Non-normal modal logics: a challenge to proof theory

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Abstract: A general procedure that follows the guidelines of inferentialism is presented for generating G3-style sequent calculi for non-normal modal logics on the basis of neighbourhood semantics.

Keywords: Neighbourhood semantics, classical modal logics, non-normal modal logics, labelled sequent calculus.

1 Introduction

The early decades of the modern study of modal logic were marked by the advent of possible worlds semantics. Earlier axiomatic studies of modal concepts were replaced by a solid and uniform semantic method that displayed the connections between modal axioms and conditions on the accessibility relation between possible worlds. The success of the semantic method, however, was not directly followed by equally powerful syntactic theories of modal and conditional concepts and reasoning and the literature until the 1990's shows a striking contrast between the generality of the semantic method and the scattered, goal-directed developments of the proof-theoretic method. Traditional Gentzen systems failed in the establishing of basic properties such as normalization/cut-elimination and analyticity even for basic modal systems. Awareness of this gap was often expressed by defaitist statements among practitioners in the field.

The insufficiency of traditional Gentzen systems to meet the challenge of the development of a proof theory for modal and non-classical logic has led to the development of alternative formalisms which, in one way or other, extend the syntax of sequent calculus. In the proliferation of calculi beyond Gentzen systems, there have been two main lines of development, one that enriches the structure of sequents (display calculi, hypersequents, nested sequents, tree-hypersequents, deep inference), another that maintains their simple structure but adds *labels* and relations in the form of variables and atomic formulas.

Through the conversion of what are known as geometric implications into rules that extend sequent calculus in a way that maintains the admissibility of structural rules, it has been possible to obtain a uniform presentation of a large family of modal logics, all those characterised by first-order frame conditions, and provability logics such as GL and Grz[1]. Parallel to the formal developments and contributions to widening the scope of such calculi, methodological reflections have been directed to the questions on how well they respond to the central issues of inferentialism (as analyzed in Negri and von Plato 2015, Read 2015), a discussion that goes under the wide umbrella of the extension of proof-theoretic semantics to non-classical logics.

Despite a wide range of applications, the powerful methods of possible worlds semantics are not a universal tool in the analysis of philosophical logics: they impose the straitjacket of *normality*, i.e. validity of the rule of necessitation, from $\vdash A$ to infer $\vdash \Box A$, and of the K axiom, $\Box(A \supset B) \supset (\Box A \supset \Box B)$. The limitative character of these imposed validities becomes clear in many of the logics that one encounters in the ever-expanding domains of applications of modal logic (in mathematics, philosophy, computer science, linguistics, cognitive science, social science). For instance, with the epistemic reading of the modality as a knowledge operator, an agent knows A if A holds in all the epistemic states available to her, and then the properties have the consequence that (1) *whatever has been proved is known* and (2) *an agent knows all the logical consequences of what she knows.* This leads to *logical omniscience*, clearly inadequate for cognitive agents with human capabilities, and thus to the rejection of both requirements. The same limitation is clear in the interpretation of the modality as a likelihood operator where one sees that the normal modal-logical validity of $\Box A \& \Box B \supset \Box(A \& B)$ should be avoided (Pacuit 2007).

Another limitation in systems based on a typical Kripke-style semantics is that the propositional base is classical or intuitionistic logic. In both cases one is forced to material implication, shown since the analysis of C.I. Lewis to be an inadequate form of conditional if logical analysis is to be pursued in other venues than mathematics: the classical propositional base of modal logic is insufficient to treat conditionals beyond material or strict implication, as shown in David Lewis' *Counterfactuals* (1973).

Among non-normal modal logics, *classical* modal logics are those obtained by requiring that the modality respects logical equivalence, that is closure under the rule $\frac{A \supset\subset B}{\Box A \supset\subset \Box B}$. One can then obtain other systems below

[1]Cf. Negri 2005, 2015; Dyckhoff and Negri 2015, 2016.

the normal modal logic **K** by removing the normality axiom and the necessitation rule and adding other axioms; combinations of the axiom schemes M, C, N give a lattice of eight different logics (cf. the diagram on p. 237 of Chellas 1980). It is known that non-normal modal logics can be simulated through an appropriate translation by a normal modal logic with three modalities (cf. Gasquet and Herzig 1996, Kracht and Wolter 1999), so that their proof theory can be approached indirectly through the translation: in the system proposed by Gilbert and Maffezioli (2015), the translation from non-normal to normal modal logics is used to define labelled sequent calculi with non-local *systems of rules* (in the sense of Negri 2016) for basic systems of *classical* modal logics.

Rather than reducing non-normal modal logics to normal ones, we shall develop proof systems for them in a way that parallels the generation of labelled calculi for systems based on possible world semantics. To this end, we shall use the more general *neighbourhood semantics* which was introduced in the 1970's to provide a uniform semantic framework for philosophical logics that cannot be accommodated within the normal modal logic setting. Instead of an accessibility relation on a set of possible worlds one has for every possible world w a family of sets of possible worlds called neighbourhoods of w. The correspondence between relational frames and certain specific types of neighbourhood frames shows that neighbourhood semantics is a generalisation of the earlier possible world semantics. Further, it gives a way to transfer the intuition from one semantics to the other: roughly, worlds in a neighbourhood of w replace worlds accessible from w.

Our goal is to set the grounds for a proof theory of non-normal modal systems based on neighbourhood semantics, to achieve this directly, i.e. without the use of translations, with local rules, and in a way that makes possible extensions in various directions.

The goal will be accomplished by following the guidelines of inferentialism, that is, by starting from the meaning explanations of logical constants and converting them into well-behaved rules of a calculus through a five-stage procedure. In the systems obtained, all the logical rules are invertible and all structural rules admissible. On the one hand, these properties makes the proof of metatheorems such as completeness a straightforward task; on the other, thay make the calculi obtained suitable for proof search.

We concentrate here on the procedure of generation of calculi based on neighbourhood semantics and only mention the properties that these calculi enjoy. Detailed statements and proofs of the structural properties of such calculi as well as a more comprehensive bibliography will be given in Negri

(2017). The development of calculi for conditional logics based on neighbourhood semantics and their application to the decision problem for such logics is presented in Negri and Olivetti (2015) and Girlando et al. (2016).

2 Neighbourhood Semantics

A *neighbourhood frame* is a pair $\mathcal{F} \equiv (W, I)$, where W is a set of worlds (states) and I is a neighbourhood function

$$I : W \longrightarrow \mathcal{P}(\mathcal{P}(W))$$

that assigns a collection of sets of worlds to each world in W. A *neighbourhood model* is then a pair $\mathcal{M} \equiv (\mathcal{F}, \mathcal{V})$, where $\mathcal{F}$ is a neighbourhood frame and $\mathcal{V}$ a propositional valuation, i.e. a map $\mathcal{V} : \mathtt{Atm} \longrightarrow \mathcal{P}(W)$.

Worlds in a neighbourhood are the substitute, in this more general semantics, of accessible worlds. The inductive clauses for truth of a formula in a model are the usual ones for the propositional clauses; for the modal operator we have

$$\mathcal{M}, w \Vdash \Box A \equiv \textit{ext}(A) \textit{is in } I(w),$$

where $\textit{ext}(A) \equiv \{u \in W | \mathcal{M}, u \Vdash A\}$.

Given a relational frame (W, R), one can define a neighbourhood frame by taking as neighbourhoods of a world x the supersets of worlds accessible from x

$$I^R(x) \equiv \{a \mid R(x) \subseteq a\}$$

Conversely, given a neighbourhood frame (W, I) one can define a relational frame by

$$xR^I y \equiv y \in \bigcap I(x)$$

A neighbourhood frame is *augmented* if for all x, $\bigcap I(x) \in I(x)$ and is supplemented, i.e. closed under supersets. Relational frames correspond to augmented neighbourhood frames, in the sense that given a relational frame, there is an augmented neighbourhood frame that validates the same formulas, and viceversa. [2]

[2] Details of the correspondence are found in Chellas (1980, p. 221). For an extensive survey on neighbourhood semantics see Pacuit (2007).

3 Five steps to good sequent calculi

In this section we shall present in detail the way to the determination of the rules of a G3-style sequent calculus that internalizes neighbourhood semantics. Much of the rationale is common to the determination of the rules of a G3-style sequent calculus based on possible worlds semantics, but there are new important elements that need to be taken into consideration when moving from possible worlds to the more general neighbourhood semantics.

The stages in the determination of the rules can be summarized as follows; each item will be further detailed together with the illustration of the procedure:

1. Turn the *semantic explanation* of logical constants into introduction rules of natural deduction.

2. Through *inversion principles*, find the corresponding elimination rules.

3. Translate the natural deduction system thus obtained into a sequent calculus. The resulting calculus is a sequent calculus with independent contexts.

4. Refine the calculus into a G3-style sequent calculus.

We observe that the above explanation is not really specific to the determination of labelled sequent calculi, but rather parametric and with an end-result that depends on the semantic explanation one starts with. With the BHK explanation of logical constants, the recipe gives the standard G3 sequent calculi, and in fact this is the route followed in ch. 1 of Negri and von Plato (2001). With possible worlds semantics, the meaning also depends on certain properties of an accessibility relation between worlds, so one has an additional step:

5. To obtain specific systems (e.g. intermediate logics) we add the rules for the accessibility relation following the method of "axioms as rules" (Negri and von Plato 1998, Negri 2003) for universal and geometric frame conditions.

The procedure for obtaining labelled calculi through this 5-stage explanation is carried through both for intuitionistic logic and basic modal logics in Negri and von Plato (2015); the resulting calculi are those which have been investigated, respectively, in Dyckhoff and Negri (2012) and in Negri

(2005). We remark that one does not need to stop at geometric frame conditions, but one can expand the spectrum of frame conditions that can be dealt with: *generalised geometric implications* can be treated by the method of *systems of rules* as developed in Negri (2016) without extending the language but at the expense of locality. With the method of *geometrization of first-order logic* (Dyckhoff and Negri 2015) and more specifically by the addition of new primitives together with a semidefinitional conservative extension, one obtains a splitting of the rules for frame conditions into possibly several geometric rules; the resulting sequent calculi allow to capture logics with *arbitrary* first-order conditions on their Kripke frames.

Next, we detail the method and the new step needed for neighbourhood semantics.

1. Convert semantic explanations into introduction rules.
We start with the truth condition for the necessity modality in terms of neighbourhood semantics

$$x \Vdash \Box A \equiv \textit{for some } a \textit{ in } I(x).a = \textit{ext}(A)$$

i.e.

$$x \Vdash \Box A \equiv \exists a \in I(x).(\forall x(x \in a \to x \Vdash A) \,\&\, \forall x(x \Vdash A \to x \in a))$$

This cannot be converted into a local rule in a way similar to the condition in terms of relational semantics because of the nesting of quantifiers. To proceed we need a further step:

0. Add new primitives (definitional extension) and their rules.
The new primitives are the relation of "local" forcing, a forcing relation between neighbourhoods and formulas (here local is opposed to the pointwise forcing of possible worlds) and of "cover" between a formula and a neighbourhood, to express the mutual inclusions between the neighbourhood a and the extension of formula A.

$$a \Vdash^{\forall} A \equiv \forall x(x \in a \to x \Vdash A) \qquad A \lhd a \equiv \forall x(x \Vdash A \to x \in a)$$

Steps 0 and 1 give the following introduction rules:

$$\frac{\begin{array}{c}[x \in a]\\ \vdots\\ x : A\end{array}}{a \Vdash^{\forall} A} \Vdash^{\forall} I,\ x\textit{ fresh} \qquad \frac{\begin{array}{c}[x : A]\\ \vdots\\ x \in a\end{array}}{A \lhd a} \lhd I,\ x\textit{ fresh}$$

$$\frac{a \in I(x) \quad a \Vdash^{\forall} A \quad A \lhd a}{x : \Box A}\ \Box I$$

2. Obtain elimination rules through the *inversion principle*. The rule is determined in two stages: first, the elimination rule should be in accordance with the inversion principle; this states that whatever follows from the grounds for deriving a proposition must follows from that proposition (cf. Negri and von Plato 2001, p. 6). Application of this principle to $x : \Box A$ is unproblematic and gives the rule $\Box E$ below. For $a \Vdash^{\forall} A$ and $A \lhd a$ some further comments are in order. The grounds for deriving the proposition $a \Vdash^{\forall} A$ are given by a derivation of $x : A$ from $x \in a$ where x is arbitrary. Similarly for $A \lhd a$. In this case we thus have that a direct application of the inversion principle would take to higher-level elimination rules, where assumptions in derivations are given by other derivations. It is possible to avoid the recourse to higher-level rules by proceeding as for the elimination rule for implication in the system of natural deduction with general elimination rules and obtain the following elimination rules for $a \Vdash^{\forall} A$ and $A \lhd a$:[3]

$$\frac{a \Vdash^{\forall} A \quad x \in a \quad \begin{matrix}[x : A] \\ \vdots \\ D\end{matrix}}{D}\ \Vdash^{\forall} E \qquad \frac{A \lhd a \quad x : A \quad \begin{matrix}[x \in a] \\ \vdots \\ D\end{matrix}}{D}\ \lhd E$$

$$\frac{x : \Box A \quad \begin{matrix}[a \in I(x), a \Vdash^{\forall} A, A \lhd a] \\ \vdots \\ D\end{matrix}}{D}\ \Box E$$

3. Translate ND rules to sequent calculus rules. This is a part of the general procedure for transforming a system of natural deduction into one of sequent calculus. First, the deducibility relation is internalized with an explicit notation for it, the sequent arrow in place of vertical dots; then the rules are made local by having the open assumptions listed in the left hand

[3] The procedure that replaces a higher level rule that discharges derivations with an ordinary general elimination rule that discharges formulas has been called *flattening* and the limitation of its scope investigated by Olkhovikov and Schroeder-Heister (2014).

side of sequents at each step of a derivation. The introduction rules become right rules and the elimination rules left rules. The format of rules that one obtains by this translation is that of a *sequent calculus with independent contexts* (cf. von Plato 2001):

$$\Vdash^{\forall} I \quad \rightsquigarrow \quad \frac{x \in a, \Gamma \Rightarrow x : A}{\Gamma \Rightarrow a \Vdash^{\forall} A} \; R \Vdash^{\forall},\ x \textit{ fresh}$$

$$\Vdash^{\forall} E \quad \rightsquigarrow \quad \frac{\Gamma \Rightarrow x \in a \quad \Gamma', x : A \Rightarrow D}{a \Vdash^{\forall} A, \Gamma, \Gamma' \Rightarrow D} \; L \Vdash^{\forall}$$

$$\lhd I \quad \rightsquigarrow \quad \frac{x : A, \Gamma \Rightarrow x \in a}{\Gamma \Rightarrow A \lhd a} \; R \lhd,\ x \textit{ fresh}$$

$$\lhd E \quad \rightsquigarrow \quad \frac{\Gamma \Rightarrow x : A \quad \Gamma', x \in a \Rightarrow D}{A \lhd a, \Gamma, \Gamma' \Rightarrow D} \; L \lhd$$

$$\Box I \quad \rightsquigarrow \quad \frac{\Gamma \Rightarrow a \in I(x) \quad \Gamma' \Rightarrow a \Vdash^{\forall} A \quad \Gamma'' \Rightarrow A \lhd a}{\Gamma, \Gamma', \Gamma'' \Rightarrow x : \Box A} \; R\Box$$

$$\Box E \quad \rightsquigarrow \quad \frac{a \in I(x), a \Vdash^{\forall} A, A \lhd a, \Gamma \Rightarrow D}{x : \Box A, \Gamma \Rightarrow D} \; L\Box$$

4. Adapt the sequent calculus rules obtained in 3 to the G3 style. First, all rules are brought to the *shared context* form, that is, in two-premiss rules the same multisets of formulas appear in the contexts of both premisses. Second, the calculus has to be *multisuccedent*, so there are arbitrary multisets as contexts in the succedents of sequents. Third, rules that are not already invertible are made so by the repetition of the principal formulas in the premisses. Some optimization to reduce the number of premisses is also possible: rule $R\Box$ is rewritten as an equivalent two-premiss rule by having the formula $a \in I(x)$ in the antecedent of the conclusion so that one of the premisses becomes superfluous.

$$\frac{x \in a, \Gamma \Rightarrow \Delta, x : A}{\Gamma \Rightarrow \Delta, a \Vdash^{\forall} A} \; R \Vdash^{\forall},\ x \textit{ fresh} \qquad \frac{x \in a, x : A, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta} \; L \Vdash^{\forall}$$

$$\frac{y \in a, A \lhd a, y : A, \Gamma \Rightarrow \Delta}{A \lhd a, y : A, \Gamma \Rightarrow \Delta} \; L \lhd \qquad \frac{y : A, \Gamma \Rightarrow \Delta, y \in a}{\Gamma \Rightarrow \Delta, A \lhd a} \; R \lhd,\ y \textit{ fresh}$$

$$\frac{a \in I(x), a \Vdash^{\forall} A, A \lhd a, \Gamma \Rightarrow \Delta}{x : \Box A, \Gamma \Rightarrow \Delta} \; L\Box, \; a \textit{ fresh}$$

$$\frac{a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A, a \Vdash^{\forall} A \quad a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A, A \lhd a}{a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A} \; R\Box$$

Finally, as in standard G3 calculi, initial sequents have only labelled atomic formulas (of the form $x : P$) and neighbourhood atoms (a priori, of the form $x \in a$ or $a \in I(x)$) as principal. We observe that formulas of the form $a \in I(x)$ are never active in the right-hand side of the rules we have listed above, and therefore the corresponding initial sequents can be dispensed with. So the only initial sequents actually needed in the calculus are

$$x : P, \Gamma \Rightarrow \Delta, x : P \qquad\qquad x \in a, \Gamma \Rightarrow \Delta, x \in a$$

By the procedure described and the addition of the propositional part of the labelled calculus G3K (cf. Negri 2005) a G3 sequent calculus for the basic system **E** is obtained.

5. Extensions of system **E** (known as systems of *classical modal logics*) are obtained by adding axioms such as the following:

(M) $\Box(A \& B) \supset \Box A \& \Box B$

(C) $\Box A \& \Box B \supset \Box(A \& B)$

(N) $\Box\top$

To obtain complete sequent calculi for the classical systems defined by each of the above axioms, we incorporate in the basic calculus **G3E** the rules originated the neighbourhood-semantic conditions that correspond to the axioms. Such rules are found through known correspondences in neighbourhood semantics and conversion into rules, or directly by a method of abduction from proof search in the basic calculus.

This process may involve the need for new primitives. The neighbourhood condition that corresponds to the first axiom states that supersets of a neighbourhood of x are themselves neighbourhoods of x; the condition that corresponds to the second axiom is closure under intersection: the intersection of two neighbourhoods of x is a neighbourhood of x. The third conditions requires that the set of all possible worlds (called the *unit*) is itself a neighbourhood of x. The table below summarizes for each system the modal axiom, the neighbourhood condition, and the corresponding rule. To complete the process, one needs rules for the new primitives of formal inclusion, intersection, and unit.

Axiom	**NS property**	**Rule**
(M) $\Box(A\&B) \supset \Box A\&\Box B$	$a \in I(x)$ & $a \subseteq b \rightarrow b \in I(x)$	$\dfrac{a \in I(x), a \subseteq b, b \in I(x), \Gamma \Rightarrow \Delta}{a \in I(x), a \subseteq b, \Gamma \Rightarrow \Delta}$
(C) $\Box A\&\Box B \supset \Box(A\&B)$	$a \in I(x)$ & $b \in I(x) \rightarrow a \cap b \in I(x)$	$\dfrac{a \in I(x), b \in I(x), a \cap b \in I(x), \Gamma \Rightarrow \Delta}{a \in I(x), b \in I(x), \Gamma \Rightarrow \Delta}$
(N) $\Box\top$	$W \in I(x)$	$\dfrac{W \in I(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$

Table 1: From modal axioms to NS rules

Formal inclusion between two neighbourhoods a, b is defined by[4]

$$a \subseteq b \equiv \forall x(x \in a \supset x \in b)$$

and has the sequent calculus rules

$$\frac{x \in a, \Gamma \Rightarrow \Delta, x \in b}{\Gamma \Rightarrow \Delta, a \subseteq b}\ R\subseteq,\ x\,\textit{fresh} \qquad \frac{x \in b, x \in a, a \subseteq b, \Gamma \Rightarrow \Delta}{x \in a, a \subseteq b, \Gamma \Rightarrow \Delta}\ L\subseteq$$

Formal intersection has the rules

$$\frac{x \in a, x \in b, x \in a \cap b, \Gamma \Rightarrow \Delta}{x \in a \cap b, \Gamma \Rightarrow \Delta}\ L\cap$$

$$\frac{\Gamma \Rightarrow \Delta, x \in a \cap b, x \in a \quad \Gamma \Rightarrow \Delta, x \in a \cap b, x \in b}{\Gamma \Rightarrow \Delta, x \in a \cap b}\ R\cap$$

Finally, the rule that defines the unit is simply

$$\frac{x \in W, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ W$$

[4]Observe that to keep the notation simpler we use the same symbols ($\in, \subseteq, W$) both at the semantic and syntactic levels.

4 Streamlining

The method we have outlined in its main points produces complete calculi for the logical systems under examination, but can still be improved. In some cases, instead of adding extra neighbourhood properties, it is convenient to modify the forcing conditions in such a way that they become in-built. A similar move worked for the condition of Noetherianity for the provability logics GL and Grz (Negri 2005, Negri 2014, Dyckhoff and Negri 2016). Unlike in that case, where the move was forced by the fact that Noetherianity is not expressible as a rule because it is not a first-order frame condition, here the move is optional, but it has a double advantage: avoid the addition of some neighbourhood rules and simplify the modal rules.

In the presence of monotonicity, the following forcing conditions give the same class of valid formulas (see Negri 2017 for a proof):

1. $x \Vdash^1 \Box A \equiv \exists a \in I(x)(a \Vdash^\forall A \,\&\, \forall y(y \Vdash A \to y \in a))$
2. $x \Vdash^2 \Box A \equiv \exists a \in I(x).a \Vdash^\forall A$

In logical systems closed under monotonicity, the rules for the necessity operator can thus be simplified into the following form with no added rule for monotonicity required:

$$\frac{a \in I(x), a \Vdash^\forall A, \Gamma \Rightarrow \Delta}{x : \Box A, \Gamma \Rightarrow \Delta}\; L\Box', a\ fresh$$

$$\frac{a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A, a \Vdash^\forall A}{a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A}\; R\Box'$$

5 Other modalities

It is often useful to have primitive rules also for modalities which can be defined though duality, such as the possibility modality. For this purpose, in addition to the universal forcing $\Vdash^\forall$ it is useful to consider another relation of local forcing, the existential one

$a \Vdash^\exists A$ is true iff there is some world x in a such that $x \Vdash A$

The corresponding rules are

$$\frac{x \in a, \Gamma \Rightarrow \Delta, x : A, a \Vdash^\exists A}{x \in a, \Gamma \Rightarrow \Delta, a \Vdash^\exists A}\; R\Vdash^\exists \qquad \frac{x \in a, x : A, \Gamma \Rightarrow \Delta}{a \Vdash^\exists A, \Gamma \Rightarrow \Delta}\; L\Vdash^\exists,\ x\, fresh$$

The rules that one obtains by unfolding the definition of forcing for the dual of necessity and by applying the process described above are

$$\frac{a \in I(x), x : \Diamond A, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta \quad a \in I(x), x : \Diamond A, \Gamma \Rightarrow \Delta, \neg A \lhd a}{a \in I(x), x : \Diamond A, \Gamma \Rightarrow \Delta} \; L\Diamond$$

$$\frac{a \in I(x), \neg A \lhd a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}{\Gamma \Rightarrow \Delta, x : \Diamond A} \; R\Diamond, \; a \textit{ fresh}$$

In the presence of monotonicity, the simplified rules are as follows:

$$\frac{a \in I(x), x : \Diamond A, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}{a \in I(x), x : \Diamond A, \Gamma \Rightarrow \Delta} \; L\Diamond'$$

$$\frac{a \in I(x), \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}{\Gamma \Rightarrow \Delta, x : \Diamond A} \; R\Diamond'$$

The use of neighbourhood semantics in place of the relational semantics gives a splitting of the standard alethic modalities into four modalities, $[\,]$, $\langle\,]$, $[\,\rangle$, $\langle\,\rangle$, that correspond to the four different combinations of quantifiers in the semantic explanation:

$x \Vdash [\,]A$ iff *for every neighbourhood a of x, $a \Vdash^{\forall} A$*

$x \Vdash \langle\,]A$ iff *there is some neighbourhood a of x such that $a \Vdash^{\forall} A$*

$x \Vdash [\,\rangle A$ iff *for every neighbourhood a of x, $a \Vdash^{\exists} A$*

$x \Vdash \langle\,\rangle A$ iff *there is some neighbourhood a of x such that $a \Vdash^{\exists} A$*

It is then an easy exercise to convert the above semantic explanation into rules

Rules for NS-alethic modalities:

$$\frac{a \in I(x), \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}{\Gamma \Rightarrow \Delta, x : [\,]A} \; R[\,], \; a \textit{ fresh} \qquad \frac{a \in I(x), x : [\,]A, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{a \in I(x), x : [\,]A, \Gamma \Rightarrow \Delta} \; L[\,]$$

$$\frac{a \in I(x), \Gamma \Rightarrow \Delta, x : \langle\,]A, a \Vdash^{\forall} A}{a \in I(x), \Gamma \Rightarrow \Delta, x : \langle\,]A} \; R\langle\,] \qquad \frac{a \in I(x), a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x : \langle\,]A, \Gamma \Rightarrow \Delta} \; L\langle\,], \; a \textit{ fresh}$$

$$\frac{a \in I(x), \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}{\Gamma \Rightarrow \Delta, x : [\,\rangle A} \; R[\,\rangle, \; a \textit{ fresh} \qquad \frac{a \in I(x), x : [\,\rangle A, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}{a \in I(x), x : [\,\rangle A, \Gamma \Rightarrow \Delta} \; L[\,\rangle$$

$$\frac{a \in I(x), \Gamma \Rightarrow \Delta, x : \langle\ \rangle A, a \Vdash^{\exists} A}{a \in I(x), \Gamma \Rightarrow \Delta, x : \langle\ \rangle A} R\langle\ \rangle \quad \frac{a \in I(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}{x : \langle\ \rangle A, \Gamma \Rightarrow \Delta} L\langle\ \rangle,\ a\,\textit{fresh}$$

As a further example we consider the *ought* modality of deontic logic, perhaps the best known example of a logic for which the standard normal modal logic setting is inadequate.

We recall that the standard axiomatization of deontic logic is obtained by adding the axiom $\neg\mathcal{O}\bot$ to the axiomatisation of **K** and that the normal modal base leads to well known deontic paradoxes (e.g. the *gentle murder* of Forrester 1984).

Non-normal systems of deontic logic have been proposed as a way out from paradoxes (see Orlandelli 2014). They are obtained as extensions of classical modal logics. System **ED**, **MD**, **RD**, and **KD** are obtained, respectively, as extensions of systems **E**, **M**, **N**, and **C** with the deontic axiom $\neg\,\mathcal{O}\bot$. The latter axiom corresponds to the rule

$$\frac{y \in a, a \in I(x), \Gamma \Rightarrow \Delta}{a \in I(x), \Gamma \Rightarrow \Delta} D,\ y\ \textit{fresh}$$

G3MD has the modal rules:

$$\frac{a \in I(x), a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x : \mathcal{O}A, \Gamma \Rightarrow \Delta} L\mathcal{O}, a\,\textit{fresh}$$

$$\frac{a \in I(x), \Gamma \Rightarrow \Delta, x : \mathcal{O}A, a \Vdash^{\forall} A}{a \in I(x), \Gamma \Rightarrow \Delta, x : \mathcal{O}A} R\mathcal{O}$$

Other systems of deontic logic are obtained by adding the rules that correspond to each neighbourhood property, as in Table 1. For the non-monotonic system, the rule for the deontic modality follows the general form of the rule for the alethic modality of system **E**.

6 Properties of NS-sequent calculi

The structural properties are established in a uniform way for any set of modalities and neighbourhood properties; the guidelines set for relational semantics require some additions and modifications. First, besides world labels, one has neighbourhood labels which are treated in the same way as world labels with respect to substitutions. Secondly, a suitable definition of formula weight is needed to reflect the nesting of the local forcing relations in the meaning explanation and the subsequent layering of the modal rules.

The following results are then established (see Negri 2017 for details):

1. Substitution is height-preserving admissible:

   (a) If $\vdash_n \Gamma \Rightarrow \Delta$, then $\vdash_n \Gamma(y/x) \Rightarrow \Delta(y/x)$;

   (b) If $\vdash_n \Gamma \Rightarrow \Delta$, then $\vdash_n \Gamma(b/a) \Rightarrow \Delta(b/a)$.

2. All the rules are height-preserving invertible.

3. The rules of left and right weakening are contraction are height-preserving admissible.

4. Cut is admissible.

5. The calculus is shown complete indirectly through equivalence to the axiomatic systems and known completeness results.

6. Direct completeness proof: All the rules are sound with respect to neighbourhood models and for every sequent, either there is a derivation or a countermodel in the intended class of neighbourhood models.

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