Correction note to "Proof theory for non-normal modal logics: The neighbourhood formalism" and basic results"

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The sequent calculus **G3n** for system **E** presented in my paper *Proof theory for non-normal modal logics: The neighbourhood formalism and basic results* (this Journal, vol. 4(4), pp. 1241–1286, 2017) is not cut free. This can be seen by showing that the valid sequent $x : \Box(A\&B) \Rightarrow x : \Box(B\&A)$ is not derivable without a cut. The reason for this problem is the form of the left rule for $\lhd$, with the formula $y : A$ in the antecedent of the conclusion

$$\frac{y \in a, y : A, A \lhd a, \Gamma \Rightarrow \Delta}{y : A, A \lhd a, \Gamma \Rightarrow \Delta}\ L\lhd$$

A similar form, used for example for $L\Box$ allows to reduce the number of premisses (from two to one): so instead of the rule

$$\frac{x : \Box A, \Gamma \Rightarrow \Delta, xRy \quad y : A, x : \Box A, \Gamma \Rightarrow \Delta}{x : \Box A, \Gamma \Rightarrow \Delta}\ L\Box$$

one can use the equivalent rule

$$\frac{y : A, xRy, x : \Box A, \Gamma \Rightarrow \Delta}{xRy, x : \Box A, \Gamma \Rightarrow \Delta}\ L\Box$$

In this way, an application of rule $L\Box$ is licenced just when we have an accessibility atom of the form xRy in the antecedent of the conclusion. The reason why a similar reduction doesn't work in the case under discussion is that the formula $y : A$ doesn't behave like a relational atom: it can be principal in a right rule and therefore a cut

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with left premiss derived by a right rule with $y : A$ principal and left premiss derived by $L \lhd$ with $y : A, A \lhd a$ in the conclusion cannot be permuted.

We can obtain a cut-free sequent calculus for system of **E** by just avoiding the simplification step used for $L\Box$ and using the rule with two premisses

$$\frac{A \lhd a, \Gamma \Rightarrow \Delta, y : A \quad y \in a, A \lhd a, \Gamma \Rightarrow \Delta}{A \lhd a, \Gamma \Rightarrow \Delta} L \lhd$$

Here y is an arbitrary label, but it is enough—by the usual argument that shows analyticity as an application of height-preserving substitution of labels—to restrict the rule to labels in the conclusion.

All the results stated in the paper hold with the two-premiss version of the rule; obvious modifications to account for the new form of the rule are needed in Lemma 3.3, Lemma 4.2, Lemma 4.5, Theorem 4.9, Theorem 5.3, Definition 5.4, and Lemma 5.5. For completeness, these modifications are detailed below.

Lemma 3.3. *Rule RE is admissible in* **G3E***.*

Proof. By the following derivation:

$$\cfrac{\cfrac{\cfrac{x : A \Rightarrow x : B}{a \Vdash^{\forall} A \Rightarrow a \Vdash^{\forall} B} 3.2 \qquad \cfrac{y : B \ldots \Rightarrow \ldots y : A \quad y \in a \ldots \Rightarrow \ldots y \in a}{y : B, a \in I(x), a \Vdash^{\forall} A, A \lhd a \Rightarrow x : \Box B, y \in a} L \lhd}{a \in I(x), a \Vdash^{\forall} A, A \lhd a \Rightarrow x : \Box B, a \Vdash^{\forall} B \quad a \in I(x), a \Vdash^{\forall} A, A \lhd a \Rightarrow x : \Box B, B \lhd a} R \lhd}{\cfrac{a \in I(x), a \Vdash^{\forall} A, A \lhd a \Rightarrow x : \Box B}{x : \Box A \Rightarrow x : \Box B} L\Box} R\Box$$

QED

Lemma 4.2(2). Sequents of the following form are derivable in $\mathbf{G3n}^*$ for arbitrary formulas A and B in the propositional modal language of $\mathbf{G3n}^*$:

2. $A \lhd a, \Gamma \Rightarrow \Delta, A \lhd a$

Proof. 2. By the following derivation

$$\cfrac{\cfrac{x : A, A \lhd a, \Gamma \Rightarrow \Delta, x \in a, x : A \quad x \in a, x : A, A \lhd a, \Gamma \Rightarrow \Delta, x \in a}{x : A, A \lhd a, \Gamma \Rightarrow \Delta, x \in a} L \lhd}{A \lhd a, \Gamma \Rightarrow \Delta, A \lhd a} R \lhd$$

where one topsequent is derivable by inductive hypothesis and the other is initial. QED

Lemma 4.5(13).

13. If $\vdash_n A \lhd a, \Gamma \Rightarrow \Delta$ then $\vdash_n A \lhd a, \Gamma \Rightarrow \Delta, y : A$ and $\vdash_n y \in a, A \lhd a, \Gamma \Rightarrow \Delta$.

Theorem 4.9. Cut is admissible in **G3n***.

Proof. 4. The cut formula is $A \lhd a$, principal in both premisses of cut. We have:

$$\dfrac{\dfrac{\overset{\mathcal{D}}{x : A, \Gamma \Rightarrow \Delta, x \in a}}{\Gamma \Rightarrow \Delta, A \lhd a}\, R\lhd \qquad \dfrac{A \lhd a, \Gamma' \Rightarrow \Delta', y : A \quad y \in a, A \lhd a, \Gamma' \Rightarrow \Delta'}{A \lhd a, \Gamma' \Rightarrow \Delta'}\, L\lhd}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, \mathit{Cut}$$

The cut is converted as follows:

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, A \lhd a \quad A \lhd a, \Gamma' \Rightarrow \Delta', y : A}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', y : A}\, \mathit{Cut} \qquad \dfrac{\overset{\mathcal{D}(y/x)}{y : A, \Gamma \Rightarrow \Delta, y \in a} \quad \dfrac{\Gamma \Rightarrow \Delta, A \lhd a \quad y \in a, A \lhd a, \Gamma' \Rightarrow \Delta'}{y \in a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, \mathit{Cut}}{y : A, \Gamma^2, \Gamma' \Rightarrow \Delta^2, \Delta'}\, \mathit{Cut}}{\Gamma^3, \Gamma'^2 \Rightarrow \Delta^3, \Delta'^2}\, \mathit{Cut}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, \mathit{Ctr}^*$$

where the two upper cuts are of reduced cut height and the lower ones of reduced weight of cut formula because $\mathtt{w}(y \in a) < \mathtt{w}(A \lhd a)$, $\mathtt{w}(y : A) < \mathtt{w}(A \lhd a)$. QED

Theorem 5.3. If $\Gamma \Rightarrow \Delta$ is derivable in **G3n*** (respectively **G3nM***, **G3nC***, **G3nN***), then it is valid in the class of neighbourhood frames (respectively neighbourhood frames which are supplemented, closed under intersection, containing the unit) with the * properties.

Proof. If the last rule is $L \lhd$, assume that the premisses $A \lhd a, \Gamma \Rightarrow \Delta, y : A$, $y \in a, A \lhd a, \Gamma \Rightarrow \Delta$ are valid, and let (ρ, σ) be an arbitrary SN-realisation with (1) $\mathcal{M} \models_{\rho,\sigma} A \lhd a, \Gamma \Rightarrow \Delta, y : A$ and (2) $\mathcal{M} \models_{\rho,\sigma} y \in a, A \lhd a, \Gamma \Rightarrow \Delta$ and assume $\mathcal{M} \models_{\rho,\sigma} A \lhd a, \Gamma$. If (1) gives that there is B in Δ such that $\mathcal{M} \models_{\rho,\sigma} B$ we are done. Else we have $\rho(y) \in [A]$; since by assumption $[A] \subseteq \sigma(a)$, we have $\rho(y) \in \sigma(a)$, thus $\mathcal{M} \models_{\rho,\sigma} y \in a$. From (2) and it follows that there is B in Δ such that $\mathcal{M} \models_{\rho,\sigma} B$. QED

Definition 5.4($L \lhd$). We say that a branch in a proof search from the endsequent up to a sequent $\Gamma \Rightarrow \Delta$ is *saturated* with respect to rule $L \lhd$ if the following condition holds

($L \lhd$) If $A \lhd a$ and y are in $\downarrow\Gamma$, then $y \in a$ is in Γ or $y : A$ is in Δ.

Lemma 5.5(d). Let $\mathcal{B} \equiv \{\Gamma_i \Rightarrow \Delta_i\}$ be a saturated branch in a proof-search tree for $\Gamma \Rightarrow \Delta$. Then there exists a countermodel $\mathcal{M}$ to $\Gamma \Rightarrow \Delta$, which makes all the formulas in $\mathbf{\Gamma}$ true, and all the formulas in $\mathbf{\Delta}$ false.

Proof. (d) If $A \lhd a$ is in $\mathbf{\Gamma}$, let y be an arbitrary world in the model, that is, by definition of $\mathcal{M}$, a label in $\downarrow\Gamma$. Then by saturation $y \in a$ is in Γ or $y : A$ is in Δ, so by inductive hypothesis $\mathcal{M} \not\models_{\rho,\sigma} y : A$ or $\mathcal{M} \models_{\rho,\sigma} y \in a$. Overall, this means that $\mathcal{M} \models_{\rho,\sigma} A \lhd a$. QED

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