On the duality of proofs and countermodels in labelled sequent calculi

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The duality of proofs and counterexamples, or more generally, refutations, is ubiquitous in science, but involves distinctions often blurred by the rethoric of argumentation. More crisp distinctions between proofs and refutations are found in mathematics, especially in well defined formalized fragments.

Every working mathematician knows that finding a proof and looking for a counterexample are two very different activities that cannot be carried on simultaneusly. Usually the latter starts when the hope to find a proof is fading away, and the failed attempts will serve as an implicit guide to chart the territory in which to look for a counterexample. No general recipe is, however, gained from the failures, and a leap of creativity is required to find a counterxample, if such is at all obtained.

In logic, things are more regimented because of the possibility to reason within formal *analytic calculi* that reduce the proving of theorems to *automatic* tasks. Usually one can rest upon a completeness theorem that guarantees a perfect duality between proofs and countermodels. So in theory. In practice, we are encountered with obstacles: completeness proofs are often non-effective (non-constructive) and countermodels are artificially built from Henkin sets or Lindenbaum algebras, and thus far away from what we regard as counterexamples. Furthermore, the canonical countermodels provided by traditional completeness proofs may fall out of the intended classes and need a model-theoretic fine tuning with such procedures as unravelling and bulldozing.

The question naturally arises as to whether we can find in some sense "concrete" countermodels in the same automated way in which we find proofs. *Refutation calculi* (as those found in [5, 9, 20, 23]) produce refutations rather than proofs and can be used as a basis for building countermodels. These calculi are separate from the direct inferential systems, their rules are not invertible (root-first, the rules give only sufficient conditions of non-validity) and sometimes the decision method through countermodel constructions uses a pre-processing of formulas into a suitable normal form (as in [11]). As pointed out in [10] in the presentation of a combination of a derivation and a refutation calculus for bi-intuitionistic logic, these calculi often depart from Gentzen's original systems, because the sequent calculus LI or its contraction-free variant LJT [2] have rules that are not invertible; thus, while preserving validity, they do not preserve refutability. *Prefixed tableaux* in the style of Fitting, on the other hand, restrict the refutations to relational models, and countermodels can be read off from failed proof search. As remarked in [6], the tree structure inherent in these calculi makes them suitable to a relatively restricted family of logics and, furthermore, the non-locality of the rules makes the extraction of the countermodel not an immediate task.

We shall present a method for unifying proof search and countermodel construction that is a synthesis of a generation of calculi with internalized semantics (as presented in [14] and in chapter 11 of [19]), a Tait-Schütte-Takeuti style completeness proof [15] and, finally, a procedure to finitize the countermodel construction. This final part is obtained either through the search of a minimal, or irredundant, derivation (a procedure employed to establish decidability of basic modal logics in [14] and formalized in [7] for a labelled sequent system for intuitionistic logic), a pruning of infinite branches in search trees through a suitable syntactic counterpart of semantic filtration (a method employed in [1] for Priorean linear time logic and in [8] for multimodal logics) or through a proof-theoretic embedding into an appropriate provability logic that internalizes finiteness in its rules, as in [4].

The emphasis here is on the *methodology*, so we shall present the three stages in detail for the case of intuitionistic logic. Our starting point is **G3I**, a labelled contraction- and cut-free intuitionistic multi-succedent calculus in which *all rules are invertible*. The calculus is obtained through the internalization of Kripke semantics for intuitionistic logic: the rules for the logical constants are obtained by unfolding the inductive definition of truth at a world and the properties of the accessibility relation are added as rules, following the method of "axioms as rules" to encode axioms into a sequent calculus while preserving the structural properties of the basic logical calculus [18, 13]. The structural properties guarantee a root-first determinism, with the consequence that there is no need of backtracking in proof search. Notably for our purpose, all the rules of the calculus preserve countermodels because of invertibility, and thus any terminal node in a failed proof search gives a Kripke countermodel.

The methodology of generation of complete analytic countermodel-producing calculi covers in addition the following (families of) logics and extensions:

Intermediate logics and their **modal companions**: These are obtained as extensions of **G3I** and of the labelled calculus for basic modal logic **G3K** by the addition of geometric frame rules. Because of the uniformity of generation of these calculi, proofs of faithfulness of the modal translation between the respective logical systems are achieved in a modular and simple way [3].

Provability logics: The condition of Noetherian frames, though not first order, is internalized through suitable formulations of the right rule for the modality. By choosing *harmonious* rules (as in [14]), a syntactic completeness proof for Gödel-Löb provability logic was obtained. Through a variant of the calculus obtained by giving up harmony, we achieve instead a semantic completeness proof which gives at the same time also decidability and the finite model property [17]. Completeness for Grzegorczyk provability logic **Grz** is obtained in a similar semantic way and prepares the ground for a syntactic embedding of **Int** into **Grz** and thus for an indirect decision procedure for intuitionistic logic [4].

Knowability logic: This logic has been in the focus of recent literature on the investigation of paradoxes that arise from the principles of the verificationist theory of truth [21]. By the methods of proof analysis, it has been possible to pinpoint how the ground logic is responsible for the paradoxical consequences of these principles. A study focused on the well known Church-Fitch paradox brought forward a new challenge to the method of conversion of axioms into rules. The *knowability principle*, which states that whatever is true can be known, is rendered in a standard multimodal alethic/epistemic language by the axiom $A \supset \Diamond \mathcal{K} A$. This axiom corresponds, in turn, to the frame property

$$\forall x \exists y (xRy \,\&\, \forall z (yR_{\mathcal{K}}z \supset x \leqslant z))$$

Here R, $R_{\mathcal{K}}$, and $\leqslant$ are the alethic, epistemic, and intuitionistic accessibility relations, respectively. This frame property goes beyond the scheme of geometric implication and therefore the conversion into rules cannot be carried through with the usual rule scheme for geometric implications. In this specific case, we succeeded with a combination of two rules linked together by a side condition on the eigenvariable. The resulting calculus has all the structural properties of the ground logical system and leads to definite answers to the questions raised by the Church-Fitch paradox by means of a complete control over the structure of derivations for knowability logic [12].

Extensions beyond geometric theories: The generalization and systematization of the method of system of rules allows the treatment of axiomatic theories and of logics characterized by frame properties expressible through *generalized geometric implications* that admit arbitrary quantifier alternations and a more complex propositional structure than that of geometric implications [16]. The class of generalized geometric implications is defined as follows: We start from a *geometric axiom* (i.e. a conjunct in the canonical form of a geometric implication [22], where the P_i range over a finite set of atomic formulas and all the M_j are conjunctions of atomic formulas and the variables y_j are not free in the P_i)

$$GA_0 \equiv \forall \overline{x} (\& P_i \supset \exists y_1 M_1 \vee \ldots \vee \exists y_n M_n)$$

We take GA_0 as the base case in the inductive definition of a *generalized geometric axiom*. We then define

$$GA_1 \equiv \forall \overline{x} (\,\& \, P_i \supset \exists y_1 \,\&\, GA_0 \vee \ldots \vee \exists y_m \,\&\, GA_0)$$

Next we define by induction

$$GA_{n+1} \equiv \forall \overline{x} (\,\& \, P_i \supset \exists y_1 \,\&\, GA_{k_1} \vee \ldots \vee \exists y_m \,\&\, GA_{k_m})$$

Here $\& \, GA_i$ denotes a conjunction of GA_i-axioms and $k_1, \ldots, k_m \leqslant n$.

Through an operative conversion to normal form, generalized geometric implications can also be characterized in terms of *Glivenko classes* as those first-order formulas that do not contain implications or universal quantifiers in their negative parts.

The equivalence, established in [13], between the axiomatic systems based on geometric axioms and contraction- and cut-free sequent systems with geometric rules, is extended by a suitable definition of *systems of rules* for generalized geometric axioms. Here the word "system" is used in the same sense as in linear algebra where there are systems of equations with variables in common, and each equation is meaningful and can be solved only if considered together with the other equations of the system. In the same way, the systems of rules considered in this context consist of rules connected to each other by some variables and subject in addition to the condition of appearing in a certain order in a derivation.

The precise form of system of rules, the structural properties for the resulting extensions of sequent calculus (admissibility of cut, weakening, and contraction), a generalization of Barr's theorem, examples from axiomatic theories and applications to the proof theory of non-classical logics through a proof of completeness of the proof systems obtained, are all detailed in [16].

We shall conclude with some open problems and further directions.

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