have a derivation of the form

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B, x : A} \quad \overset{\mathcal{D}_2}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B, x : \Box_w(A \supset B)}}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B}{\scriptstyle R\Box\!\!\rightarrow_2} \quad \dfrac{\overset{\mathcal{D}_3}{w : A \Box\!\!\rightarrow B, \Gamma' \Rightarrow \Delta', z : A} \quad \overset{\mathcal{D}_4}{y : A, y : \Box_w(A \supset B), \Gamma' \Rightarrow \Delta'}}{w : A \Box\!\!\rightarrow B, \Gamma' \Rightarrow \Delta'}{\scriptstyle L\Box\!\!\rightarrow}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{\scriptstyle Cut}$$

The cut is converted into four cuts of reduced height or weight of cut formula as follows: First we have the derivation (call it $\mathcal{D}_5$)

$$\dfrac{\overset{\mathcal{D}_1}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B, x : A} \quad \dfrac{\overset{\mathcal{D}_3}{w : A \Box\!\!\rightarrow B, \Gamma' \Rightarrow \Delta', z : A} \quad \overset{\mathcal{D}_4}{y : A, y : \Box_w(A \supset B), \Gamma' \Rightarrow \Delta'}}{w : A \Box\!\!\rightarrow B, \Gamma' \Rightarrow \Delta'}{\scriptstyle L\Box\!\!\rightarrow}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x : A}{\scriptstyle Cut}$$

with a cut of reduced height. We also have the derivation (call it $\mathcal{D}_6$)

$$\dfrac{\dfrac{\overset{\mathcal{D}_2}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B, x : \Box_w(A \supset B)} \quad \dfrac{\overset{\mathcal{D}_3}{w : A \Box\!\!\rightarrow B, \Gamma' \Rightarrow \Delta', z : A} \quad \overset{\mathcal{D}_4}{y : A, y : \Box_w(A \supset B), \Gamma' \Rightarrow \Delta'}}{w : A \Box\!\!\rightarrow B, \Gamma' \Rightarrow \Delta'}{\scriptstyle L\Box\!\!\rightarrow}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x : \Box_w(A \supset B)}{\scriptstyle Cut} \quad \overset{\mathcal{D}_4(x/y)}{x : A, x : \Box_w(A \supset B), \Gamma' \Rightarrow \Delta'}}{x : A, \Gamma, \Gamma'^2 \Rightarrow \Delta, \Delta'^2}{\scriptstyle Cut}$$

with two cuts, the upper of reduced height, and the lower of reduced weight; finally we obtain the derivation

$$\dfrac{\dfrac{\mathcal{D}_5 \qquad \mathcal{D}_6}{\Gamma^2, \Gamma'^3 \Rightarrow \Delta^2, \Delta'^3}{\scriptstyle Cut}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{\scriptstyle Ctr^*}$$

with a cut of reduced weight and repeated applications of contraction. □

To ensure the consequences of cut elimination we need to establish another crucial property of our system. We say that a labelled system has the *subterm property* if every variable occurring in any derivation is either an eigenvariable or occurs in the conclusion.[13] Clearly, the rules of **G3LC** do not, as they stand, satisfy the subterm property, but we can prove that we can, without loss of generality, restrict proof search to derivations that have the subterm property.

PROPOSITION 3.8. *Every sequent derivable in* **G3LC** *is derivable by a derivation that satisfies the subterm property.*

Proof. By induction on the height of the derivation. For the inductive step, the conclusion follows immediately if the last step is one of the rules in which all the labels in the premisses satisfy the subterm property. For the other rules (in this specific calculus, rules *Ref* and $R \Box\!\!\rightarrow_1$), we consider the violating cases in which the premisses contain a label which is not in the conclusion. Using hp-substitution, we replace it to a label in the conclusion and obtain a derivation of the same height that satisfies the subterm property. □

By the above result, in the following we shall always restrict attention to derivations with the subterm property.

§4. Lewis's axioms and rules. In this section we shall prove that Lewis's axiomatic system for counterfactuals **VC**, regarded as Lewis's *official logic of counterfactuals*, is

[13] This property is called *analyticity* in Dyckhoff and Negri (2012).

captured by our system **G3LC** by showing that Lewis's axioms are provable in **G3LC** and that the inference rules of **VC** are admissible.[14] Together with a proof of soundness of our rules with respect to Lewis's semantics for counterfactuals based on a ternary accessibility relation, this result will give an indirect completeness proof for our system. We shall however present in Section 5 a direct completeness proof for **G3LC** with respect to the same semantics. For brevity, in the following proofs the repetition of the main formula in the premisses is not indicated.

As discussed in natural language examples at the beginning, the counterfactual conditional is not, in general, transitive, and therefore a rule of the form

$$\frac{\vdash D \boxright A \quad \vdash A \boxright B}{\vdash D \boxright B} \tag{1}$$

should not be taken as part of the corresponding axiomatic system. A form of transitivity does however hold, namely

$$\frac{\vdash D \boxright A \quad \vdash A \supset B}{\vdash D \boxright B} \tag{2}$$

Such a rule follows through modus ponens from

$$\frac{\vdash A \supset B}{\vdash (D \boxright A) \supset (D \boxright B)} \tag{3}$$

In the other direction, observe that 3 *does not* follow from 2 by the deduction theorem because the antecedent in question is a derivation, not an assumption.[15] However, something stronger than 2 does hold in a system assumed to be complete with respect to Lewis semantics, namely, for an arbitrary world w we have, directly from the definitions,

$$\frac{w \Vdash D \boxright A \quad \vdash A \supset B}{w \Vdash D \boxright B} \tag{4}$$

In turn, 3 is obtained from 4 by definition of validity of material implication and completeness. The generalisation in 3 to an antecedent of implication formed by an n-ary conjunctions is called *deduction within conditionals* and is one of the three rules of the Hilbert system for Lewis counterfactual. The other rules are modus ponens and interchange of logical equivalents.

PROPOSITION 4.1. *The following rules are admissible in* **G3LC***:*

1. *Modus Ponens:*

$$\frac{\vdash A \quad \vdash A \supset B}{\vdash B}$$

2. *Deduction within Conditionals: for any* $n \geq 1$

$$\frac{\vdash A_1 \& \ldots \& A_n \supset B}{\vdash ((D \boxright A_1) \& \ldots \& (D \boxright A_n)) \supset (D \boxright B)}$$

3. *Interchange of logical equivalents: if* $\vdash A \supset\subset B$ *and* $\vdash \Phi(A)$ *then* $\vdash \Phi(B)$*, where* Φ *is an arbitrary formula in the language.*

[14] Cf. p. 132 of *Counterfactuals*.

[15] For a discussion on these distinctions for the deduction theorem, its validity, and claims of failure, especially in relation to modal systems, see Hakli & Negri (2012).

Proof. (1) The rule of modus ponens of an axiomatic system is translated as the sequent calculus rule

$$\frac{\Rightarrow x : A \quad \Rightarrow x : A \supset B}{\Rightarrow x : B}$$

The rule is shown admissible in **G3LC** using invertibility of $R\supset$ and admissibility of cut.

(2) Assume $\Rightarrow z : A_1 \& \ldots \& A_n \supset B$ for an arbitrary z. By invertibility of $R\supset$ and $L\&$ we obtain $z : A_1, \ldots, z : A_n \Rightarrow z : B$. Then for an arbitrary x we have to derive $\Rightarrow x : ((D \,\Box\!\!\rightarrow A_1) \& \ldots \& (D \,\Box\!\!\rightarrow A_n)) \supset (D \,\Box\!\!\rightarrow B)$. Consider the following proof search (divided in two parts for lack of page width):[16]

$$\dfrac{\dfrac{\dfrac{\dfrac{z : D \Rightarrow z : D \quad \begin{matrix}\vdots \\ y_1 : D, \ldots, y_n : D, y_1 : \Box_x(D \supset A_1), \ldots, y_n : \Box_x(D \supset A_n) \Rightarrow x : D \,\Box\!\!\rightarrow B\end{matrix}}{z : D, x : D \,\Box\!\!\rightarrow A_1, \ldots, x : D \,\Box\!\!\rightarrow A_n \Rightarrow x : D \,\Box\!\!\rightarrow B}\, L\Box\!\!\rightarrow,\ n\ times}{x : D \,\Box\!\!\rightarrow A_1, \ldots, x : D \,\Box\!\!\rightarrow A_n \Rightarrow x : D \,\Box\!\!\rightarrow B}\, R\Box\!\!\rightarrow_1}{x : (D \,\Box\!\!\rightarrow A_1) \& \ldots \& (D \,\Box\!\!\rightarrow A_n) \Rightarrow x : D \,\Box\!\!\rightarrow B}\, L\&,\ n\ times}{\Rightarrow x : ((D \,\Box\!\!\rightarrow A_1) \& \ldots \& (D \,\Box\!\!\rightarrow A_n)) \supset (D \,\Box\!\!\rightarrow B)}\, R\supset$$

$$\dfrac{\dfrac{\dfrac{y_1 R_x w_1, \ldots, y_n R_x w_n, y_1 : \Box_x(D \supset A_1), \ldots, y_n : \Box_x(D \supset A_n), w_1 : D, \ldots, w_n : D \Rightarrow w_1 : B, \ldots, w_n : B}{y_1 R_x w_1, \ldots, y_n R_x w_n, y_1 : \Box_x(D \supset A_1), \ldots, y_n : \Box_x(D \supset A_n) \Rightarrow w_1 : D \supset B, \ldots, w_n : D \supset B}\, R\supset,\ n\ times}{y_1 : \Box_x(D \supset A_1), \ldots, y_n : \Box_x(D \supset A_n) \Rightarrow y_1 : \Box_x(D \supset B), \ldots, y_n : \Box_x(D \supset B)}\, R\Box_x,\ n\ times}{y_1 : D, \ldots, y_n : D, y_1 : \Box_x(D \supset A_1), \ldots, y_n : \Box_x(D \supset A_n) \Rightarrow x : D \,\Box\!\!\rightarrow B}\, R\Box\!\!\rightarrow_2,\ n\ times$$

Observe that we cannot conclude by applications of $L\Box_x$ to obtain a sequent derivable from the assumption because the formulas in the right topsequent of the above tree have different labels by the freshness conditions. Instead we apply at most n times *SConn* with the following pattern on the right premiss:

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ y_1 R_x y_3, y_1 R_x y_2, \Gamma'' \Rightarrow \Delta\end{matrix} \quad \begin{matrix}\vdots \\ y_3 R_x y_1, y_1 R_x y_2, \Gamma'' \Rightarrow \Delta\end{matrix}}{y_1 R_x y_2, \Gamma' \Rightarrow \Delta} \quad \dfrac{\begin{matrix}\vdots \\ y_2 R_x y_3, y_2 R_x y_1, \Gamma''' \Rightarrow \Delta\end{matrix} \quad \begin{matrix}\vdots \\ y_3 R_x y_2, y_2 R_x y_1, \Gamma''' \Rightarrow \Delta\end{matrix}}{y_2 R_x y_1, \Gamma' \Rightarrow \Delta}}{y_1 R_x w_1, \ldots, y_n R_x w_n, \Gamma \Rightarrow \Delta}$$

Thus by *Trans* at each topsequent there is a y_h such that $y_1 R_x y_h, \ldots, y_n R_x y_h$. Correspondingly, there is a w_h such that (again by n applications of *Trans*), $y_1 R_x w_h, \ldots, y_n R_x w_h$. On this w_h we apply n times $L\Box_x$, obtaining the following (the relational atoms are omitted for brevity):

$$\dfrac{\dfrac{w_h : D \Rightarrow w_h : D \quad w_h : A_1, \ldots, w_h : A_n \Rightarrow w_h : B}{w_h : D \supset A_1, \ldots, w_h : D \supset A_n, w_h : D \Rightarrow w_h : B}\, L\supset,\ n\ times}{y_1 : \Box_x(D \supset A_1), \ldots, y_n : \Box_x(D \supset A_n), w_h : D \Rightarrow w_h : B}\, L\Box_x,\ n\ times$$

The right premiss is now derivable from the assumption (with the arbitrary label z chosen to be w_h).

(3) By induction on the structure of Φ. If Φ is $\bot$ or an atomic formula the claim is trivial; if it is a conjunction, disjunction, implication, or an indexed modality, corresponding steps and invertibility of $R\supset$ give the claim. If Φ is a conditional and the interchange is done in the succedent, the conclusion follows from (2). If the interchange is in the antecedent, the

[16] For brevity, some of the repetitions of principal formulas in the premisses as well as the derivable left premisses of $R\Box\!\!\rightarrow_2$ have been omitted. Use of indexes for formulas should be obvious.

claim follows from admissibility of

$$\dfrac{\vdash A \supset\subset B}{A \mathrel{\Box\!\!\rightarrow} C \vdash B \mathrel{\Box\!\!\rightarrow} C}$$

First, we observe that from the assumption we have that for arbitrary y, z the sequents $y : A \Rightarrow y : B$ and $z : B \Rightarrow z : A$ are derivable, hence also the sequent $y : \Box_w(A \supset C) \Rightarrow y : \Box_w(B \supset C)$ is. We then have the following derivation

$$\dfrac{\dfrac{z : B \Rightarrow z : A \qquad \dfrac{\dots y : A \Rightarrow y : B \dots \qquad \dots y : \Box_w(A \supset C) \Rightarrow y : \Box_w(B \supset C) \dots}{y : B, y : \Box_w(A \supset C) \Rightarrow w : B \mathrel{\Box\!\!\rightarrow} C}\, R{\Box\!\!\rightarrow_2}}{z : B, w : A \mathrel{\Box\!\!\rightarrow} C \Rightarrow w : B \mathrel{\Box\!\!\rightarrow} C}\, L{\Box\!\!\rightarrow}}{w : A \mathrel{\Box\!\!\rightarrow} C \Rightarrow w : B \mathrel{\Box\!\!\rightarrow} C}\, R{\Box\!\!\rightarrow_1}$$

□

Next, we prove that all the axioms of **VC** are derivable in **G3LC**, i.e. for each axiom A the sequent $\Rightarrow x : A$ is derivable in the calculus where x is an arbitrary label.

PROPOSITION 4.2. *The following are derivable in* **G3LC***:*

1. *Propositonal tautologies,*
2. $A \mathrel{\Box\!\!\rightarrow} A$,
3. $(\neg A \mathrel{\Box\!\!\rightarrow} A) \supset (B \mathrel{\Box\!\!\rightarrow} A)$,
4. $(A \mathrel{\Box\!\!\rightarrow} \neg B) \vee (((A \& B) \mathrel{\Box\!\!\rightarrow} C) \supset\subset (A \mathrel{\Box\!\!\rightarrow} (B \supset C)))$,
5. $(A \mathrel{\Box\!\!\rightarrow} B) \supset (A \supset B)$,
6. $(A \& B) \supset (A \mathrel{\Box\!\!\rightarrow} B)$.

Proof. All propositional tautologies are clearly derivable because **G3LC** is an extension of a complete calculus for classical propositional logic. For items 2–6 we have the following derivations:

(2)

$$\dfrac{\dfrac{z : A \Rightarrow z : A \qquad \dfrac{\dfrac{zR_x y, z : A, y : A \Rightarrow y : A}{zR_x y, z : A \Rightarrow y : A \supset A}\, R{\supset}}{z : A \Rightarrow z : \Box_x(A \supset A)}\, R\Box_x}{z : A \Rightarrow x : A \mathrel{\Box\!\!\rightarrow} A}\, R{\Box\!\!\rightarrow_2}}{\Rightarrow x : A \mathrel{\Box\!\!\rightarrow} A}\, R{\Box\!\!\rightarrow_1}$$

(3)

$$\dfrac{\dfrac{\dfrac{z : B \Rightarrow z : B \qquad \dfrac{\dfrac{\dfrac{\dfrac{y : A \Rightarrow y : A}{\Rightarrow y : A, y : \neg A}\, R{\supset} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{w : A \Rightarrow w : A}{\Rightarrow w : \neg A, w : A}\, R{\supset}}{w : \neg A \Rightarrow w : \neg A}\, L{\supset} \qquad \dfrac{w : A \Rightarrow w : A}{w : \neg A, w : A \Rightarrow}\, L{\supset}}{wR_x w, w : \neg A, w : \neg A \supset A \Rightarrow}\, L{\supset}}{wR_x w, w : \neg A, w : \Box_x(\neg A \supset A) \Rightarrow}\, L\Box_x}{w : \neg A, w : \Box_x(\neg A \supset A) \Rightarrow}\, Ref}{zR_x y, y : B, x : \neg A \mathrel{\Box\!\!\rightarrow} A \Rightarrow y : A}\, L{\Box\!\!\rightarrow}}{zR_x y, x : \neg A \mathrel{\Box\!\!\rightarrow} A \Rightarrow y : B \supset A}\, R{\supset}}{x : \neg A \mathrel{\Box\!\!\rightarrow} A \Rightarrow z : \Box_x(B \supset A)}\, R\Box_x}{z : B, x : \neg A \mathrel{\Box\!\!\rightarrow} A \Rightarrow x : B \mathrel{\Box\!\!\rightarrow} A}\, R{\Box\!\!\rightarrow_2}}{x : \neg A \mathrel{\Box\!\!\rightarrow} A \Rightarrow x : B \mathrel{\Box\!\!\rightarrow} A}\, R{\Box\!\!\rightarrow_1}}{\Rightarrow x : (\neg A \mathrel{\Box\!\!\rightarrow} A) \supset (B \mathrel{\Box\!\!\rightarrow} A)}\, R{\supset}$$

(4)

$$\begin{array}{cl} \vdots & \\ \hline \Rightarrow x : A \Box\!\!\rightarrow \neg B, x : A\&B \Box\!\!\rightarrow C \supset\subset A \Box\!\!\rightarrow (B \supset C) & R\& \\ \hline \Rightarrow x : (A \Box\!\!\rightarrow \neg B) \vee (A\&B \Box\!\!\rightarrow C \supset\subset A \Box\!\!\rightarrow (B \supset C)) & R\vee \end{array}$$

The proof branches; we continue below with the two premisses

$$\Rightarrow x : A \Box\!\!\rightarrow \neg B, x : ((A\&B) \Box\!\!\rightarrow C) \supset (A \Box\!\!\rightarrow (B \supset C))$$

and

$$\Rightarrow x : A \Box\!\!\rightarrow \neg B, x : (A \Box\!\!\rightarrow (B \supset C)) \supset ((A\&B) \Box\!\!\rightarrow C)$$

Proof of the former:[17]

$$\begin{array}{cl} t : A \Rightarrow t : A \quad t : B \Rightarrow t : B & \\ \hline t : A, t : B \Rightarrow t : A\&B \qquad t : C \Rightarrow t : C & R\& \\ \hline t : A\&B \supset C, t : A, t : B \Rightarrow t : C & L\supset \\ \hline t : A\&B \supset C, t : A \Rightarrow t : B \supset C & R\supset \\ \hline wR_x t, t : A\&B \supset C \Rightarrow t : A \supset (B \supset C) & R\supset \\ \hline wR_x t, w : \Box_x(A\&B \supset C) \Rightarrow t : A \supset (B \supset C) & L\Box_x \\ \hline w : A, w : B, w : \Box_x(A\&B \supset C) \Rightarrow w : \Box_x(A \supset (B \supset C)) & R\Box_x \\ \hline w : A, w : B, w : \Box_x(A\&B \supset C) \Rightarrow x : A \Box\!\!\rightarrow (B \supset C) & R\Box\!\!\rightarrow_2 \\ \hline w : A\&B, w : \Box_x(A\&B \supset C) \Rightarrow x : A \Box\!\!\rightarrow (B \supset C) & L\& \end{array}$$

$$\begin{array}{cl} y : A \Rightarrow y : A \quad y : B \Rightarrow y : B & \\ \hline y : A, y : B \Rightarrow y : A\&B & R\& \end{array}$$

$$\begin{array}{cl} \hline y : A, y : B, x : A\&B \Box\!\!\rightarrow C \Rightarrow x : A \Box\!\!\rightarrow (B \supset C) & L\Box\!\!\rightarrow \\ \hline y : A, x : A\&B \Box\!\!\rightarrow C \Rightarrow y : \neg B, x : A \Box\!\!\rightarrow (B \supset C) & R\supset \\ \hline zR_x y, x : A\&B \Box\!\!\rightarrow C \Rightarrow y : A \supset \neg B, x : A \Box\!\!\rightarrow (B \supset C) & R\supset \\ \hline x : A\&B \Box\!\!\rightarrow C \Rightarrow z : \Box_x(A \supset \neg B), x : A \Box\!\!\rightarrow (B \supset C) & R\Box_x \\ \hline z : A, x : A\&B \Box\!\!\rightarrow C \Rightarrow x : A \Box\!\!\rightarrow \neg B, x : A \Box\!\!\rightarrow (B \supset C) & R\Box\!\!\rightarrow_2 \\ \hline x : A\&B \Box\!\!\rightarrow C \Rightarrow x : A \Box\!\!\rightarrow \neg B, x : A \Box\!\!\rightarrow (B \supset C) & R\Box\!\!\rightarrow_1 \\ \hline \Rightarrow x : A \Box\!\!\rightarrow \neg B, x : A\&B \Box\!\!\rightarrow C \supset A \Box\!\!\rightarrow (B \supset C) & R\supset \end{array}$$

Proof of the latter:[18]

$$\begin{array}{cl} z : B \Rightarrow z : B \quad z : C \Rightarrow z : C & \\ \hline z : A \Rightarrow z : A \qquad z : B, z : B \supset C \Rightarrow z : C & L\supset \\ \hline z : A, z : B, z : A \supset (B \supset C) \Rightarrow z : C & L\supset \\ \hline wR_x z, wR_x y, yR_x z, z : A\&B, z : A \supset (B \supset C) \Rightarrow z : C & L\& \\ \hline wR_x z, wR_x y, yR_x z, z : A\&B, w : \Box_x(A \supset (B \supset C)) \Rightarrow z : C & L\Box_x \\ \hline wR_x y, yR_x z, z : A\&B, w : \Box_x(A \supset (B \supset C)) \Rightarrow z : C & Trans \\ \hline wR_x y, yR_x z, w : \Box_x(A \supset (B \supset C)) \Rightarrow z : A\&B \supset C & R\supset \\ \hline wR_x y, w : \Box_x(A \supset (B \supset C)) \Rightarrow y : \Box_x(A\&B \supset C) & R\Box_x \end{array}$$

$$\begin{array}{cl} y : A \Rightarrow y : A \quad y : B \Rightarrow y : B & \\ \hline y : A, y : B \Rightarrow y : A\&B & R\& \end{array}$$

$$\begin{array}{cl} \hline wR_x y, y : A, y : B, w : \Box_x(A \supset (B \supset C)) \Rightarrow x : A\&B \Box\!\!\rightarrow C & R\Box\!\!\rightarrow_2 \\ \hline wR_x y, y : A, w : \Box_x(A \supset (B \supset C)) \Rightarrow y : \neg B, x : A\&B \Box\!\!\rightarrow C & R\supset \\ \hline wR_x y, w : \Box_x(A \supset (B \supset C)) \Rightarrow y : A \supset \neg B, x : A\&B \Box\!\!\rightarrow C & R\supset \\ \hline w : A, w : \Box_x(A \supset (B \supset C)) \Rightarrow w : \Box_x(A \supset \neg B), x : A\&B \Box\!\!\rightarrow C & R\Box_x \\ \hline w : A, w : \Box_x(A \supset (B \supset C)) \Rightarrow x : A \Box\!\!\rightarrow \neg B, x : A\&B \Box\!\!\rightarrow C & R\Box\!\!\rightarrow_2 \\ \hline t : A, x : A \Box\!\!\rightarrow (B \supset C) \Rightarrow x : A \Box\!\!\rightarrow \neg B, x : A\&B \Box\!\!\rightarrow C & L\Box\!\!\rightarrow \\ \hline x : A \Box\!\!\rightarrow (B \supset C) \Rightarrow x : A \Box\!\!\rightarrow \neg B, x : A\&B \Box\!\!\rightarrow C & R\Box\!\!\rightarrow_1 \\ \hline \Rightarrow x : A \Box\!\!\rightarrow \neg B, x : A \Box\!\!\rightarrow (B \supset C) \supset A\&B \Box\!\!\rightarrow C & R\supset \end{array}$$

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17 Here an below, to save space, the (derivable) premisses of $R\Box\!\!\rightarrow_2$ and $L\Box\!\!\rightarrow$ that have the same formula in the antecedent and succedent have been omitted.

18 Observe that this derivation, among others, shows that the left premiss of $R\Box\!\!\rightarrow_2$ is not dispensable.

(5)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{w : A \Rightarrow w : A \quad w : B \Rightarrow w : B}{w : A \supset B, w : A \Rightarrow w : B}\,{\scriptstyle L\supset}}{wR_x w, w : \Box_x(A \supset B), w : A \Rightarrow w : B}\,{\scriptstyle L\Box_x}}{x = w, xR_x w, w : \Box_x(A \supset B), x : A \Rightarrow x : B}\,{\scriptstyle Repl}}{xR_x w, w : \Box_x(A \supset B), x : A \Rightarrow x : B}\,{\scriptstyle LMin} \qquad \dfrac{\dfrac{x : A \Rightarrow x : A \quad x : B \Rightarrow x : B}{x : A \supset B, x : A \Rightarrow x : B}\,{\scriptstyle L\supset}}{wR_x x, w : \Box_x(A \supset B), x : A \Rightarrow x : B}\,{\scriptstyle L\Box_x}}{\dfrac{w : A, w : \Box_x(A \supset B), x : A \Rightarrow x : B}{\dfrac{x : A \mathrel{\Box\!\!\!\rightarrow} B, x : A \Rightarrow x : B}{\dfrac{x : A \mathrel{\Box\!\!\!\rightarrow} B \Rightarrow x : A \supset B}{\Rightarrow x : A \mathrel{\Box\!\!\!\rightarrow} B \supset (A \supset B)}\,{\scriptstyle R\supset}}\,{\scriptstyle R\supset}}\,{\scriptstyle L\Box\!\!\!\rightarrow}}\,{\scriptstyle SConn}$$

(6)

$$\dfrac{x : A \Rightarrow x : A \qquad \dfrac{\dfrac{\dfrac{\dfrac{y : A, y : A, y : B \Rightarrow y : B}{y : A, y : B \Rightarrow y : A \supset B}\,{\scriptstyle R\supset}}{x = y, xR_x y, x : A, x : B \Rightarrow y : A \supset B}\,{\scriptstyle Repl}}{xR_x y, x : A, x : B \Rightarrow y : A \supset B}\,{\scriptstyle LMin}}{x : A, x : B \Rightarrow x : \Box_x(A \supset B)}\,{\scriptstyle R\Box_x}}{\dfrac{x : A, x : B \Rightarrow x : A \mathrel{\Box\!\!\!\rightarrow} B}{\dfrac{x : A\&B \Rightarrow x : A \mathrel{\Box\!\!\!\rightarrow} B}{\Rightarrow x : (A\&B) \supset (A \mathrel{\Box\!\!\!\rightarrow} B)}\,{\scriptstyle L\supset}}\,{\scriptstyle L\&}}\,{\scriptstyle R\Box\!\!\!\rightarrow_2}$$

□

4.1. Equivalence with L-SC. We can show that the rules of **G3LC** are interderivable with Lewis's defining semantic condition for the counterfactual, suitably formalized in a first-order language with variables ranging over possible worlds as follows:

$$w : A \mathrel{\Box\!\!\!\rightarrow} B \supset\subset \forall z(z : \neg A) \vee \exists x(x : A \wedge x : \Box_w(A \supset B)) \qquad \textbf{L-SC}$$

For brevity, we will shorten the right-hand side of the biconditional **L-SC** with the letter K.

Observe that the proofs below use a hybrid language with metalinguistic elements which are not part of the calculus that we have introduced. The derivations have to be read as a formal account of the meta-reasoning that justifies the rules in view of the semantic explanation of the counterfactual, and vice versa. As possible worlds can be regarded as first-order entities, we take the usual rules for the quantifiers for that purpose.

Given the above constraints, we show:

PROPOSITION 4.3. *The rules of* **G3LC** *are interderivable with* **L-SC**.

Proof. To save space, we shall not copy the main formula in the premisses and omit mentioning Lemma 2.2 for the derivability of initial sequents with arbitrary formulas. The following is a proof that $L\mathrel{\Box\!\!\!\rightarrow}$ gives the left-to-right direction of **L-SC**:

$$\dfrac{z : A \Rightarrow z : A \qquad \dfrac{\dfrac{\dfrac{x : A, x : \Box_w(A \supset B) \Rightarrow x : A \quad x : A, x : \Box_w(A \supset B) \Rightarrow x : \Box_w(A \supset B)}{x : A, x : \Box_w(A \supset B) \Rightarrow x : A \wedge x : \Box_w(A \supset B)}\,{\scriptstyle R\wedge}}{x : A, x : \Box_w(A \supset B) \Rightarrow \exists x(x : A \wedge x : \Box_w(A \supset B))}\,{\scriptstyle R\exists}}{x : A, x : \Box_w(A \supset B), z : A \Rightarrow \exists x(x : A \wedge x : \Box_w(A \supset B))}\,{\scriptstyle LW}}{\dfrac{w : A \mathrel{\Box\!\!\!\rightarrow} B, z : A \Rightarrow \exists x(x : A \wedge x : \Box_w(A \supset B))}{\dfrac{w : A \mathrel{\Box\!\!\!\rightarrow} B \Rightarrow z : \neg A, \exists x(x : A \wedge x : \Box_w(A \supset B))}{\dfrac{w : A \mathrel{\Box\!\!\!\rightarrow} B \Rightarrow \forall z(z : \neg A), \exists x(x : A \wedge x : \Box_w(A \supset B))}{\dfrac{w : A \mathrel{\Box\!\!\!\rightarrow} B \Rightarrow K}{\Rightarrow w : A \mathrel{\Box\!\!\!\rightarrow} B \supset K}\,{\scriptstyle R\supset}}\,{\scriptstyle R\vee}}\,{\scriptstyle R\forall}}\,{\scriptstyle R\neg}}\,{\scriptstyle L\Box\!\!\!\rightarrow}$$

The following is a derivation of the right-to-left direction of **L-SC** that uses rules $R\Box\!\!\rightarrow_1$ and $R\Box\!\!\rightarrow_2$:

$$\dfrac{\dfrac{\dfrac{\dfrac{y : A \Rightarrow y : A}{y : \neg A, y : A \Rightarrow}\,{\scriptstyle L\neg}}{\forall z(z : \neg A), y : A \Rightarrow}\,{\scriptstyle L\forall}}{\forall z(z : \neg A) \Rightarrow w : A \Box\!\!\rightarrow B}\,{\scriptstyle R\Box\!\rightarrow_1} \qquad \dfrac{\dfrac{\dfrac{x : A \Rightarrow x : A \quad x : A, x : \Box_w(A \supset B) \Rightarrow x : \Box_w(A \supset B)}{x : A, x : \Box_w(A \supset B) \Rightarrow w : A \Box\!\!\rightarrow B}\,{\scriptstyle R\Box\!\rightarrow_2}}{x : A \wedge x : \Box_w(A \supset B) \Rightarrow w : A \Box\!\!\rightarrow B}\,{\scriptstyle L\wedge}}{\exists x(x : A \wedge x : \Box_w(A \supset B)) \Rightarrow w : A \Box\!\!\rightarrow B}\,{\scriptstyle L\exists}}{\dfrac{K \Rightarrow w : A \Box\!\!\rightarrow B}{\Rightarrow K \supset w : A \Box\!\!\rightarrow B}\,{\scriptstyle L\supset}}\,{\scriptstyle L\vee}$$

For the remaining derivations that prove the equivalence, we use the right-to-left direction of **L-SC** to get first a derivation of rule $R\Box\!\!\rightarrow_1$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{z : A, \Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B}{\Gamma \Rightarrow \Delta, z : \neg A, w : A \Box\!\!\rightarrow B}\,{\scriptstyle R\neg}}{\Gamma \Rightarrow \Delta, \forall z(z : \neg A), w : A \Box\!\!\rightarrow B}\,{\scriptstyle R\forall}}{\Gamma \Rightarrow \Delta, \forall z(z : \neg A), \exists x(x : A \wedge x : \Box_w(A \supset B)), w : A \Box\!\!\rightarrow B}\,{\scriptstyle RW}}{\Gamma \Rightarrow \Delta, K, w : A \Box\!\!\rightarrow B}\,{\scriptstyle R\vee} \qquad K \Rightarrow w : A \Box\!\!\rightarrow B}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B}\,{\scriptstyle Cut}$$

and then a derivation of $R\Box\!\!\rightarrow_2$ by the second disjunct

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x : A, w : A \Box\!\!\rightarrow B \quad \Gamma \Rightarrow \Delta, x : \Box_w(A \supset B), w : A \Box\!\!\rightarrow B}{\Gamma \Rightarrow \Delta, x : A \wedge x : \Box_w(A \supset B), w : A \Box\!\!\rightarrow B}\,{\scriptstyle R\wedge}}{\Gamma \Rightarrow \Delta, \exists x(x : A \wedge x : \Box_w(A \supset B)), w : A \Box\!\!\rightarrow B}\,{\scriptstyle R\exists}}{\Gamma \Rightarrow \Delta, \forall z(z : \neg A), \exists x(x : A \wedge x : \Box_w(A \supset B)), w : A \Box\!\!\rightarrow B}\,{\scriptstyle RW}}{\Gamma \Rightarrow \Delta, K, w : A \Box\!\!\rightarrow B}\,{\scriptstyle R\vee} \qquad K \Rightarrow w : A \Box\!\!\rightarrow B}{\Gamma \Rightarrow \Delta, w : A \Box\!\!\rightarrow B}\,{\scriptstyle Cut}$$

Finally we have a derivation of $L\Box\!\!\rightarrow$ using the left-to-right direction of **L-SC**

$$\dfrac{w : A \Box\!\!\rightarrow B \Rightarrow K \qquad \dfrac{\dfrac{\dfrac{w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta, z : A}{w : A \Box\!\!\rightarrow B, z : \neg A, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\neg}}{w : A \Box\!\!\rightarrow B, \forall z(z : \neg A), \Gamma \Rightarrow \Delta}\,{\scriptstyle L\forall} \quad \dfrac{\dfrac{x : A, x : \Box_w(A \supset B), \Gamma \Rightarrow \Delta}{x : A \wedge x : \Box_w(A \supset B), \Gamma \Rightarrow \Delta}\,{\scriptstyle L\wedge}}{\exists x(x : A \wedge x : \Box_w(A \supset B)), \Gamma \Rightarrow \Delta}\,{\scriptstyle L\exists}}{w : A \Box\!\!\rightarrow B, K, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\vee}}{w : A \Box\!\!\rightarrow B, \Gamma \Rightarrow \Delta}\,{\scriptstyle Cut}$$

□

§5. Completeness. In this section we shall give a direct completeness proof for **G3LC** with respect to Lewis semantics. The proof has the overall structure of the completeness proof for labelled systems for modal and non-classical logics given in Negri (2009, 2014a), but the semantics is here based on comparative similarity systems rather than on Kripke models.

DEFINITION 5.1. *A* **comparative similarity system** $\mathcal{S}$ *is given by is a set W of possible worlds and for every $w \in W$ a set of two-place relations $\preceq_w$ which satisfy the conditions:*

1. *Transitivity: If $x \preceq_w y$ and $y \preceq_w z$ then $x \preceq_w z$,*
2. *Strong connectedness: Either $x \preceq_w y$ or $y \preceq_w x$,*
3. *$\preceq_w$-Minimality: If $x \preceq_w w$ then $x = w$.*

An **interpretation** *of a set of labels L in W is a map $[\![\cdot]\!] : L \rightarrow W$. A* **valuation** *of atomic formulas in $\mathcal{S}$ is a map $\mathcal{V} : AtFrm \rightarrow \mathcal{P}(W)$. Instead of writing $[\![w]\!] \in \mathcal{V}(P)$, we adopt the standard notation $[\![w]\!] \Vdash P$.*

Valuations are extended to arbitrary formulas by the following inductive clauses:

$\mathcal{V}_{\bot}$: $x \Vdash \bot$ for no x.

$\mathcal{V}_{\&}$: $x \Vdash A \& B$ iff $x \Vdash A$ and $x \Vdash B$.

$\mathcal{V}_{\vee}$: $x \Vdash A \vee B$ iff $x \Vdash A$ or $x \Vdash B$.

$\mathcal{V}_{\supset}$: $x \Vdash A \supset B$ iff if $x \Vdash A$ then $x \Vdash B$.

$\mathcal{V}_{\Box_w}$: $x \Vdash \Box_w A$ iff for all y, if $y \preceq_w x$ then $y \Vdash A$.

$\mathcal{V}_{\Box\!\!\rightarrow}$: $x \Vdash A \Box\!\!\rightarrow B$ iff either $z \Vdash A$ for no z, or $y \Vdash A$ and $y \Vdash \Box_x(A \supset B)$ for some y.

DEFINITION 5.2. *A labelled formula $x : A$ (resp. a relational atom $xR_w y$) is* **true** *for an interpretation $[\![\cdot]\!]$ and a valuation $\mathcal{V}$ in a system $\mathcal{S}$ iff $[\![x]\!] \Vdash A$ (resp. $[\![y]\!] \preceq_{[\![w]\!]} [\![x]\!]$). A sequent $\Gamma \Rightarrow \Delta$ is true for an interpretation $[\![\cdot]\!]$ and a valuation $\mathcal{V}$ in a system $\mathcal{S}$ if, whenever for all labelled formulas $x : A$ and relational atom $xR_w y$ in Γ, if it is the case that $[\![x]\!] \Vdash A$ and $[\![y]\!] \preceq_{[\![w]\!]} [\![x]\!]$, then for some $w : B$ in Δ, $[\![w]\!] \Vdash B$. A sequent is* **valid** *in a system $\mathcal{S}$ iff it is true for every interpretation and valuation in $\mathcal{S}$.*

THEOREM 5.3 (**Soundness**). *If a sequent is derivable in* **G3LC** *then it is valid in every comparative similarity system $\mathcal{S}$.*

Proof. The proof is by induction on the height of the derivation and follows the methodology employed for modal and non-classical logics in Negri & von Plato (2011), Negri (2009) and Negri (2014a). We use Φ' to denote the set Φ without the principal formula of the inference step considered, which should be everywhere clear from the context.

Assume that $\Gamma \Rightarrow \Delta$ is derivable and that $[\![\cdot]\!]$ makes true all the formulas in Γ.

If $\Gamma \Rightarrow \Delta$ is an initial sequent, it is valid since both Γ and Δ contain some formula $x : A$, and then the claim is obvious. Likewise, if it is the conclusion of $L\bot$, since no interpretation would then make all the formulas in Γ true.

If $\Gamma \Rightarrow \Delta$ is the conclusion of $L\&$, then $[\![x]\!] \Vdash A\&B$, whence $[\![x]\!] \Vdash A$ and $[\![x]\!] \Vdash B$ by $\mathcal{V}_{\&}$. By inductive hypothesis, $x : A, x : B, \Gamma' \Rightarrow \Delta$ is valid, and so $[\![\cdot]\!]$ makes true some formula in Δ. Hence $\Gamma \Rightarrow \Delta$ is valid. The argument is routine for the remaining propositional rules, and similarly for *Ref* and $Repl_{At}$.

If $\Gamma \Rightarrow \Delta$ is the conclusion of *Trans*, then $[\![y]\!] \preceq_{[\![w]\!]} [\![x]\!]$ and $[\![z]\!] \preceq_{[\![w]\!]} [\![y]\!]$, whence $[\![z]\!] \preceq_{[\![w]\!]} [\![x]\!]$ by *Transitivity* of $\mathcal{S}$. By inductive hypothesis, $xR_w z, xR_w y, yR_w z, \Gamma' \Rightarrow \Delta$ is valid, and so $[\![\cdot]\!]$ makes true some formula in Δ. Hence $\Gamma \Rightarrow \Delta$ is valid.

If $\Gamma \Rightarrow \Delta$ is the conclusion of *SConn*, then for any w, x, y, either $[\![x]\!] \preceq_{[\![w]\!]} [\![y]\!]$ or $[\![y]\!] \preceq_{[\![w]\!]} [\![x]\!]$, by *Strong connectedness* of $\mathcal{S}$. By inductive hypothesis, both (i) $xR_w y, \Gamma \Rightarrow \Delta$ and (ii) $yR_w x, \Gamma \Rightarrow \Delta$ are valid. So if $[\![x]\!] \preceq_{[\![w]\!]} [\![y]\!]$, then by (i) $[\![\cdot]\!]$ makes true some formula in Δ. Otherwise if $[\![y]\!] \preceq_{[\![w]\!]} [\![x]\!]$, then by (ii) $[\![\cdot]\!]$ makes true some formula in Δ. In either case, $\Gamma \Rightarrow \Delta$ is valid.

If $\Gamma \Rightarrow \Delta$ is the conclusion of *LMin*, then $[\![x]\!] \preceq_{[\![w]\!]} [\![w]\!]$, whence $[\![x]\!] = [\![w]\!]$ by $\preceq_w$*-Minimality* of $\mathcal{S}$. By inductive hypothesis, $x = w, wR_w x, \Gamma' \Rightarrow \Delta$ is valid, and so $[\![\cdot]\!]$ makes true some formula in Δ. Hence $\Gamma \Rightarrow \Delta$ is valid.

If $\Gamma \Rightarrow \Delta$ is the conclusion of $R\Box_w$, let z be an arbitrary element of $\mathcal{W}$ such that $z \preceq_{[\![w]\!]} [\![x]\!]$ holds in $\mathcal{S}$. Let $[\![\cdot]\!]'$ be the same as $[\![\cdot]\!]$ except possibly on $[\![y]\!]$, where we set $[\![y]\!]' = z$. Then $[\![\cdot]\!]'$ makes all the formulas in $xR_w y, \Gamma$ true. By inductive hypothesis, $xR_w y, \Gamma \Rightarrow \Delta', y : A$ is valid, so either some formula in Δ' or $y : A$ is true under $[\![\cdot]\!]'$. If the former holds, then $\Gamma \Rightarrow \Delta$ is valid independently of the choice of $[\![\cdot]\!]'$. If the latter, then $[\![y]\!]' \Vdash A$ for arbitrary $[\![y]\!]'$. Hence $[\![x]\!]' \Vdash \Box_w A$ by $\mathcal{V}_{\Box_w}$. But $[\![\cdot]\!]'$ be the same as $[\![\cdot]\!]$ w.r.t. $[\![x]\!]$, and so $[\![x]\!] \Vdash \Box_w A$. Either case, $\Gamma \Rightarrow \Delta$ is valid.

If $\Gamma \Rightarrow \Delta$ is the conclusion of $R\Box_w$, then $[\![y]\!] \preceq_{[\![w]\!]} [\![x]\!]$ and $[\![x]\!] \Vdash \Box_w A$, whence $[\![y]\!] \Vdash A$ by $\mathcal{V}_{\Box_w}$. By inductive hypothesis, $xR_w y, x : \Box_w A, y : A, \Gamma \Rightarrow \Delta$ is valid, and so $[\![\cdot]\!]$ makes true some formula in Δ. Hence $\Gamma \Rightarrow \Delta$ is valid.

If $\Gamma \Rightarrow \Delta$ is the conclusion of $R\Box\!\!\rightarrow_1$, choose $z \notin \Gamma, \Delta$ and let $[\![\cdot]\!]'$ be the same as $[\![\cdot]\!]$ except possibly on $[\![z]\!]$, which we set $[\![z]\!]' = y$ for some arbitrary $y \in \mathcal{W}$ such that $y \Vdash A$. By inductive hypothesis, $z : A, \Gamma \Rightarrow \Delta', w : A \Box\!\!\rightarrow B$ is valid. The case in which there is such a y is obvious. If there is no such y, notice that $[\![\cdot]\!]' \equiv [\![\cdot]\!]$ and $[\![w]\!] \Vdash A \Box\!\!\rightarrow B$ by $\mathcal{V}_{\Box\!\!\rightarrow}$. Hence $\Gamma \Rightarrow \Delta$ is valid.

If $\Gamma \Rightarrow \Delta$ is the conclusion of $R\Box\!\!\rightarrow_2$, then both $\Gamma \Rightarrow \Delta, x : A$ and $\Gamma \Rightarrow \Delta, x : \Box_w(A \supset B)$ are valid by inductive hypothesis. If $[\![\cdot]\!]$ makes some formula in Δ true, then $\Gamma \Rightarrow \Delta$ is valid. Otherwise, $[\![x]\!] \Vdash A$ and $[\![x]\!] \Vdash \Box_w(A \supset B)$. So by $\mathcal{V}_{\Box\!\!\rightarrow}$, $[\![w]\!] \Vdash A \Box\!\!\rightarrow B$. Hence $\Gamma \Rightarrow \Delta$ is valid.

If $\Gamma \Rightarrow \Delta$ is the conclusion of $L\Box\!\!\rightarrow$, then $[\![w]\!] \Vdash A \Box\!\!\rightarrow B$, whence either for no z, $[\![z]\!] \Vdash A$, or for some x, $[\![x]\!] \Vdash A$ and $[\![x]\!] \Vdash \Box_w(A \supset B)$, by $\mathcal{V}_{\Box\!\!\rightarrow}$. Suppose the former. By inductive hypothesis, $\Gamma \Rightarrow \Delta, z : A$ is valid, so $[\![\cdot]\!]$ makes true some formula in Δ, since $[\![z]\!] \Vdash A$ is ruled out. Hence $\Gamma \Rightarrow \Delta$ is valid. Suppose now the latter. By inductive hypothesis, $x : A, x : \Box_w(A \supset B), \Gamma' \Rightarrow \Delta$ is valid, and so $[\![\cdot]\!]$ makes true some formula in Δ. Hence $\Gamma \Rightarrow \Delta$ is valid. □

THEOREM 5.4 (**Completeness**). *Let $\Gamma \Rightarrow \Delta$ be a sequent in the language of* **G3LC**. *If it is valid in every comparative similarity system, it is derivable in* **G3LC**.

Proof. Immediate by Proposition 4.1, Proposition 4.2, Theorem 5.3 and Lewis's own completeness proof (cf. *Counterfactuals*, pp. 118-134). □

Completeness can be established also as a corollary of the following:

THEOREM 5.5. *Let $\Gamma \Rightarrow \Delta$ be a sequent in the language of* **G3LC**. *Then either it is derivable in* **G3LC** *or it has a countermodel in the class of comparative similarity systems.*

Proof. We define inductively a reduction tree for an arbitrary sequent $\Gamma \Rightarrow \Delta$ in the language of **G3LC** by applying the rules of **G3LC** root first in every possible way. If the construction terminates, we have a proof of the sequent. If it doesn't, then by König's lemma the reduction tree has an infinite branch which is used to define a countermodel to $\Gamma \Rightarrow \Delta$.

1. *Construction of the reduction tree.* Let us take an arbitrary sequent $\Gamma \Rightarrow \Delta$ and take it as the root of the tree (stage 0). At stage n, we distinguish two cases:

Either all topmost sequents are an initial sequent or a conclusion of $L\bot$, in which case the construction terminates.

Otherwise, we continue the construction by applying every possible rule to all topmost sequents in a given order. There are $11 + 5$ different stages, 11 for the rules of **G3LC**, 5 for the similarity rules. At stage $n = 11 + 5 + 1$ we repeat stage 1, at stage $n = 11 + 5 + 2$ we repeat stage 2, and so on.

We start for $n = 1$ with $L\&$. Suppose all topmost sequents have the form

$$x_1 : B_1 \& C_1, ..., x_m : B_m \& C_m, \Gamma' \Rightarrow \Delta$$

On top of each, we write

$$x_1 : B_1, x_1 : C_1, ..., x_m : B_m, x_m : C_m, \Gamma' \Rightarrow \Delta$$

This step corresponds to applying root first m times rule $L\&$.

For $n = 2$ we consider all sequents of the form

$$\Gamma \Rightarrow x_1 : B_1 \& C_1, ..., x_m : B_m \& C_m, \Delta'$$

We write on top of each 2^m sequents of the form

$$\Gamma \Rightarrow x_1 : D_1, ..., x_m : D_m, \Delta'$$

where D_i is either B_i or C_i and all possible choices are taken. This step corresponds to applying root first $R\&$ for each conjunction in the right-hand side.

For $n = 3$ we consider all sequents of the form

$$x_1 : B_1 \vee C_1, ..., x_m : B_m \vee C_m, \Gamma' \Rightarrow \Delta$$

We write on top of each 2^m sequents of the form

$$x_1 : D_1, ..., x_m : D_m, \Gamma' \Rightarrow \Delta$$

This case is analogous to step $n = 2$ and corresponds to the application root first of $L\vee$.

For $n = 4$ we consider all sequents of the form

$$\Gamma \Rightarrow x_1 : B_1 \vee C_1, ..., x_m : B_m \vee C_m, \Delta'$$

On top of each, we write

$$\Gamma \Rightarrow x_1 : B_1, x_1 : C_1, ..., x_m : B_m, x_m : C_m, \Delta'$$

This step corresponds to applying root first m times rule $R\vee$.

For $n = 5$ we consider all sequents of the form

$$x_1 : B_1 \supset C_1, ..., x_m : B_m \supset C_m, \Gamma' \Rightarrow \Delta$$

On top of each, we write 2^m sequents of the form

$$x_{i_1} : C_{i_1}, ..., x_{i_k} : C_{i_k}, \Gamma' \Rightarrow x_{j_{k+1}} : B_{j_{k+1}}, ..., x_{j_m} : B_{j_m}, \Delta$$

where $i_1, ...i_k \in \{1, ..., m\}$ and $j_{k+1}, ..., j_m \in \{1, ..., m\} - \{i_1, ..., i_k\}$. This step correspond to the root first application of $L \supset$.

For $n = 6$ we consider all sequents of the form

$$\Gamma \Rightarrow x_1 : B_1 \supset C_1, ..., x_m : B_m \supset C_m, \Delta'$$

On top of each, we write

$$x_1 : B_1, ..., x_m : B_m, \Gamma \Rightarrow C_1, ..., x_m : C_m, \Delta'$$

This step corresponds to applying root first m times rule $R \supset$.

For $n = 7$ we consider all sequents with relational atoms $x_1 R_w y_1, ..., x_m R_w y_m$ and formulas $x_1 : \Box_w B_1, ..., x_m : \Box_w B_m$ on the left-hand side. On top of each, we write

$$x_1 R_w y_1, ..., x_m R_w y_m, x_1 : \Box_w B_1, ..., x_m : \Box_w B_m, y_1 : B_1, ..., y_m : B_m, \Gamma' \Rightarrow \Delta$$

This step corresponds to applying root first m times rule $L\Box_w$.

For $n = 8$ we consider all sequents of the form

$$\Gamma \Rightarrow x_1 : \Box_w B_1, ..., x_m : \Box_w B_m, \Delta'$$

Let $y_1, ..., y_m$ be fresh variables, not yet used in the reduction tree. Then on top of each sequent we write

$$x_1 R_w y_1, ..., x_m R_w y_m, \Gamma \Rightarrow y_1 : B_1, ..., y_m : B_m, \Delta'$$

This step corresponds to applying root first m times rule $R\Box_w$.

For $n = 9$ we consider all sequents of the form

$$x_1 : B_1 \Box\!\!\to C_1, ..., x_m : B_m \Box\!\!\to C_m, \Gamma' \Rightarrow \Delta$$

Let $y_1, ..., y_m$ be fresh variables, not yet used in the reduction tree, and suppose $z_1, ..., z_m$ are any variables already used in the reduction tree, hence disjoint from the y's. Then on top of each sequent, we write 2^m sequents with formulas $y_{i_1} : B_{i_1}, ..., y_{i_k} : B_{i_k}$ together with $y_{i_1} : \Box_{x_{i_1}}(B_{i_1} \supset C_{i_1}), ..., y_{i_k} : \Box_{x_{i_k}}(B_{i_k} \supset C_{i_k})$, as well as $x_{j_{k+1}} : B_{j_{k+1}} \Box\!\!\to C_{j_{k+1}}, ..., x_{j_m} : B_{j_m} \Box\!\!\to C_{j_m}$, on the left-hand side, and formulas $z_{j_{k+1}} : B_{j_{k+1}}, ..., z_{j_m} : B_{j_m}$ on the right-hand side, where $i_1, ...i_k \in \{1, ..., m\}$ and $j_{k+1}, ..., j_m \in \{1, ..., m\} - \{i_1, ..., i_k\}$. This step correspond to the root first application of $L\Box\!\!\to$.

For $n = 10$, we consider all sequents with formulas $x_1 : B_1 \Box\!\!\to C_1, ..., x_m : B_m \Box\!\!\to C_m$ on the right-hand side. Let $y_1, ..., y_m$ be fresh variables, not yet used in the reduction tree. We write on top of each the sequent

$$y_1 : B_1, ...y_m : B_m, \Gamma \Rightarrow x_1 : B_1 \Box\!\!\to C_1, ..., x_m : B_m \Box\!\!\to C_m, \Delta'$$

This step corresponds to applying root first m times rule $R\Box\!\!\to_1$.

For $n = 11$, we consider again all sequents with formulas $x_1 : B_1 \Box\!\!\to C_1, ..., x_m : B_m \Box\!\!\to C_m$ on the right-hand side. Then suppose $z_1, ..., z_m$ are any variables already used in the reduction tree and write 2^m sequents of the form

$$\Gamma \Rightarrow x_1 : B_1 \Box\!\!\to C_1, ..., x_m : B_m \Box\!\!\to C_m, z_1 : D_1, ..., z_m : D_m, \Delta'$$

on top of each, where D_i is either $z_i : B_i$ or $z_i : \Box_{x_i}(B_i \supset C_i)$, and all possible choices are taken. This step corresponds to applying root first m times rule $R\Box\!\!\to_2$.

Finally, for $n = 11 + j$ we consider the generic case of a frame rule, viz. rules for the relation R_w and identity. Because of the subterm property, these rules need to be instantiated only on terms in the conclusion (notice there are no frame rules with eigenvariables). Thus, e.g., for *Ref*, a step corresponding to the rule consists in adding to the left-hand side all atoms $x = x$ for all x in $\Gamma \Rightarrow \Delta$ and then writing the sequent thus obtained on top of each topmost sequent.

For any n, for sequents that are neither initial nor conclusions of $L\bot$ nor treatable by any one of the above reductions, we copy the sequent above itself. This step is needed to treat uniformly the failure of proof search in the following two cases: the case in which the search goes on for ever because new rules always become applicable, and the case in which a sequent is reached which is not a conclusion of any rule nor an initial sequent.

If the reduction tree is finite, all its leaves are initial or conclusions of $L\bot$, and the tree read from the leaves to the root, yields a derivation by spelling out in individual rule applications the simultaneous reduction steps.

2. *Construction of the countermodel.* If the reduction tree is infinite, it has an infinite branch. Let $\Gamma_0 \Rightarrow \Delta_0 \equiv \Gamma \Rightarrow \Delta, \Gamma_1 \Rightarrow \Delta_1, ..., \Gamma_i \Rightarrow \Delta_i, ...$ be one such branch. Consider the set of labelled formulas and relational atoms

$$\mathbf{\Gamma} \equiv \bigcup_{i>0} \Gamma_i \qquad \mathbf{\Delta} \equiv \bigcup_{i>0} \Delta_i$$

We define a model that forces all the formulas in $\mathbf{\Gamma}$ and no formula in $\mathbf{\Delta}$ and is therefore a countermodel to the sequent $\Gamma \Rightarrow \Delta$. Consider the system $\mathcal{S}$ whose elements are all the worlds denoted by the labels appearing in $\mathbf{\Gamma}$ and whose relations are all those expressed by the relations xR_wy in $\mathbf{\Gamma}$. The system $\mathcal{S}$ is transitive, strongly connected and $\preceq_w$-minimal by construction. The model is defined as follows: for all atomic formulas P such as $x : P$ is in $\mathbf{\Gamma}$, we stipulate that $x \Vdash P$.

We now show inductively on the weight of formulas that A is forced at world x if $x : A$ is in $\mathbf{\Gamma}$ and is not forced at world x if $x : A$ is in $\mathbf{\Delta}$.

If $x : A \equiv x : \bot$, $x : A$ cannot be in $\mathbf{\Gamma}$ since no sequent in $\Gamma_0 \Rightarrow \Delta_0$ has $x : \bot$ on the left-hand side, so it is not forced at any world of the model.

The antecedent atomic case has already been taken care of by definition, and the succedent one follows from the fact that no initial sequent is in the branch.

If $x : A \equiv x : B\&C$ is in $\mathbf{\Gamma}$, there exists i such that $x : B\&C$ appears first in Γ_i and therefore, for some $g \geq 0$, $x : B$ and $x : C$ appear in Γ_{i+g}. By inductive hypothesis, $x \Vdash B$ and $x \Vdash C$, hence $x \Vdash B\&C$.

If $x : A \equiv x : B\&C$ is in $\mathbf{\Delta}$, consider the step i in which the reduction for $x : B\&C$ applies. This gives a branching, such that one of the branches belongs to $\Gamma_0 \Rightarrow \Delta_0$. So either $x : B$ or $x : C$ is in $\mathbf{\Delta}$, and so by inductive hypothesis, either $x \nVdash B$ or $x \nVdash C$. Hence $x \nVdash B\&C$.

If $x : A \equiv x : B \vee C$ is in $\mathbf{\Gamma}$, consider the step i in which the reduction for $x : B \vee C$ applies. This gives a branching, such that one of the branches belongs to $\Gamma_0 \Rightarrow \Delta_0$. So either $x : B$ or $x : C$ is in $\mathbf{\Gamma}$, and so by inductive hypothesis, either $x \Vdash B$ or $x \Vdash C$. Hence $x \Vdash B \vee C$.

If $x : A \equiv x : B \vee C$ is in $\mathbf{\Delta}$, there exists i such that $x : B \vee C$ appears first in Δ_i and therefore, for some $g \geq 0$, $x : B$ and $x : C$ appear in Δ_{i+g}. By inductive hypothesis, $x \nVdash B$ and $x \nVdash C$, hence $x \nVdash B \vee C$.

If $x : A \equiv x : B \supset C$ is in $\mathbf{\Gamma}$, consider the step i in which the reduction for $x : B \supset C$ applies. This gives a branching, and one of the premisses belongs to the infinite branch. So either $x : B$ is in $\mathbf{\Delta}$ or $x : C$ is in $\mathbf{\Gamma}$. In the former case, by inductive hypothesis we have $x \nVdash B$ and so $x \Vdash B \supset C$. Otherwise, by inductive hypothesis we have $x \Vdash C$ and so $x \Vdash B \supset C$. Hence $x \Vdash B \supset C$.

If $x : A \equiv x : B \supset C$ is in $\mathbf{\Delta}$, there exists i such that $x : B \supset C$ appears first in Δ_i and therefore, for some $g \geq 0$, $x : B$ appears in Γ_{i+g} and $x : C$ in Δ_{i+g}. By inductive hypothesis, $x \Vdash B$ and $x \nVdash C$, hence $x \nVdash B \supset C$.

If $x : A \equiv x : \Box_w B$ is in $\mathbf{\Gamma}$, we consider all the relational atoms $xR_w y$ that occur in $\mathbf{\Gamma}$ (notice that at least there occurs the atom $xR_w x$). For any such atom, we find an occurrence of $y : B$ in $\mathbf{\Gamma}$ by construction of the reduction tree. So by inductive hypothesis, $y \Vdash B$, hence $x \Vdash \Box_w B$.

If $x : A \equiv x : \Box_w B$ is in $\mathbf{\Delta}$, consider the step i in which the reduction for $x : \Box_w B$ applies. Then for some $g \geq 0$, we find $y : B$ in Δ_{i+g} for some $xR_w y$ in Γ_{i+g}. So by inductive hypothesis, $y \nVdash B$, hence $x \nVdash \Box_w B$.

If $x : A \equiv x : B \Box\!\!\rightarrow C$ is in $\mathbf{\Gamma}$, consider the step i in which the reduction for $x : B \Box\!\!\rightarrow C$ applies. This gives a branching, such that one of the premisses belongs to $\Gamma_0 \Rightarrow \Delta_0$. So either $z : B$ is in $\mathbf{\Delta}$ for any given z already occurring in the reduction tree, or else we find some $y : B$ and $y : \Box_x(B \supset C)$ in $\mathbf{\Gamma}$. In the latter case, by inductive hypothesis we have $y \Vdash B$ and $y \Vdash \Box_x(B \supset C)$, and so $x \Vdash B \Box\!\!\rightarrow C$. In the former case we have by inductive hypothesis $z \nVdash B$, which does not yet allow us to infer $x \Vdash B \Box\!\!\rightarrow C$. However, $x : B \Box\!\!\rightarrow C$ always gets repeated in the premiss and for each branching determined by an application of $L \Box\!\!\rightarrow$ either the first or the second premiss belongs to the infinite branch. It is enough that the right premiss belongs to the infinite branch once to conclude that $x \Vdash B \Box\!\!\rightarrow C$. In the opposite case, we have that the left premiss always belongs to the infinite branch, and therefore $z \nVdash B$ for each available label. So, by the forcing condition for $\Box\!\!\rightarrow$, we have that $x \Vdash B \Box\!\!\rightarrow C$ because the antecedent is never satisfied.

If $x : A \equiv x : B \mathrel{\Box\!\!\rightarrow} C$ is in $\mathbf{\Delta}$, consider the step i in which the first reduction for $x : B \mathrel{\Box\!\!\rightarrow} C$ applies. Then $z : B$ appears in Γ_i. So by inductive hypothesis, $z \Vdash B$. This is not sufficient to conclude that $x \nVdash B \mathrel{\Box\!\!\rightarrow} C$, but proves that the first disjunct in the semantic clause for $\Box\!\!\rightarrow$ is false. We need to prove that also the second disjunct is false, i.e. that there is no y in the model such that $y \Vdash B$ and $y \Vdash \Box_x(B \supset C)$. By the saturation clause for $R \Box\!\!\rightarrow_2$ we have that for all available labels y, either $y : B$ or $y : \Box_x(B \supset C)$ are in the succedent in the infinite branch. In case the former holds for all available labels, then in particular we would have that $z : B$ would be in the succedent in the branch, with $z : B$ in the antecedent introduced by the clause for $R \Box\!\!\rightarrow_1$. Since the infinite branch cannot contain an initial (or derivable) sequent, this is excluded. So for at least one label the latter case holds and we obtain by inductive hypothesis that $y \nVdash \Box_x(B \supset C)$. Since y was an arbitrary y in the model such that $y \Vdash A$, together with the first part, we have the conclusion, namely $x \nVdash B \mathrel{\Box\!\!\rightarrow} C$. □

To illustrate the direct completeness proof, consider again Stalnaker's example given in the introduction:

If Hoover had been a Communist, he would have been a traitor.
If Hoover had been born in Russia, he would have been a Communist.

∴ If Hoover had been born in Russia, he would have been a traitor.

This inference corresponds to transitivity of the counterfactual and we can show how the calculus accounts for its failure by showing how to construct a countermodel for the sequent

(T) $\quad w : P \mathrel{\Box\!\!\rightarrow} Q, w : Q \mathrel{\Box\!\!\rightarrow} R \Rightarrow w : P \mathrel{\Box\!\!\rightarrow} R$

from a failed proof search. In the following, we avoid tedious application of unnecessary rules. We also consider only the non-terminating branch, for brevity. After a few steps, the (simplified) reduction tree looks like this:

$$
\begin{array}{cl}
\vdots & \\
xR_w y, s : P, x : P, y : P, y : Q, z : Q, z : \Box_w(Q \supset R), x : \Box_w(P \supset Q) \Rightarrow w : P \mathrel{\Box\!\!\rightarrow} R, y : R & \\
\hline
xR_w y, s : P, x : P, y : P, y : Q, x : \Box_w(P \supset Q), w : Q \mathrel{\Box\!\!\rightarrow} R \Rightarrow w : P \mathrel{\Box\!\!\rightarrow} R, y : R & L \Box\!\!\rightarrow \\
\hline
xR_w y, s : P, x : P, y : P, y : P \supset Q, x : \Box_w(P \supset Q), w : Q \mathrel{\Box\!\!\rightarrow} R \Rightarrow w : P \mathrel{\Box\!\!\rightarrow} R, y : R & L \supset \\
\hline
xR_w y, s : P, x : P, y : P, x : \Box_w(P \supset Q), w : Q \mathrel{\Box\!\!\rightarrow} R \Rightarrow w : P \mathrel{\Box\!\!\rightarrow} R, y : R & L\Box_w \\
\hline
xR_w y, s : P, x : P, x : \Box_w(P \supset Q), w : Q \mathrel{\Box\!\!\rightarrow} R \Rightarrow w : P \mathrel{\Box\!\!\rightarrow} R, y : P \supset R & R \supset \\
\hline
s : P, x : P, x : \Box_w(P \supset Q), w : Q \mathrel{\Box\!\!\rightarrow} R \Rightarrow w : P \mathrel{\Box\!\!\rightarrow} R, x : \Box_w(P \supset R) & R\Box_w \\
\hline
s : P, x : P, x : \Box_w(P \supset Q), w : Q \mathrel{\Box\!\!\rightarrow} R \Rightarrow w : P \mathrel{\Box\!\!\rightarrow} R & R \Box\!\!\rightarrow_2 \\
\hline
s : P, w : P \mathrel{\Box\!\!\rightarrow} Q, w : Q \mathrel{\Box\!\!\rightarrow} R \Rightarrow w : P \mathrel{\Box\!\!\rightarrow} R & L \Box\!\!\rightarrow \\
\hline
w : P \mathrel{\Box\!\!\rightarrow} Q, w : Q \mathrel{\Box\!\!\rightarrow} R \Rightarrow w : P \mathrel{\Box\!\!\rightarrow} R & R \Box\!\!\rightarrow_1
\end{array}
$$

It should be clear that application of other rules or that other orders of application wouldn't improve the proof search. The topmost sequent of the branch above is not an initial sequent nor conclusion of $L\bot$. Moreover, there is no relational atom of the form $zR_w v$ for some v, so we cannot apply $L\Box_w$ to $z : \Box_w(Q \supset R)$. Hence the only sensible move is to apply $R \Box\!\!\rightarrow_2$ to $w : P \mathrel{\Box\!\!\rightarrow} R$, which could introduce such atom via a subsequent application of $R\Box_w$. The rule $R \Box\!\!\rightarrow_2$ however introduces a branching: if the rightmost branch is to append the formula $z : \Box_w(P \supset R)$ to the right of the sequent arrow,

$$xR_w y, s : P, x : P, y : P, y : Q, z : Q, z : \Box_w(Q \supset R), x : \Box_w(P \supset Q) \Rightarrow w : P \mathrel{\Box\!\!\rightarrow} R, y : R, z : \Box_w(P \supset R)$$

thus giving a terminating branch (it suffices to apply a step of *SConn*), the leftmost branch appends the formula $z : P$ to the right of the sequent arrow,

$$xR_w y, s : P, x : P, y : P, y : Q, z : Q, z : \Box_w(Q \supset R), x : \Box_w(P \supset Q) \Rightarrow w : P \Box\!\!\rightarrow R, y : R, z : P$$

and this branch does not terminate. Clearly we could keep applying rules (since $R \Box\!\!\rightarrow$-rules copy the main formula, they can be applied indefinitely many times), but we can already "read off" a countermodel from this branch. We follow the steps for the construction of a countermodel given in the course of the proof of the above theorem. A countermodel to (T) is defined by the following clauses:

- $y \preceq_w x$;
- $s \Vdash P$;
- $x \Vdash P, x \Vdash Q$;
- $y \Vdash P, y \Vdash Q, y \nVdash R$;
- $z \Vdash Q, z \Vdash R$.

In $\mathcal{S}_C$, $x \Vdash P$ and $x \Vdash \Box_w(P \supset Q)$ so $w \Vdash P \Box\!\!\rightarrow Q$ for clause 2 in Lewis's truth conditions of the counterfactual conditional. So in (T), the first formula to the left of the sequent arrow is true (which correspond to the first premise in an instance of an argument by Transitivity). Moreover, $z \Vdash Q$ and $z \Vdash \Box_w(Q \supset R)$, so $w \Vdash Q \Box\!\!\rightarrow R$ in $\mathcal{S}_C$ for clause 2 in Lewis's truth conditions of the counterfactual conditional. So in (T), the second formula to the left of the sequent arrow is true (which correspond to the second premise in an instance of an argument by Transitivity). However, let y be such that $y \preceq_w x$ and let $y \Vdash P$ and $y \nVdash R$. Then $w \nVdash P \Box\!\!\rightarrow R$ (to visualize such model, suppose further that e.g. $z \preceq_w y$ and $z \nVdash P$). Hence $\mathcal{S}_C$ is a countermodel to (T).

§6. Decidability. In general cut elimination alone does not ensure terminating proof search in a given calculus. The exhaustive proof search used in the proof of Theorem 5 is not a decision method nor an effective method of finding countermodels when proof search fails, as it may produce infinite branches and therefore infinite countermodels. By way of example, consider the following branch in the search for a proof of the sequent $\Rightarrow w : \Box_x\neg\Box_x A \supset \Box_x B$ (this is analogous to the case for S4 discussed in Section 11.5 of Negri & von Plato 2011):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{wR_x y, yR_x z, wR_x z, zR_x t, w : \Box_x\neg\Box_x A \Rightarrow t : A, z : A, y : B}}{wR_x y, yR_x z, wR_x z, w : \Box_x\neg\Box_x A \Rightarrow z : \Box_x A, z : A, y : B}\,R\Box_x}{wR_x y, yR_x z, wR_x z, w : \Box_x\neg\Box_x A, z : \neg\Box_x A \Rightarrow z : A, y : B}\,L\supset}{wR_x y, yR_x z, wR_x z, w : \Box_x\neg\Box_x A \Rightarrow z : A, y : B}\,L\Box_x}{wR_x y, yR_x z, w : \Box_x\neg\Box_x A \Rightarrow z : A, y : B}\,Trans}{wR_x y, w : \Box_x\neg\Box_x A \Rightarrow y : \Box_x A, y : B}\,R\Box_x}{wR_x y, w : \Box_x\neg\Box_x A, y : \neg\Box_x A \Rightarrow y : B}\,L\supset}{wR_x y, w : \Box_x\neg\Box_x A \Rightarrow y : B}\,L\Box_x}{w : \Box_x\neg\Box_x A \Rightarrow w : \Box_x B}\,R\Box_x}{\Rightarrow w : \Box_x\neg\Box_x A \supset \Box_x B}\,R\supset$$

Clearly the search goes on forever because of the new accessibility relations that are created by the right rules for the indexed modalities, $R\Box_x$, together with *Trans*. To see that this search does not end up in a derivation, we may nevertheless exhibit a finite countermodel by a suitable truncation of the otherwise infinite countermodel provided by the completeness proof.

Following the method of finitization of countermodels generated by proof search in a labelled calculus, presented for intuitionistic propositional logic in Negri (2014a) and for multi-modal logics in Garg et al. (2012), we define a *saturation* condition for branches on a reduction tree. Intuitively, a branch is saturated when its leaf is not an initial sequent nor a conclusion of $L\bot$, and when it is closed under all the rules except for $R\Box_x$ in case it generates a loop modulo new labelling. To obtain the finite countermodel, we define a partial order through the reflexive and transitive closure of the similarity relation together with a relation that witnesses such loops. Let $\downarrow \Gamma (\downarrow \Delta)$ be the union of the antecedents (succedents) in a branch from the endsequent up to $\Gamma \Rightarrow \Delta$.

Let us define the following sets of formulas:

$$\mathcal{F}^1_{\Gamma\Rightarrow\Delta}(w) \equiv \{A \mid w : A \in \downarrow\Gamma\} \cup \{\Box_x A \mid y : \Box_x A, yR_x w \in \Gamma\}$$
$$\mathcal{F}^2_{\Gamma\Rightarrow\Delta}(w) \equiv \{A \mid w : A \in \downarrow\Delta\}$$

and let $w \leq_{\Gamma\Rightarrow\Delta} y$ iff $\mathcal{F}^i_{\Gamma\Rightarrow\Delta}(w) \subseteq \mathcal{F}^i_{\Gamma\Rightarrow\Delta}(y)$ for $i = 1, 2$.

DEFINITION 6.1. *A branch in a proof search up to a sequent $\Gamma \Rightarrow \Delta$ is* **saturated** *if the following conditions are satisfied:*[19]

1. *If w is a label in Γ, Δ, then $w = w$ and $wR_x w$ are in Γ.*
2. *If $wR_x y$ and $yR_x z$ are in Γ, then $wR_x z$ is.*
3. *If $wR_w x$ is in Γ, then $x = w$ is.*
4. *If w, x, y are labels in Γ, Δ, then either $wR_x y$ or $yR_x w$ is in Γ*
5. *There is no w such that $w : \bot$ is in Γ.*
6. *If $w : A\&B$ is in $\downarrow\Gamma$, then $w : A$ and $w : B$ are in $\downarrow\Gamma$.*
7. *If $w : A\&B$ is in $\downarrow\Delta$, then either $w : A$ or $w : B$ is in $\downarrow\Delta$.*
8. *If $w : A \vee B$ is in $\downarrow\Gamma$, then either $w : A$ or $w : B$ is in $\downarrow\Gamma$.*
9. *If $w : A \vee B$ is in $\downarrow\Delta$, then $w : A$ and $w : B$ are in $\downarrow\Delta$.*
10. *If $w : A \supset B$ is in $\downarrow\Gamma$, then either $w : A$ is in $\downarrow\Delta$ or $w : B$ is in $\downarrow\Gamma$.*
11. *If $w : A \supset B$ is in $\downarrow\Delta$, then $w : A$ is in $\downarrow\Gamma$ and $w : B$ is in $\downarrow\Delta$.*
12. *If $w : \Box_x A$ and $wR_x y$ are in Γ, then $y : A$ is in $\downarrow\Gamma$.*
13. *If $w : \Box_x A$ is in $\downarrow\Delta$, then either*
    a. *for some y, there is $wR_x y$ in Γ and $y : A$ is in $\downarrow\Delta$, or*
    b. *for some y such that $y \neq w$, there is $yR_x w$ in Γ and $w \leq_{\Gamma\Rightarrow\Delta} y$.*
14. *If $w : A \,\Box\!\!\rightarrow B$ is in Γ, then either $z : A$ is in $\downarrow\Delta$ for z in Γ, Δ, or for some y, $y : A, y : \Box_w(A \supset B)$ is in Γ.*
15. *If $w : A \,\Box\!\!\rightarrow B$ is in $\downarrow\Delta$, then $y : A$ is in $\downarrow\Gamma$ and either $z : A$ or $z : \Box_w(A \supset B)$ is in $\downarrow\Delta$ for z in Γ, Δ.*

Notice that this definition blocks the proof search in the example above when it produces the formula $t : \Box_x A$ because of clause 13.b (since we then have $t \leq_{\Gamma\Rightarrow\Delta} z$). The finite countermodel is defined as in the proof of Theorem 5.5 starting from the sets $\downarrow\Gamma, \downarrow\Delta$.

[19] Observe that some of the clauses take into account the cumulative character of some rules of the calculus, with the repetition of the principal formulas in the premisses an thus the avoidance of the downward closure in the respective saturation clauses.

PROPOSITION 6.2. *The finite countermodel defined by the saturation procedure is a comparative similarity system.*

Proof. We need to show (i) that the saturated branch provides a countermodel, and (ii) that the countermodel thus obtained is a comparative similarity system. For a given $\Gamma \Rightarrow \Delta$, let $\mathcal{W}$ be the set of labels in Γ, the relation $\mathcal{R}_w$ be the reflexive and transitive closure of R_w together with $\leq_{\Gamma \Rightarrow \Delta}$, the definition of interpretation and valuation being as usual. Then:

(i) $\langle \mathcal{W}, \mathcal{R}_w \rangle$ is a countermodel to $\Gamma \Rightarrow \Delta$ iff $x \Vdash A$ for all $x : A$ in Γ, and $x \nVdash A$ for all $x : A$ in Δ. Notice that this is a consequence of the following:

(a) If A is in $\mathcal{F}^1(x)$, then $x \Vdash A$.

(b) If A is in $\mathcal{F}^2(x)$, then $x \nVdash A$.

The two claims are proved by induction on the length of the formula. If it is atomic, they hold by definition of $\Vdash$. If it is a conjunction, disjunction, implication or conditional, they hold by the corresponding saturation clauses and the inductive hypothesis. If $A \equiv \Box_w B$ is in $\mathcal{F}^1(x)$, let $x \equiv x_0 \mathcal{R}_w \ldots \mathcal{R}_w x_n \equiv y$. For $n = 0$, either $x : \Box_w B$ is in $\downarrow \Gamma$, hence in Γ, or for some y, $y : \Box_w B$ and $y R_w x$ are in Γ. In both cases we get the conclusion by step (12) of the saturation procedure. For the inductive step, suppose first $A \equiv \Box_w B$ is in $\mathcal{F}^2(x)$. Then either for some y we have $x R_w y$ in Γ and $y : B$ in $\downarrow \Delta$, or there is a y distinct form x and such that $y R_w x$ is in Γ and $x \leq y$. From the former, we have $B \in \mathcal{F}^2(y)$ and so by inductive hypothesis, $y \nVdash B$, whence $x \nVdash \Box_w B$. From the latter, $\mathcal{F}^2(x) \subseteq \mathcal{F}^2(y)$, and so $\Box_x B \in \mathcal{F}^2(y)$. By inductive hypothesis (y is a smaller label in the R_w-ordering), $y \nVdash \Box_w B$, whence $x \nVdash \Box_w B$.

(ii) $\langle \mathcal{W}, \mathcal{R}_w \rangle$ is a comparative similarity system iff $\mathcal{R}_w$ is transitive, strongly connected and $\preceq_w$-minimal. The result follows immediately from points (2), (3) and (4) of the definition of saturation. □

We further observe that by the subterm property the number of distinct formulas in the sequents of an attempted proof is bounded. Since duplication of the same labelled formulas is not possible by hp-admissibility of contraction, we can prove the following result:

THEOREM 6.3. *The system* **G3LC** *allows a terminating proof search.*

Proof. Let F be the set of (unlabelled) subformulas of the endsequent and consider a string of labels $w_0 R_x w_1, w_1 R_x w_2, w_2 R_x w_3, \ldots$ generated by the saturation procedure. For an arbitrary w_j consider the values of the sets $\mathcal{F}^i(w_k)$ for $k < j$ at the step in which w_j was introduced. Clearly $\mathcal{F}^i(w_j) \nsubseteq \mathcal{F}^i(w_k)$ or else w_j would not have been introduced. So each new label corresponds to a new subset of $F \times F$. Since the number of these subsets is finite, also the length of each chain of labels must be finite. □

§7. Concluding remarks and related work. We presented **G3LC**, a Gentzen-style sequent calculus for David Lewis's logic of counterfactuals **VC**, and proved it sound and complete with respect to Lewis's semantics. In **G3LC**, substitution of labels and left and right weakening and contraction are height-preserving admissible and cut is admissible. Moreover, all the rules are invertible. Finally, we proved a decidability result based on a bounded procedure of root-first proof search that for any given sequent either provides a derivation or a countermodel.

What remains to be spelled out in detail are similar results for the extension with an alethic modality which we presented in Section 2. In his book *Counterfactuals*, Lewis presents a class **V** of axomatic systems for conditional logics, among which is **VC**.

We leave for further work a detailed deductive analysis of the entire class, as well as of conditional logics that are based on alternative versions of Lewis's semantics. We expect these further results to be achieved by application of the method developed here. Implementation of the calculi presented is also left for future work.

The first tableau proof systems for counterfactuals have been presented by de Swart (1983). These systems can be read either as Beth-tableaux systems, with rules for signed formulas, or as sequent systems, and they cover Stalnaker's system **VCS** and Lewis's system **VC**. The primitive connective chosen in de Swart's work is $\leq$, with the formula $A \leq B$ read as "*A is at least as possible as B*". We use instead the counterfactual conditional $A \mathrel{\Box\!\!\rightarrow} B$, read as "*If A were the case, then B would be the case*". These two connectives are interdefinable, as shown by Lewis, but a different choice of the primitive connective clearly gives origin to different proof systems. De Swart gives direct and constructive completeness proofs by using the calculi for defining a systematic proof search procedure that either gives a proof or a finite countermodel. Also in our system the completeness proof is direct and constructive, but the countermodel is constructed directly from the syntactic elements contained in a failed proof search branch, whereas in the Beth-tableaux approach the possible worlds are defined by nodes in the open search tree. There are other important differences which highlight the usefulness of the labelled approach that we have followed.

De Swart's system has, in addition to the standard classical propositional rules, a number $m \cdot n$ of distinct rules $F \leq (m, n)$ for each m, n, where m and n are positive integers that denote, respectively, the number of signed formulas of the form $F(A \leq B)$ or of the form $T(A \leq B)$ considered as principal formulas of the rule. Each such rule has the effect of discarding all the other formulas, which results in a lack of invertibility. It follows that in the proof search procedure what needs to be explored is not a single tree, but a set of trees.[20] Lastly, in our approach the rules are motivated through a robust meaning explanation that respects the general guidelines of inferentialism, as emphasized in Negri & von Plato (2015). On the contrary, the rules of the unlabelled approach seem to involve a not fully explicable genesis, being found "by the method of trial and error" (cf. de Swart 1983, p. 6). The inherent risk in the lack of a full methodological transparency became evident in a later correction by Gent (1992), who gave an example of a valid formula not derivable in de Swart's system and proposed an alternative sound a complete system for **VC**, while maintaining the main features of de Swart's original system.

Lellman & Pattinson (2012) present an unlabelled sequent calculus for Lewis's basic system and some of its extension with both the connective $\leq$ and the strong conditional[21] as primitive. The calculi are obtained through a procedure of *cut-elimination by saturation* which consists in closing a given set of rules under cut adding new rules. As a result, an optimal PSPACE complexity and Craig interpolation are established. Similar results are stated (without details) for the hybrid version of counterfactual logics proposed by Sano (2009). Although the latter work is not explicitly proof-theoretical, one may expect that an approach on the proof theory of counterfactuals based on hybrid logic[22] could be successfully developed. In hybrid logic the accessibility relation is not primitive but

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[20] In an example detailed in de Swart (1983, p. 10–11), a proof search for a sequent that contains only two formulas of the form $F(A \leq B)$ and $F(B \leq D)$ results, because of all the combinatorial possibilities, in the construction of 24 different partial trees.

[21] The strong conditional $\mathrel{\Box\!\!\Rightarrow}$ can be defined in terms of $\leq$ by $A \mathrel{\Box\!\!\Rightarrow} B \equiv \neg(A \& \neg B) \leq (A \& B)$.

[22] See Blackburn (2000) for a good survey presentation of the motivations and main results of the methodology of hybrid logic.

absorbed at the level of the object language in a different way from ours through the extension of the scope of the modalities to act also on indices for possible worlds. In particular, among the systems proposed for the proof theory of hybrid logic, the Seligman-style approach (for which results of completeness and termination have been recently established in Blackburn et al. 2015) appears as a good candidate for further generalization.

The work by Olivetti et al. (2007) presents a labelled sequent calculus for Lewis conditional logics using the overall methodology of Negri (2005). It thus is methodologically close to our approach, but rests crucially on the limit assumption, which results in the avoidance of the problematic nesting of quantifiers in the truth condition for the conditional. In so far as Lewis's preferred interpretation of the counterfactual conditional rejects the limit assumption, the strategy followed in the present paper appears to be a more faithful proof-theoretic analysis of Lewis's work.

Indexed modalities have already been used in the treatment of conditionals, in different ways. The idea of conditionality as a "sententially indexed modality" was hinted at already by Lewis (1971, p. 80) and further explicated by Chellas (1975, Section 5) in a proposal of using propositionally indexed modalities as a unifying framework for conditionals: "*If A, then B* means that the proposition expressed by B is in some way necessary with respect to that expressed by A". The conditional is thus suggestively rewritten as $[A]B$. Equivalent sentences A and A' give equivalent propositional indexed modalities $[A]$ and $[A']$ and each $[A]$ satisfies a normality condition written as the rule (for $n \geq 0$)

$$\frac{B_1 \& \dots \& B_n \supset B}{[A]B_1 \& \dots \& [A]B_n \supset [A]B} \; RCK$$

Modalities indexed by possible worlds represent a different notion which is used in our work to factorize the explanation of the conditional with a family of ordinary modalities obtained in the usual way from possible worlds and accessibility relations. Indexed modalities (with indices ranging among possible worlds) are used by Giordano et al. (2008) in association to a preferential semantics for conditionals. Their explanation rests on the limit assumption and indexed modalities are used to identify the closest worlds to world x that satisfy the antecedent of the conditional, denoted $Min_x(A)$, and thus appear in the antecedent of its truth condition; the resulting calculus is therefore quite different from ours, not just for being a tableau style system that uses the limit assumption. Indexed modalities are found also in other venues, such as the formalization of deontic reasoning (Gabbay & Governatori 1998), but there seem to be no general theory of indexed modalities on their own.

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