Proof theory for non-normal modal logics: The neighbourhood formalism and basic results

Sara Negri
University of Helsinki
Finland

1 Introduction

The advent of Kripke semantics marked a decisive turning point for philosophical logic: earlier axiomatic studies of modal concepts were replaced by a solid semantic method that displayed the connections between modal axioms and conditions on the accessibility relation between possible worlds. However, the success of the semantic method was not followed by equally powerful syntactic theories of modal and conditional concepts and reasoning: Concerning the former, the situation was depicted by Melvin Fitting in his survey in the *Handbook of Modal Logic* [7] as: "No proof procedure suffices for every normal modal logic determined by a class of frames"; In the chapter on tableau systems for conditional logics, Graham Priest stated that "there are presently no known tableau systems of the kind used in this book for S" (Lewis' logic for counterfactuals) ([40], p. 93).

The insufficiency of traditional Gentzen systems to meet the challenge of the development of a proof theory for modal and non-classical logic has led to alternative formalisms which, in one way or another, extend the syntax of sequent calculus. There have been two main lines of development, one that enriches the structure of sequents (display calculi, hypersequents, nested sequents, tree-hypersequents, deep

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inference), another that maintains their simple structure but adds *labels*, thus internalizing the possible worlds semantics within the proof system. In particular, for the proof theory of conditional logics there have been several contributions in the literature from both approaches [1, 44, 10, 12, 20, 30, 34, 35, 36].[1].

In his work of 1997, Grisha Mints has been among the forerunners[2] of the latter approach to the sequent calculus proof theory of modal logic.[3] In [23], he showed how one can obtain sequent calculi for normal modal logics with any combination of *reflexivity*, *transitivity*, and *symmetry* in their Kripke frames. Possible worlds were represented as prefixes, in fact, finite sequences of natural numbers, with the properties of the accessibility relations of a Kripke frame implicit in the management of prefixes in the logical rules. By this approach, it was possible to give a proof of cut elimination that can be considered as a formalization of Kripke's original completeness proof.

By making explicit the accessibility relation and by using variables, rather than sequences for possible worlds, it is possible to capture a much wider range of modal logics, in particular those characterised by *geometric frame conditions*, with properties such as seriality or directness of the accessibility relation; by using the conversion of geometric implications into rules that extend sequent calculus in a way that maintains the admissibility of structural rules [24], it has been possible to obtain a uniform presentation of a large family of modal logics, including provability logic, with modular proofs of their structural properties [25] and direct semantic completeness proofs [26].

Later, this labelled sequent calculus approach to the proof theory of modal logic has been extended to wider frame classes [27], and in further work it has been shown how the method can capture *any* logic characterized by first-order frame conditions in its relational semantics [5]; the reason is that arbitrary first-order theories can be given an analytic treatment through the extension of G3-style sequent calculi with geometric rules. Notably, in these calculi, all the rules are invertible and a strong form of completeness holds, with a simultaneous construction of formal proofs, for derivable sequents, or countermodels, for underivable ones, as shown in [28].

Despite their wide range of applications, the powerful methods of Kripke semantics are not a universal tool in the analysis of philosophical logics: they impose the straitjacket of *normality*, i.e., validity of the rule of necessitation, from $\vdash A$ to infer $\vdash \Box A$, and of the K axiom, $\Box(A \supset B) \supset (\Box A \supset \Box B)$. The limitative character of these imposed validities becomes clear in epistemic logic: with the epistemic reading

[1]See the conclusion of [34] for a discussion and comparison of these different formalisms.

[2]See also the extensive studies of labelled systems for modal logics and non-classical logics in A. Simpson's PhD dissertation from 1994 [42] and L. Viganò's monograph from 2000 [45].

[3]Labelled tableaux, on the other hand, were developed since the early 1970's, cf. [6].

of the modality, an agent knows A if A holds in all the epistemic states available to her, and then the normality properties yield that (1) *whatever has been proved is known* and that (2) *an agent knows all the logical consequences of what she knows.* This leads to *logical omniscience*, clearly inadequate for cognitive agents with human capabilities, and thus to the rejection of both requirements. The same limitation is clear in the interpretation of the modality as a likelihood operator where one sees that the normal modal logic validity $\Box A \& \Box B \supset \Box(A \& B)$ should be avoided.

Another limitation in systems based on a Kripke-style semantics is that the propositional base is classical or intuitionistic logic. In both cases, one is forced to an implication which has been shown since the analysis of C.I. Lewis to be an inadequate form of conditional if a logical analysis is to be pursued in other venues than mathematics: the classical propositional base of modal logic is insufficient to treat conditionals beyond material or strict implication, as shown by David Lewis' path-breaking book *Counterfactuals* [22], and intuitionistic implication shares many of the undesired properties of (classical) material implication.

The early literature on the semantics of conditional logic started with an attempt, in the work of Stalnaker, to reduce the reading of the conditional to a standard possible worlds semantics through the notion of *limit* and *selection functions* [43]. This approach has been criticized as inadequate in many cases: first, the aforementioned limit might not exist (as shown by Lewis in [22]), second, it can be too difficult to achieve and so cannot be taken as a standard basis for a formalization (as in the perfectly moral life of deontic systems), third, it could be impossible to define as in situations with more than one ordering, or, more concretely, conflicting obligations. The inadequacy of a normal modal base as a general framework for modal logic has also been shown in the case of the modal formalization of deontic notions by a series of paradoxes, such as the paradox of the gentle murder [8], that were used in a revisionist way to motivate non-normal modal logics.

The more general *neighbourhood semantics* was introduced in the 1970's to provide a uniform semantic framework for philosophical logics that cannot be accommodated within the setting of normal modal logic. Instead of an accessibility relation on a set of possible worlds, one has for every possible world a family of neighbourhoods, i.e., a collection of some special subsets of the set of possible worlds.

As is usual when a new semantics is introduced, its relationship with the earlier one is investigated. In this case, one can prove that there is a precise link between neighbourhood and relational semantics, in the sense that there is a way to define a neighbourhood frame from a given relational frame, and conversely, a relational frame from a neighbourhood frame. Given a relational frame (W, R), one can define a neighbourhood frame by taking as neighbourhoods of a world w the supersets of the set of worlds accessible from w. Conversely, given a neighbourhood frame

(W, I) one can define a relational frame by identfying the worlds accessible from w as the intersection of all the neighbourhoods of w. Neighbourhood frames are more general than relational frames, and in fact the correspondence is a bijection over a certain class of neighbourhood frames called *augmented* ones, those that contains the intersection of all their members and are closed under supersets (cf. [38] for details). The correspondence between relational and neighbourhood frames can be seen also as a way to transfer an intuitive explanation from one semantics to the other: roughly, worlds in a neighbourhood of w replace worlds accessible from w, and correspondingly the intuition on what it means to be an element of a neighbourhood or of the intersection of all the neighbourhoods of a worlds will depend on the kind of modality or conditional that is being modelled; the intuition thus varies from the properties of indistinguishability of worlds as epistemic states to that of plausibility of worlds as factual scenarios.

Among non-normal modal logics, *classical* modal logics are those obtained by requiring that the modality respects logical equivalence, that is, closure under the rule $\frac{A \supset\subset B}{\Box A \supset\subset \Box B}$. One can then obtain other systems below the normal modal logic **K** by removing the normality axiom and the necessitation rule and adding the axiom schemas M, C, N and their combinations. A lattice of eight different logics is obtained (cf. the diagram on p. 237 of [4]). On the logical side, it has been shown by Gasquet and Herzig [9] and Kracht and Wolter [18] that non-normal modal logics can be simulated through an appropriate translation by a normal modal logic with three modalities. This translation has been used by Gilbert and Maffezioli [11] to define modular labelled sequent calculi for the basic classical modal logics. Since the frame conditions considered go beyond the geometric class, *systems of rules* (in the sense of [27]) have been used.

Our goal is to set the grounds for a proof theory of non-normal modal systems based on neighbourhood semantics, to achieve this directly, i.e., without the use of translations, with local rules, and in a modular way, open to extensions in various interweaving directions[4].

The goal will be accomplished through the guidelines of *inferentialism*, that is, by starting from the meaning explanation of logical constants and by converting it into well-behaved rules of a calculus, as detailed in [29].

The paper is organized as follows: In Section 2, after having recalled the basic definitions of neighbourhood semantics, we show how it naturally gives rise to the four distinct modalities $[\,]$, $\langle\,]$, $[\,\rangle$, and $\langle\,\rangle$; the nesting of quantifiers in their semantic explanation is factorized with the help of *local* forcing relations, i.e., relations

[4]Such flexibility is already witnessed by developments of the labelled proof theory based on neighbourhood semantics for preferential conditional logic and conditional doxastic logic in [30, 13].

between (formal) neighbourhoods and formulas. Correspondingly, we have sequent calculus rules for such relations and for the modalities defined upon them. We then show how the basic calculus so obtained can be used to find the rules that correspond to additional properties of the neighbourhoods and the relations between such properties and the normality conditions for the $\langle\,]$ modality. It is also shown how the rules obtained validate $\langle\,](A\&B) \supset \langle\,]A \,\&\, \langle\,]B$, and how a modified forcing condition for the modality gives a more general explanation. The link between the two is given by the operation of supplementation in minimal models. In the determination of the rules of the systems we use a sort of 'bootstrapping' procedure, as we use the basic rules of the calculus to find other rules. We also assume at this early stage of the construction of the proof system that the structural properties are available, even if such properties are necessarily proved further on, when all the rules have been determined. In Section 3 we apply the methodology to classical modal logics and generate labelled G3-style sequent calculi for them. The structural properties, height-preserving invertibility of all the rules, height-preserving admissibility of weakening and contraction and admissibility of cut are proved in Section 4. In Section 5, we give a direct proof of completeness for these systems with respect to neighbourhood models as well as an indirect completeness proof via the axiomatic systems. Finally, in the conclusion, our approach to the proof theory of non-normal modal logic is related to other approaches in the literature.

2 The general framework

A *neighbourhood frame* is a pair $\mathcal{F} \equiv (W, I)$, where W is a set of worlds (states) and I is a neighbourhood function

$$I : W \longrightarrow \mathcal{P}(\mathcal{P}(W))$$

that assigns a collection of sets of worlds to each world in W. A *neighbourhood model* is then a pair $\mathcal{M} \equiv (\mathcal{F}, \mathcal{V})$, where $\mathcal{F}$ is a neighbourhood frame and $\mathcal{V}$ a propositional valuation, i.e., a map $\mathcal{V} : \mathtt{Atm} \longrightarrow \mathcal{P}(W)$ from atomic formulas to sets of possible worlds.

Worlds in a neighbourhood are the substitute, in this more general semantics, of accessible worlds. The inductive clauses for truth of a formula in a model are the usual ones for the propositional clauses; for the modal operator we have

$$\mathcal{M}, w \Vdash \Box A \equiv ext(A) \text{ is in } I(w),$$

where $ext(A) \equiv \{u \in W | \mathcal{M}, u \Vdash A\}$.[5]

[5]We observe that $ext(A)$ is also denoted by $[A]$.

Starting from the standard forcing relation between possible worlds and formulas, we extend the standard labelled language to a multi-sorted labelled language, with labels for worlds and neighbourhoods, and define two *local*, rather than pointwise, forcing relations, $\Vdash^{\exists}$ and $\Vdash^{\forall}$. These forcing relations are local because unlike the usual forcing of a formula A at a world x, they are relations between elements a of a system of neighbourhoods, that is, sets of subsets of possible worlds, and formulas. The subset a thus ranges in a family of neighbourhoods $I(x)$, which is supposed to be given for every world x. The first relation corresponds to the existence, in the neighbourhood, of a world that forces the formula, the second to the forcing for every world in the neighbourhood; here A is a formula of the propositional modal language (as we shall see below, we shall actually consider an extension of the standard propositional language with four modalities naturally arising from the semantics):

$a \Vdash^{\exists} A$ *is true* iff *there is some world* x *in* a *such that* $x \Vdash A$

$a \Vdash^{\forall} A$ *is true* iff *for any world* x *in* a, $x \Vdash A$.

The standard forcing relation can be then obtained as a special case of both existential and universal forcing through singleton sets (under the condition that they belong to the family of neighbourhoods):

$\{x\} \Vdash^{\exists} A$ iff $\{x\} \Vdash^{\forall} A$ iff $x \Vdash A$

Through the standard method of conversion of forcing clauses into sequent calculus rules [25, 32], we obtain the following rules for the local forcing relations; observe that the language of standard labelled systems is extended by the local forcing relations and has, in place of relational atoms, atoms of the form $x \in a$, $a \in I(x)$:

$$\frac{x \in a, \Gamma \Rightarrow \Delta, x : A}{\Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}\ R\Vdash^{\forall},\ x\,fresh \qquad \frac{x \in a, x : A, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}\ L\Vdash^{\forall}$$

$$\frac{x \in a, \Gamma \Rightarrow \Delta, x : A, a \Vdash^{\exists} A}{x \in a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}\ R\Vdash^{\exists} \qquad \frac{x \in a, x : A, \Gamma \Rightarrow \Delta}{a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}\ L\Vdash^{\exists},\ x\,fresh$$

Table 1: Rules for local forcing

The use of neighbourhood semantics in place of the relational semantics gives a splitting of the standard alethic modalities into four modalities, $[\ ]$, $\langle\ ]$, $[\ \rangle$, $\langle\ \rangle$ [38], corresponding to the four different combinations of quantifiers in the semantic explanation:

$x \Vdash [\ ]A$ iff *for every neighbourhood a of x, $a \Vdash^{\forall} A$*

$x \Vdash \langle\ ]A$ iff *there is some neighbourhood a of x such that $a \Vdash^{\forall} A$*

$x \Vdash [\ \rangle A$ iff *for every neighbourhood a of x, $a \Vdash^{\exists} A$*

$x \Vdash \langle\ \rangle A$ iff *there is some neighbourhood a of x such that $a \Vdash^{\exists} A$*

The semantic clauses are translated into the following rules:

$$\frac{a \in I(x), \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}{\Gamma \Rightarrow \Delta, x : [\ ]A}\ R[\ ],\ a\,fresh \qquad \frac{a \in I(x), x : [\ ]A, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{a \in I(x), x : [\ ]A, \Gamma \Rightarrow \Delta}\ L[\ ]$$

$$\frac{a \in I(x), \Gamma \Rightarrow \Delta, x : \langle\ ]A, a \Vdash^{\forall} A}{a \in I(x), \Gamma \Rightarrow \Delta, x : \langle\ ]A}\ R\langle\ ] \qquad \frac{a \in I(x), a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x : \langle\ ]A, \Gamma \Rightarrow \Delta}\ L\langle\ ],\ a\,fresh$$

$$\frac{a \in I(x), \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}{\Gamma \Rightarrow \Delta, x : [\ \rangle A}\ R[\ \rangle,\ a\,fresh \qquad \frac{a \in I(x), x : [\ \rangle A, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}{a \in I(x), x : [\ \rangle A, \Gamma \Rightarrow \Delta}\ L[\ \rangle$$

$$\frac{a \in I(x), \Gamma \Rightarrow \Delta, x : \langle\ \rangle A, a \Vdash^{\exists} A}{a \in I(x), \Gamma \Rightarrow \Delta, x : \langle\ \rangle A}\ R\langle\ \rangle \qquad \frac{a \in I(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}{x : \langle\ \rangle A, \Gamma \Rightarrow \Delta}\ L\langle\ \rangle,\ a\,fresh$$

Table 2: Rules for alethic modalities

Finally, in a G3-style labelled calculus there are two types of initial sequents, those with labelled atomic formulas, from the basic propositional base, of the form $x : P, \Gamma \Rightarrow \Delta, x : P$ and those with relational atoms. For labelled calculi based on possible worlds semantics, the latter have the form $xRy, \Gamma \Rightarrow \Delta, xRy$. As observed in [25], such sequents are not needed because none of the rules of the calculus has active relational atoms on the right-hand side, so such initial sequents cannot have an active role in derivations and can thus be dispensed with. Here we have a similar situation, with the two potential types of relational initial sequents being $x \in a, \Gamma \Rightarrow \Delta, x \in a$ and $a \in I(x), \Gamma \Rightarrow \Delta, a \in I(x)$: none of the rules introduced so far has active formulas of the form $x \in a$ or $a \in I(x)$ in the right-hand side, so such initial sequents are not needed. But there is a *caveat*, and, as we shall see, once the assumption of monotonicity which is behind the determination of the above rules is relaxed, we'll have to include rules that have relational atoms of the form $x \in a$ on the right-hand side, and consequently, initial sequents for them.

The *basic calculus for neighbourhood semantics*, **G3n**, is obtained by adding the above rules of local forcing together with the rules for alethic modalities together with the needed relational initial sequents to the standard labelled **G3c** sequent calculus (the propositional part of the calculus **G3K** [25]). For ease of the reader, we give such rules in the table below, with the added relational initial sequents; they are in parentheses since they are needed in extensions but not for the basic system with the rules presented so far.

Initial sequents:

$x : P, \Gamma \Rightarrow \Delta, x : P$ $\qquad\qquad$ $(x \in a, \Gamma \Rightarrow \Delta, x \in a)$

Propositional rules:

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A \& B, \Gamma \Rightarrow \Delta} L\& \qquad \frac{\Gamma \Rightarrow \Delta, x : A \quad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \& B} R\&$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \quad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta} L\vee \qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B} R\vee$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad x : B, \Gamma \Rightarrow \Delta}{x : A \supset B, \Gamma \Rightarrow \Delta} L\supset \qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \supset B} R\supset$$

$$\frac{}{x : \perp, \Gamma \Rightarrow \Delta} L\perp$$

Table 3: The propositional part of system G3n

As an example of the use of the system obtained, we show how to obtain a formal derivation of one of the sequents which gives the known dualities between the compound alethic modalities, namely $x : \langle\ ]A \Rightarrow x : \neg[\ \rangle\neg A$ (here and elsewhere in the paper negation is not primitive, but defined through implication); by root-first application of the rules we find the following partial derivation:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{a \in I(x), y \in A, a \Vdash^{\forall} A, y : A, x : [\ \rangle\neg A \Rightarrow x : \perp, y : A \quad x : \perp, \ldots \Rightarrow \ldots}{a \in I(x), y \in A, a \Vdash^{\forall} A, y : A, y : \neg A, x : [\ \rangle\neg A \Rightarrow x : \perp} L \supset}{a \in I(x), y \in A, a \Vdash^{\forall} A, y : \neg A, x : [\ \rangle\neg A \Rightarrow x : \perp} L \Vdash \forall}{a \in I(x), a \Vdash^{\forall} A, a \Vdash^{\exists} \neg A, x : [\ \rangle\neg A \Rightarrow x : \perp} L \Vdash \exists}{a \in I(x), a \Vdash^{\forall} A, x : [\ \rangle\neg A \Rightarrow x : \perp} L[\ \rangle}{x : \langle\ ]A, x : [\ \rangle\neg A \Rightarrow x : \perp} L\langle\ ]}{x : \langle\ ]A \Rightarrow x : \neg[\ \rangle\neg A} R \supset$$

Derivability of the left topsequent follows from Lemma 4.2 below. In a similar way we obtain the other parts of the dualities between $\langle\ ]$ and $[\ \rangle$ and between $\langle\ \rangle$ and $[\ ]$, namely we have:

Proposition 2.1. *The following sequents are derivable in* **G3n***:*

1. $x : \langle\ ]A \Rightarrow x : \neg[\ \rangle\neg A$
2. $x : \neg[\ \rangle\neg A \Rightarrow x : \langle\ ]A$
3. $x : \langle\ \rangle A \Rightarrow x : \neg[\ ]\neg A$
4. $x : \neg[\ ]\neg A \Rightarrow x : \langle\ \rangle A$

We proceed with finding the properties required of the family of neighbourhoods $I(x)$ to obtain a modality that satisfies the K axiom and necessitation[6] through application of the invertible rules of the calculus. Before doing so, we need a formal definition of inclusion between neighbourhoods. Unsurprisingly, inclusion between two neighbourhoods a, b is defined by

$$a \subseteq b \equiv \forall x(x \in a \supset x \in b)$$

with the sequent calculus rules

$$\frac{x \in a, \Gamma \Rightarrow \Delta, x \in b}{\Gamma \Rightarrow \Delta, a \subseteq b}\ R\subseteq,\ x\,\mathit{fresh} \qquad \frac{x \in b, x \in a, a \subseteq b, \Gamma \Rightarrow \Delta}{x \in a, a \subseteq b, \Gamma \Rightarrow \Delta}\ L\subseteq$$

Observe that to keep the notation simpler we use the same symbols $(\in, \subseteq)$ both at the semantic and at the syntactic level.

Definition 2.2. *A family of neighbourhoods $I(x)$ is* prebasic *if for all $a, b \in I(x)$, there exists $c \in I(x)$ such that $c \subseteq a$ and $c \subseteq b$.*

If the definition of a prebasic family of neighbourhoods is not fully unfolded to the level of worlds but left at the level of inclusion between neighbourhoods, the property of being prebasic can be translated into sequent calculus rules that follow the geometric rule scheme:

$$\frac{a \in I(x), b \in I(x), c \in I(x), c \subseteq a, c \subseteq b, \Gamma \Rightarrow \Delta}{a \in I(x), b \in I(x), \Gamma \Rightarrow \Delta}\ \mathit{Prebasic},\ c\ \mathit{fresh}$$

We have:

Lemma 2.3. *Suppose that for all x the family of neighbourhoods $I(x)$ is prebasic. Then $\langle\ ](A \supset B) \supset (\langle\ ]A \supset \langle\ ]B)$ is valid with respect to the neighbourhood semantics.*

[6]Among the four modalities introduced above, the modality now in question is $\langle\ ]$.

Proof. Validity is guaranteed by the following derivation in the labelled calculus

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{...y : A \Rightarrow y : A... \quad ...y : B \Rightarrow ...y : B}{...y : A, y : A \supset B \Rightarrow ...y : B}\; L\supset}{y \in a, y \in b, y \in c, a \in I(x), b \in I(x), c \in I(x), c \subseteq a, c \subseteq b, a \Vdash^{\forall} A, b \Vdash^{\forall} A \supset B \Rightarrow x : \langle\,]B, y : B}\; L\Vdash^{\forall}\text{ (twice)}}{y \in c, a \in I(x), b \in I(x), c \in I(x), c \subseteq a, c \subseteq b, a \Vdash^{\forall} A, b \Vdash^{\forall} A \supset B \Rightarrow x : \langle\,]B, y : B}\; L\subseteq\text{ (twice)}}{a \in I(x), b \in I(x), c \in I(x), c \subseteq a, c \subseteq b, a \Vdash^{\forall} A, b \Vdash^{\forall} A \supset B \Rightarrow x : \langle\,]B, c \Vdash^{\forall} B}\; R\Vdash^{\forall}}{a \in I(x), b \in I(x), c \in I(x), c \subseteq a, c \subseteq b, a \Vdash^{\forall} A, b \Vdash^{\forall} A \supset B \Rightarrow x : \langle\,]B}\; R\langle\,]}{a \in I(x), b \in I(x), a \Vdash^{\forall} A, b \Vdash^{\forall} A \supset B \Rightarrow x : \langle\,]B}\; \textit{Prebasic}}{x : \langle\,]A, x : \langle\,](A \supset B) \Rightarrow x : \langle\,]B}\; L\langle\,]\text{ (twice)}}{x : \langle\,](A \supset B) \Rightarrow x : \langle\,]A \supset \langle\,]B}\; R\supset}{\Rightarrow x : \langle\,](A \supset B) \supset (\langle\,]A \supset \langle\,]B)}\; R\supset
$$

where the topsequents are derivable by Lemma 4.2. To conclude the proof one needs to show that the calculus **G3n** is sound with respect to neighbourhood semantics. This will be established as a general result in Theorem 5.3 below. QED

The condition of being prebasic is not only sufficient but also necessary to validate the normality axiom, in fact if rule *prebasic* is not available proof search is limited to the rules of **G3n** and we have:

Lemma 2.4. *Proof search for the K-axiom in the calculus* **G3n** *fails and from the failed proof search it is possible to construct a countermodel in the class of neighbourhood frames.*

Proof. We apply all the rules of **G3n** with conclusion that matches the sequent; we start from the sequent $a \in I(x), b \in I(x), a \Vdash^{\forall} A, b \Vdash^{\forall} A \supset B \Rightarrow x : \langle\,]B$ of the above proof search,[7] and obtain, through two applications resp. of $R\langle\,]$, $R \Vdash^{\forall}$ and $L \Vdash^{\exists}$, the sequent $a \in I(x), b \in I(x), y \in a, z \in b, y : A, z : A \supset B, a \Vdash^{\forall} A$, $b \Vdash^{\forall} A \supset B \Rightarrow x : \langle\,]B, y : B, z : B$; next a step of $L \supset$ gives a right derivable premiss (that contains both in the left-hand side and in the right-hand side the labelled formula $z : B$) and a left premiss $a \in I(x), b \in I(x), y \in a, z \in b, y : A$, $a \Vdash^{\forall} A, b \Vdash^{\forall} A \supset B \Rightarrow x : \langle\,]B, z : A, y : B, z : B$. This is not derivable and a countermodel is obtained by taking $I(x)$ to consist of the neighbourhoods a and b inhabited by (only) the worlds in the antecedent, i.e., $a \in I(x), b \in I(x), y \in a, z \in b$ with the forcing relations $y \Vdash A$, $z \nVdash A$, $y \nVdash B$, $z \nVdash B$. Clearly, $a \Vdash^{\forall} A$, $b \Vdash^{\forall} A \supset B$, but there is no neighbourhood of x that forces universally B. QED

Observe that the above doesn't exclude the possibility that the normality axiom would be derivable in other extensions of **G3n** since proof search in these calculi

[7]Since all the rules applied are invertible, this is not restrictive.

would be different, and the countermodel constructed here might not be in the class of frames for the stronger logic.

Next, we look for the property of $I(x)$ that characterizes validity of the rule of necessitation, i.e., the rule

$$\frac{\vdash x : A}{\vdash x : \langle\ ]A}\ \mathrm{Nec}$$

If we want to apply root-first the rules of **G3n** from the sequent $\Rightarrow x : \langle\ ]A$, the only way to start is to enable the application of rule $R\langle\ ]$ by assuming the existence of $a \in I(x)$, i.e., to assume the availability of the geometric rule (with the condition that a is a fresh neighbourhood label)

$$\frac{a \in I(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Nondeg}$$

This justifies the following definition:

Definition 2.5. *A family of neighbourhoods $I(x)$ is* nondegenerate *if $I(x)$ contains at least a neighbourhood.*

Lemma 2.6. *The rule of necessitation is admissible in the calculus* **G3n** *extended with rule* Nondeg*.*

Proof. We have the following

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Rightarrow x : A}{\Rightarrow y : A}\ \mathit{hp\text{-}subst}}{a \in I(x), y \in a \Rightarrow x : \langle\ ]A, y : A}\ \mathit{LW}, \mathit{RW}}{a \in I(x), \Rightarrow x : \langle\ ]A, a \Vdash^{\forall} A}\ R\Vdash^{\forall}}{a \in I(x), \Rightarrow x : \langle\ ]A}\ R\langle\ ]}{\Rightarrow x : \langle\ ]A}\ \mathit{Nondeg}$$

Here we have used the admissible rules of height-preserving substitution and weakening (to be proved in Propositions 4.3, 4.4 below), hence the statement on admissibility rather than derivability. QED

Relation with minimal models: Neighbourhood models are also called *minimal* models in the literature.[8] Observe that the definition of forcing that we have given for the modality $\langle\ ]$ validates

$$\langle\ ](A\&B) \supset \langle\ ]A \,\&\, \langle\ ]B$$

[8]See Chapter 7 of Chellas (1980), in particular 7.1, for the definition of minimal models.

and therefore is not minimal in the sense that it automatically imposes some validities. This is avoided if the forcing is modified by requiring that the neighbourhood a not only is included, but *coincides* with the extension $ext(A)$ of A, i.e.,

$$x \Vdash^{+} \langle\ ]A \text{ iff } \textit{there is } a \textit{ in } I(x) \textit{ such that } a \Vdash^{\forall} A \text{ and } ext(A) \subseteq a$$

Then the rules for $\langle\ ]$ justified by the semantics of minimal models are as follows:

$$\frac{a \in I(x), \Gamma \Rightarrow \Delta, x : \langle\ ]A, a \Vdash^{\forall} A \quad a \in I(x), \Gamma \Rightarrow \Delta, x : \langle\ ]A, ext(A) \subseteq a}{a \in I(x), \Gamma \Rightarrow \Delta, x : \langle\ ]A} \; R\langle\ ]'$$

$$\frac{a \in I(x), a \Vdash^{\forall} A, ext(A) \subseteq a, \Gamma \Rightarrow \Delta}{x : \langle\ ]A, \Gamma \Rightarrow \Delta} \; L\langle\ ]', \textit{a fresh}$$

together with the rules for inclusion and the obvious rules for $ext(A)$, namely[9]

$$\frac{y : A, \Gamma \Rightarrow \Delta}{y \in ext(A), \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, y \in ext(A)}$$

It is easy to show that with these rules the sequent $\Rightarrow x : \langle\ ](A\&B) \supset \langle\ ]A \,\&\, \langle\ ]B$ is not derivable, the reason being that from $ext(A\&B) \subseteq a$ we cannot infer $ext(A) \subseteq a$ and $ext(B) \subseteq a$.

There is however a precise link between the two forcing conditions. We first recall a definition:[10]

Definition 2.7. *The* supplementation *of a neighbourhood model* $\mathcal{M} \equiv (W, I, \Vdash)$ *is the neighbourhood model* $\mathcal{M}^{+} \equiv (W, I^{+}, \Vdash)$ *obtained by taking the superset closure of* $I(x)$ *for each* x *in* W*, i.e.,* $a \in I(x)^{+}$ *if and only if* $a \supseteq b$ *for some* $b \in I(x)$*.*

We also recall the following:[11]

Proposition 2.8. *For all formula* A *we have*

$$\mathcal{M}, x \Vdash \langle\ ]A \textit{ if and only if } \mathcal{M}^{+}, x \Vdash^{+} \langle\ ]A$$

Proof. In one direction, if there is a in $I(x)$ such that $a \Vdash^{\forall} A$, i.e., $a \subseteq ext(A)$, then $ext(A)$ is in $I(x)^{+}$ (and $ext(A) = ext(A)$). For the converse, if there is a in $I(x)^{+}$ such that $a = ext(A)$, then $b \subseteq ext(A)$ for some b in $I(x)$. QED

[9]The rules in terms of ext are intuitively semantically motivated. We shall give below an alternative, more concise, version of the rules in which the inclusion $ext(A) \subseteq a$ is replaced by a binary predicate $A \lhd a$ with its own rules which do not require separate rules for inclusion.

[10]This is Definition 7.6 in Chellas (1980).

[11]This is essentially exercise 7.25 (b) in Chellas's book.

3 Classical and other non-normal modal logics

Classical modal logics[12] are non-normal modal logics obtained as extensions of classical propositional logic (**CL**) that contain the schema

$$\Diamond A \supset\subset \neg\Box\neg A$$

and the rule of inference

$$\frac{A \supset\subset B}{\Box A \supset\subset \Box B}\ RE$$

System **E** is the smallest classical system thus obtained. Other classical modal logics are obtained as extensions of **E**. Extensions containing the rule

$$\frac{A \supset B}{\Box A \supset \Box B}\ RM$$

are called *monotonic* logics and the smallest such system is denoted by **M**; extensions containing the rule

$$\frac{A\&B \supset C}{\Box A\&\Box B \supset \Box C}\ RR$$

are called *regular*, and the smallest such system is denoted by **C**.

It is well know (and easily provable) that every normal system is regular, every regular system is monotonic, and every monotonic system is classical.

It can be convenient to give a characterization of extensions **E** through axiom schemas. Among such extension, of particular interest are those obtained by the addition of any combination of the following:

(M) $\Box(A\&B) \supset \Box A\&\Box B$

(C) $\Box A\&\Box B \supset \Box(A\&B)$

(N) $\Box\top$

We recall from Chellas (1980, ch. 8):

Proposition 3.1. *Let Σ be an extension of* **E**. *Then*

1. *Σ is monotonic iff it contains the axiom schema* M.

[12] See ch. 8 of Chellas (1980) for a thorough treatment of classical, monotonic and regular modal logics in an axiomatic setting.

2. Σ *is regular iff it contains the axiom schema* C *and is closed under* RM.

3. Σ *is regular iff it contains the axiom schemas* C *and* M.

4. Σ *is normal iff it is regular and contains the axiom schema* N.

These logics are denoted with $\mathbf{ES}_1 \ldots \mathbf{S}_n$ or simply $\mathbf{S}_1 \ldots \mathbf{S}_n$, where $\mathbf{S}_1, \ldots, \mathbf{S}_n$ are the axiom/rule schemas added to system $\mathbf{E}$. With this notation we have $\mathbf{K} = \mathbf{RN} = \mathbf{MCN} = \mathbf{EMCN}$.

We recall that the forcing clause for the alethic modality in neighbourhood semantics is as follows:

$$x \Vdash \Box A \;\equiv\; \exists a \in I(x)(a \Vdash^{\forall} A \,\&\, \forall y(y \Vdash A \supset y \in a))$$

or equivalently

$$x \Vdash \Box A \;\equiv\; \exists a \in I(x)(a \Vdash^{\forall} A \,\&\, ext(A) \subseteq a)$$

The semantic clause is not one of the form that can be directly translated into geometric rules, but we proceed in a way similar to Skolem's definitional extension ([41], see also Section 2 of [5]) and add a new predicate $A \lhd a$ for $\forall y(y \Vdash A \supset y \in a)$ together with its definition. The definition is in turn formulated in terms of rules to be added to the calculus. When all the requirement to obtain a calculus with the desired properties are taken care of, the rules are as follows:[13]

$$\frac{y \in a, A \lhd a, y : A, \Gamma \Rightarrow \Delta}{A \lhd a, y : A, \Gamma \Rightarrow \Delta}\; L\lhd \qquad \frac{y : A, \Gamma \Rightarrow \Delta, y \in a}{\Gamma \Rightarrow \Delta, A \lhd a}\; R\lhd,\ y\,fresh$$

$$\frac{a \in I(x), a \Vdash^{\forall} A, A \lhd a, \Gamma \Rightarrow \Delta}{x : \Box A, \Gamma \Rightarrow \Delta}\; L\Box,\ a\,fresh$$

$$\frac{a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A, a \Vdash^{\forall} A \quad a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A, A \lhd a}{a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A}\; R\Box$$

Table 4: Modal rules of system E

[13]See [29] for details making explicit a procedure used to obtain such sequent rules starting from the meaning explanation in terms of neighbourhood semantics.

The complete G3-system for **E** is obtained by adding the above rules to the rules for $\Vdash^{\forall}$ of table 1 and the rules for the propositional part of **G3n** of table 3, including the initial sequents of the form $x \in a, \Gamma \Rightarrow \Delta, x \in a$.[14] We shall denote with **G3E** the resulting system.

In the proofs that follow we use admissibility of the structural rules, that will be proved in Section 4.

Lemma 3.2. *The rule*

$$\frac{x : A, \Gamma \Rightarrow \Delta, x : B}{a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, a \Vdash^{\forall} B} \; (x \notin \Gamma, \Delta)$$

is admissible in **G3E**.

Proof. By admissibility of weakening and steps of $L \Vdash^{\forall}$ and $R \Vdash^{\forall}$. QED

Neither rule *RE* nor a labelled version of the rule has to be added as a rule of **G3E**. The situation is similar to what happens with **G3K**, the sequent calculus for basic normal modal logic, where the rule of necessitation doesn't have to be added as an explicit rule because it is admissible, i.e., whenever it premiss is derivable, also its conclusion is. With the proviso of completeness (proved in Section 5), this amounts to proving that whenever $\Rightarrow x : A \supset\subset B$ is derivable for an arbitrary label x then also $\Rightarrow x : \Box A \supset\subset \Box B$ is derivable for an arbitrary label x:

Lemma 3.3. *Rule RE is admissible in* **G3E**.

Proof. By the following derivation (where we use admissible cut and weakening steps):

$$\dfrac{\dfrac{\dfrac{\dfrac{x : A \Rightarrow x : B}{a \Vdash^{\forall} A \Rightarrow a \Vdash^{\forall} B}\,3.2}{a \in I(x), a \Vdash^{\forall} A, A \lhd a \Rightarrow x : \Box B, a \Vdash^{\forall} B} \qquad \dfrac{\dfrac{y : B \Rightarrow y : A \qquad \dfrac{y \in a, \ldots \Rightarrow \ldots, y \in a}{y : A, a \in I(x), a \Vdash^{\forall} A, A \lhd a \Rightarrow x : \Box B, y \in a}\,L\lhd}{y : B, a \in I(x), a \Vdash^{\forall} A, A \lhd a \Rightarrow x : \Box B, y \in a}\,cut}{a \in I(x), a \Vdash^{\forall} A, A \lhd a \Rightarrow x : \Box B, B \lhd a}\,R\lhd}{a \in I(x), a \Vdash^{\forall} A, A \lhd a \Rightarrow x : \Box B}\,R\Box}{x : \Box A \Rightarrow x : \Box B}\,L\Box$$

Observe that the topsequents in the derivations correspond to both assumptions of rule *RE* and that it is also required that sequents of the form $x \in a, \Gamma \Rightarrow \Delta, x \in a$ are taken as initial. QED

Next we show how to use this basic calculus to find the extra rules that have to be added to obtain a G3 proof system for each of the above classical modal logics.

[14] The reason for the addition will be clear in the proof of Lemma 3.3 below.

Again, as we are "bootstrapping" to find the rules of the calculus, we assume that the desired invertibility and structural properties (to be proved in Section 4 below) are available.

We proceed by root-first proof search in the invertible sequent calculus **G3E**. By abduction we find a sufficient rule for deriving the labelled form of each axiom. Further on, we shall give all the formal definitions and prove that this heuristic method really does yield a complete sequent system for the logic in question.

First, observe that by invertibility of the rules $R\supset$ and $R\&$ the derivability of the sequent $\Rightarrow x : \Box(A\&B) \supset \Box A \,\&\, \Box B$ is equivalent to the derivability of both $\Rightarrow x : \Box(A\&B) \supset \Box A$ and $\Rightarrow x : \Box(A\&B) \supset \Box B$. Let us see how the former can be obtained with the following derivation, where we use derivability of initial sequents with arbitrary formulas, a result proved in the next section; the latter sequent is derivable *mutatis mutandis*:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Lemma 4.2}}{a \in I(x), y \in a, y : A, y : B, a \Vdash^\forall A\&B, \ldots \Rightarrow x : \Box A, y : A}}{a \in I(x), y \in a, y : A\&B, a \Vdash^\forall A\&B, \ldots \Rightarrow x : \Box A, y : A}\, L\Vdash^\forall}{a \in I(x), y \in a, a \Vdash^\forall A\&B, \ldots \Rightarrow x : \Box A, y : A}\, R\Vdash^\forall}{a \in I(x), a \Vdash^\forall A\&B, \ldots \Rightarrow x : \Box A, a \Vdash^\forall A}\, R \qquad \begin{matrix}\vdots\\ a \in I(x), a \Vdash^\forall A\&B, A\&B \lhd a \Rightarrow x : \Box A, A \lhd a\end{matrix}}{a \in I(x), a \Vdash^\forall A\&B, A\&B \lhd a \Rightarrow x : \Box A}\, R\Box}{x : \Box(A\&B) \Rightarrow x : \Box A}\, L\Box$$

with the dotted part as follows:

$$\dfrac{\dfrac{\dfrac{\text{Lemma 4.2}}{b \Vdash^\forall A, \ldots \Rightarrow \ldots, b \Vdash^\forall A} \qquad \dfrac{\text{Lemma 4.2}}{A \lhd b, \ldots \Rightarrow \ldots, A \lhd b}}{b \in I(x), b \Vdash^\forall A, A \lhd b, a \Vdash^\forall A\&B, a \in I(x), a \Vdash^\forall A\&B, A\&B \lhd a \Rightarrow x : \Box A, A \lhd a}\, R\Box}{a \in I(x), a \Vdash^\forall A\&B, A\&B \lhd a \Rightarrow x : \Box A, A \lhd a}\, R$$

The extra rule applied (R) amounts to requiring that $ext(A\&B) \in I(x)$ implies $ext(A) \in I(x)$. Since $ext(A\&B) \subseteq ext(A)$ holds by definition, this follows from the property of *monotonicity* of $I(x)$:

$$a \in I(x) \;\&\; a \subseteq b \supset b \in I(x) \quad Mon$$

As a rule, the property is expressed as

$$\dfrac{a \in I(x), a \subseteq b, b \in I(x), \Gamma \Rightarrow \Delta}{a \in I(x), a \subseteq b, \Gamma \Rightarrow \Delta}\; M$$

Lemma 3.4. *In the presence of monotonicity (Mon), the following forcing conditions give the same class of valid formulas:*

1. $x \Vdash_1 \Box A \equiv \exists a \in I(x)(a \Vdash^\forall A \,\&\, \forall y(y \Vdash A \supset y \in a))$

2. $x \Vdash_2 \Box A \equiv \exists a \in I(x).a \Vdash^{\forall} A$

Proof. Let $V(1)$ (resp. $V(2)$) be the class of valid formulas according to 1 (resp. 2). We show that $V(1) = V(2)$. We show by induction on formulas that A is in $V(1)$ if and only if A is in $V(2)$. The only non-trivial case is the one for boxed formulas, so suppose that $\models_1 \Box A$, that is, for all models $(W, I, \mathcal{V})$ and for all x we have $\exists a \in I(x)(a \Vdash^{\forall} A \,\&\, A \lhd a)$. It is then clear by first-order logic that $\exists a \in I(x).a \Vdash^{\forall} A$. Therefore $\models_2 \Box A$.

Conversely, if $\models_2 \Box A$, then for an arbitrary x we have $\exists a \in I(x).a \Vdash^{\forall} A$. Let b be $ext(A)$. By monotonicity, we have that $b \in I(x)$ and b clearly satisfies $A \lhd b$, so $x \Vdash_1 \Box A$. Since x was arbitrary, $\models_1 \Box A$. QED

It follows that in the case of logical systems closed under monotonicity the rules for the necessity operator can be simplified to the following form:

$$\frac{a \in I(x), a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x : \Box A, \Gamma \Rightarrow \Delta}\; L\Box',\; a\, \textit{fresh}$$

$$\frac{a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A, a \Vdash^{\forall} A}{a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A}\; R\Box'$$

Table 5: Modal rules of system G3M

Remark 3.5. *Whenever monotonicity is present, we shall consider the above, simplified rules for $\Box$ rather than the original ones with the addition of rule M; this is not just a choice to streamline the sequent calculus, but it follows also from the fact that rule M together with the right rule for inclusion gives a problematic case in the cut elimination procedure.*

Next, we proceed to the determination of the rule for system **C**. We have the following derivation:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\dots, a \cap b \Vdash^{\forall} A\&B, \dots \Rightarrow x : \Box(A\&B), a \cap b \Vdash^{\forall} A\&B \quad \dots, A\&B \lhd a \cap b, \dots \Rightarrow x : \Box(A\&B), A\&B \lhd a \cap b}{a \cap b \in I(x), a \in I(x), b \in I(x), a \cap b \Vdash^{\forall} A\&B, a \Vdash^{\forall} A, b \Vdash^{\forall} B, A\&B \lhd a \cap b, A \lhd a, B \lhd b \Rightarrow x : \Box(A\&B)}\,R\Box}{a \cap b \in I(x), a \in I(x), b \in I(x), a \cap b \Vdash^{\forall} A\&B, a \Vdash^{\forall} A, b \Vdash^{\forall} B, A \lhd a, B \lhd b \Rightarrow x : \Box(A\&B)}\,Adm_2}{a \cap b \in I(x), a \in I(x), b \in I(x), a \Vdash^{\forall} A, b \Vdash^{\forall} B, A \lhd a, B \lhd b \Rightarrow x : \Box(A\&B)}\,Adm_1}{a \in I(x), b \in I(x), a \Vdash^{\forall} A, b \Vdash^{\forall} B, A \lhd a, B \lhd b \Rightarrow x : \Box(A\&B)}\,Rule}{a \in I(x), a \Vdash^{\forall} A, A \lhd a, x : \Box B \Rightarrow x : \Box(A\&B)}\,L\Box}{x : \Box A, x : \Box B \Rightarrow x : \Box(A\&B)}\,L\Box}{x : \Box A\&\Box B \Rightarrow x : \Box(A\&B)}\,L\&$$

Here we have used two steps whose admissibility follows from admissibility of cut and contraction (to be proved below) and the derivability in **G3E** of the sequents

1. $a \Vdash^{\forall} A, b \Vdash^{\forall} B \Rightarrow a \cap b \Vdash^{\forall} A\&B$
2. $A \lhd a, B \lhd b \Rightarrow A\&B \lhd a \cap b$

in a system extended with the following rules for formal intersection:

$$\frac{x \in a, x \in b, x \in a \cap b, \Gamma \Rightarrow \Delta}{x \in a \cap b, \Gamma \Rightarrow \Delta}\ L\cap \qquad \frac{\Gamma \Rightarrow \Delta, x \in a \cap b, x \in a \quad \Gamma \Rightarrow \Delta, x \in a \cap b, x \in b}{\Gamma \Rightarrow \Delta, x \in a \cap b}\ R\cap$$

So the extra condition that should be required on the neighbourhoods is just

$$a \in I(x)\ \&\ b \in I(x) \rightarrow a \cap b \in I(x)$$

that is, closure of $I(x)$ under intersection. It corresponds to the rule

$$\frac{a \cap b \in I(x), a \in I(x), b \in I(x), \Gamma \Rightarrow \Delta}{a \in I(x), b \in I(x), \Gamma \Rightarrow \Delta}\ C$$

Observe that if $I(x)$ is closed under supersets, then the above condition can be equivalently replaced by the weaker

$$a \in I(x)\ \&\ b \in I(x) \rightarrow \exists c \in I(x). c \subseteq a\ \&\ c \subseteq b$$

which can be translated into the geometric rule *Prebasic* seen already in Section 2:

$$\frac{c \in I(x), a \in I(x), b \in I(x), c \subseteq a, c \subseteq b, \Gamma \Rightarrow \Delta}{a \in I(x), b \in I(x), \Gamma \Rightarrow \Delta}\ C'$$

where c is a fresh neighbourhood label.

Finally, we determine the rule needed to prove the validity of $\Box\top$. As a preliminary remark, we observe that in the calculus **G3K** which shares the propositional base with **G3E**, the constant $\top$ (for *true*) is not primitive but defined as $\bot \supset \bot$ (or $A \supset A$ for any formula A). The rule to be added for a labelled calculus with $\top$ as a primitive is the dual of the rule $L\bot$, that is, the zero-premiss rule[15]

$$\frac{}{\Gamma \Rightarrow \Delta, x : \top}\ R\top$$

We have the search tree[16]

$$\cfrac{\cfrac{a \in I(x), a \Vdash^{\forall} \top, \top \lhd a \Rightarrow x : \Box\top, a \Vdash^{\forall} \top \quad a \in I(x), a \Vdash^{\forall} \top, \top \lhd a \Rightarrow x : \Box\top, \top \lhd a}{a \in I(x), a \Vdash^{\forall} \top, \top \lhd a \Rightarrow x : \Box\top}\ R\Box}{\Rightarrow x : \Box\top}\ rule$$

[15]Observe that the rule is actually derivable with $\top$ defined as $\bot \supset \bot$.

[16]Since the topsequents are derivable, the proof search is a derivation once the step indicated by *rule* is taken to be a rule of the system; here, as elsewhere, proof search in the basic calculus is used to determine which additional rules have to be included in the system to make certain sequents derivable.

The extra rule correponds to the following property of neighbourhoods

$$(1.)\ \exists a \in I(x). a \Vdash^{\forall} \top \ \&\ \top \lhd a$$

which is clearly equivalent to

$$(1.')\ \exists a \in I(x). \top \lhd a$$

and corresponds to the rule

$$\frac{a \in I(x), \top \lhd a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ N$$

with a fresh. In the presence of monotonicity rule *Nondeg* (Definition 2.4) suffices, because we have

$$\cfrac{\cfrac{a \in I(x) \Rightarrow x : \Box\top, a \Vdash^{\forall} \top}{a \in I(x) \Rightarrow x : \Box\top}\ R\Box'}{\Rightarrow x : \Box\top}\ \mathit{Nondeg}$$

with topsequent clearly derivable.

3.1 Adding $\Diamond$

The possibility modality is defined in classical modal logic, as in normal modal logic, as the dual of necessity (cf. [4])

$$\Diamond A \equiv \neg\Box\neg A$$

and therefore it is not usually considered as a modality with its own rules. It is however convenient, for the same reasons why it is convenient to have classical logic with all the connectives, not just two (or even one) of them, to have primitive rules for possibility. The rules are found by imposing the above duality and using the rules of $\Box$ and the duality between $\Vdash^{\forall}$ and $\Vdash^{\exists}$. In practice, to find the left and right rules for $\Diamond$ we start with the sequents $x : \Diamond A, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, x : \Diamond A$, replace them with $x : \neg\Box\neg A, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, x : \neg\Box\neg A$, respectively, and apply the rules for $\neg$ and $\Box$. It becomes clear that the former sequent needs also $a \in I(x)$ in the antecedent, else $R\Box$ cannot be applied. The decomposition then gives $a \Vdash^{\forall} \neg A$ in the succedent (resp. antecedent) which is replaced by the equivalent $a \Vdash^{\exists} A$ in the antecedent (resp. succedent). The formula $\neg A \lhd a$ instead cannot be moved to the other side with negation removed because the scope of the negation is A, not $A \lhd a$. In the end, the rules for the possibility modality are as follows:

$$\frac{a \in I(x), x : \Diamond A, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta \quad a \in I(x), x : \Diamond A, \Gamma \Rightarrow \Delta, \neg A \lhd a}{a \in I(x), x : \Diamond A, \Gamma \Rightarrow \Delta}\ L\Diamond$$

$$\frac{a \in I(x), \neg A \lhd a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}{\Gamma \Rightarrow \Delta, x : \Diamond A}\; R\Diamond$$

In $R\Diamond$, a is a fresh neighbourhood label.

To see the rules at work, we can use them to verify the duality between the two alethic modalities, where both topsequents are derivable[17]:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \in I(x), x : \Diamond A, y \in a, y : A, y : \neg A, a \Vdash^{\forall} \neg A, \neg A \lhd a \Rightarrow x : \bot}}{a \in I(x), x : \Diamond A, y \in a, y : A, a \Vdash^{\forall} \neg A, \neg A \lhd a \Rightarrow x : \bot}\; L\Vdash^{\forall}}{a \in I(x), x : \Diamond A, a \Vdash^{\exists} A, a \Vdash^{\forall} \neg A, \neg A \lhd a \Rightarrow x : \bot}\; L\Vdash^{\exists} \qquad \dfrac{}{a \in I(x), x : \Diamond A, a \Vdash^{\forall} \neg A, \neg A \lhd a \Rightarrow x : \bot, \neg A \lhd a}}{a \in I(x), x : \Diamond A, a \Vdash^{\forall} \neg A, \neg A \lhd a \Rightarrow x : \bot}\; L\Diamond}{x : \Diamond A, x : \Box\neg A \Rightarrow x : \bot}\; L\Box}{\dfrac{x : \Diamond A \Rightarrow x : \neg\Box\neg A}{\Rightarrow x : \Diamond A \supset \neg\Box\neg A}\; R\supset}\; R\neg$$

The derivation of the other direction of the duality, namely $\Rightarrow x : \neg\Box\neg A \supset \Diamond A$, is found in a similar way using the rules for negation, the alethic modalities and the local forcing relations.

If monotonicity is absorbed into the modal rules, also the rules for $\Diamond$ get modified (and simplified). The monotonic version of the rules for $\Diamond$ is as follows:

$$\frac{a \in I(x), x : \Diamond A, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta}{a \in I(x), x : \Diamond A, \Gamma \Rightarrow \Delta}\; L\Diamond'$$

$$\frac{a \in I(x), \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A}{\Gamma \Rightarrow \Delta, x : \Diamond A}\; R\Diamond'$$

We remark here that all the results below continue to hold when $\Diamond$ is added as an explicit modality, rather than a defined one, in the calculus.

4 Structural properties

In this section we shall give detailed proofs of the structural properties for the systems based on neighbourhood semantics that we have considered. Rather than giving specific proofs for specific systems, we shall indicate how the structural properties can be established by following a generalization of the guidelines presented in [25] and [32], section 11.4. There are some important non-trivial extra considerations caused by the layering of rules for the modalities defined in terms of neighbourhood semantics, which gives a quantifier alternation more complex than in the Kripke-style semantics. Likewise, some preliminary results are needed, namely height-preserving

[17]This is a consequence of Lemma 4.2 proved in the following section.

admissibility of substitution (in short, hp-substitution) and height-preserving invertibility (in short, hp-invertibility) of the rules. We recall that the *height* of a derivation is its height as a tree, i.e., the length of its longest branch, and that $\vdash_n$ denotes derivability with derivation height bounded by n in a given system.

In the following we shall denote with **G3n*** any extension of the basic system **G3n** with rules for the modalities $[\,]$, $\langle\,]$, $[\,\rangle$, and $\langle\,\rangle$, $\Box$[18] and with extra (mathematical) rules. This extension is intended to follow the standard *closure condition* for extensions of contraction-free labelled sequent calculi (cf. [25]) to guarantee admissibility of contraction in the resulting system.

As observed above, in the light of Remark 3.5, we can obtain system **G3nM** by modifying the rules $L\Box$, $R\Box$ to the form $L\Box'$ and $R\Box'$; for extensions, we can take in place of C and N the rules C' and *Nondeg*.

In many proofs we shall use an induction on formula weight. In order to find a definition of weight that makes the induction work we have to take into account several constraints that emerge from the proofs of the structural results; the choice for this particular definition will thus become clear from the proofs to follow.

Observe that the definition extends the usual definition of weight from (pure) formulas to labelled formulas and local forcing relations, namely, to all formulas of the form $x : A$, $a \Vdash^{\forall} A$, $a \Vdash^{\exists} A$, $A \lhd a$, as well as the relational formulas $x \in a$, $a \in I(x)$, $a \subseteq b$.

Definition 4.1. *The label of formulas of the form $x : A$ is x. The label of formulas of the form $a \Vdash^{\forall} A$, $a \Vdash^{\exists} A$, $A \lhd a$ is a. The label of a formula $\mathcal{F}$ will be denoted by $l(\mathcal{F})$. The pure part of a labelled formula $\mathcal{F}$ is the part without the label and without the forcing relation, either local ($\Vdash^{\exists}$, $\Vdash^{\forall}$) or worldwise (:) and will be denoted by $p(\mathcal{F})$.*

The weight of a labelled formula *$\mathcal{F}$ is given by the pair $(\mathtt{w}(p(\mathcal{F})), \mathtt{w}(l(\mathcal{F})))$ where*

- *For all worlds labels x and all neighbourhood labels a, $\mathtt{w}(x) = 0$ and $\mathtt{w}(a) = 1 + n(\cap)$, where $n(\cap)$ is the number of formal intersections in a.*
-
  - $\mathtt{w}(P) = \mathtt{w}(\bot) = 1$,
  - $\mathtt{w}(A \circ B) = \mathtt{w}(A) + \mathtt{w}(B) + 1$ *for $\circ$ conjunction, disjunction, or implication,*
  - $\mathtt{w}(\Box A) = \mathtt{w}([\,]A) = \mathtt{w}(\langle\,]A) = \mathtt{w}([\,\rangle A) = \mathtt{w}(\langle\,\rangle A) = \mathtt{w}(A) + 1$

For formulas of the form $a \in I(x)$, $x \in a$, we stipulate $\mathtt{w}(a \in I(x)) = \mathtt{w}(x \in a) = (0, w(a))$ and for formulas of the form $a \subseteq b$, $\mathtt{w}(a \subseteq b) = (w(a), w(b))$. Weights of labelled formulas are ordered lexicographically.

[18] We assume that for each modality, the extension has to contain both the right and left rule.

From the definition of weight it is clear that the weight gets decreased if we move from a formula labelled by a neighbourhood label to the same formula labelled by a world label, or if we move (regardless the label) to a formula with a pure part of strictly smaller weight.

Lemma 4.2. *Sequents of the following form are derivable in* **G3n*** *for arbitrary formulas A and B in the propositional modal language of* **G3n****:*

1. $a \subseteq b, \Gamma \Rightarrow \Delta, a \subseteq b$
2. $A \lhd a, \Gamma \Rightarrow \Delta, A \lhd a$
3. $a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A$
4. $a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A$
5. $x : A, \Gamma \Rightarrow \Delta, x : A$

Proof. 1. By the following derivation

$$\dfrac{\dfrac{x \in b, x \in a, a \subseteq b, \Gamma \Rightarrow \Delta, x \in b}{x \in a, a \subseteq b, \Gamma \Rightarrow \Delta, x \in b}\, L\subseteq}{a \subseteq b, \Gamma \Rightarrow \Delta, a \subseteq b}\, R\subseteq$$

where the topsequent is initial.

2. By the following derivation

$$\dfrac{\dfrac{x \in a, x : A, A \lhd a, \Gamma \Rightarrow \Delta, x \in a}{x : A, A \lhd a, \Gamma \Rightarrow \Delta, x \in a}\, L\lhd}{A \lhd a, \Gamma \Rightarrow \Delta, A \lhd a}\, R\lhd$$

where the topsequent is initial.

3–5 are proved by simultaneous induction on formula weight.

3. We have the following inference

$$\dfrac{\dfrac{x : A, x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, x : A}{x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, x : A}\, L\Vdash^{\forall}}{a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}\, R\Vdash^{\forall}$$

The topsequent is derivable by induction hypothesis because $\mathtt{w}(x : A) < \mathtt{w}(a \Vdash^{\forall} A)$.

4. Similar, with $L \Vdash^{\exists}$ and $R \Vdash^{\exists}$ in place of $R \Vdash^{\forall}$ and $L \Vdash^{\forall}$, respectively, using $\mathtt{w}(x : A) < \mathtt{w}(a \Vdash^{\exists} A)$.

5. We distinguish subcases according to the structure of A. If it is atomic or $\bot$, the sequent is initial or conclusion of $L\bot$. If the outermost connective of A is a conjunction or a disjunction, or an implication, the sequent is derivable by application of the respective rules and the induction hypothesis. If it is a modality, we have the following further subcases:

5.1. $A \equiv [\ ]B$. We have the following inference

$$\dfrac{\dfrac{a \Vdash^{\forall} B, a \in I(x), x : [\ ]B, \Gamma \Rightarrow \Delta, a \Vdash^{\forall} B}{a \in I(x), x : [\ ]B, \Gamma \Rightarrow \Delta, a \Vdash^{\forall} B}\ L[\ ]}{x : [\ ]B, \Gamma \Rightarrow \Delta, x : [\ ]B}\ R[\ ]$$

where the topsequent is derivable by induction hypothesis because $\mathtt{w}(a \Vdash^{\forall} B) < \mathtt{w}(x : [\ ]B)$.

5.2. $A \equiv \langle\ ]B$. Similar with the rules $L\langle\ ]$, $R\langle\ ]$, and the inductive hypothesis on $a \Vdash^{\forall} B$, using $\mathtt{w}(a \Vdash^{\forall} B) < \mathtt{w}(x : \langle\ ]B)$.

5.3. $A \equiv [\ \rangle B$. Similar with the rules $R[\ \rangle$, $L[\ \rangle$, and the inductive hypothesis on $a \Vdash^{\exists} B$, using $\mathtt{w}(a \Vdash^{\exists} B) < \mathtt{w}(x : [\ \rangle B)$.

5.4 $A \equiv \langle\ \rangle B$. Similar with the rules $L\langle\ \rangle$, $R\langle\ \rangle$, and the inductive hypothesis on $a \Vdash^{\forall} B$.

5.5 $A \equiv \Box B$. We have the following inference

$$\dfrac{\dfrac{a \in I(x), a \Vdash^{\forall} B, B \lhd a, \Gamma \Rightarrow \Delta, x : \Box B, a \Vdash^{\forall} B \quad a \in I(x), a \Vdash^{\forall} B, B \lhd a, \Gamma \Rightarrow \Delta, x : \Box B, B \lhd a}{a \in I(x), a \Vdash^{\forall} B, B \lhd a, \Gamma \Rightarrow \Delta, x : \Box B}\ R\Box}{x : \Box B, \Gamma \Rightarrow \Delta, x : \Box B}\ L\Box$$

where the left topsequent is derivable by induction hypothesis because $\mathtt{w}(a \Vdash^{\forall} B) < \mathtt{w}(x : \Box B)$ and the right one by clause 2 above.

For extensions of **G3nM** we have the following inference:

$$\dfrac{\dfrac{a \in I(x), a \Vdash^{\forall} B, \Gamma \Rightarrow \Delta, x : \Box B, a \Vdash^{\forall} B}{a \in I(x), a \Vdash^{\forall} B, \Gamma \Rightarrow \Delta, x : \Box B}\ R\Box'}{x : \Box B, \Gamma \Rightarrow \Delta, x : \Box B}\ L\Box'$$

and we can treat this as a sub-case of the above. QED

In our system, in addition to world labels, we have neighbourhood labels. The latter are subject to similar conditions, such as the conditions of being fresh in certain rules, as the world labels. Consequently, we shall need properties of hp-substitution in our analysis. Before stating and proving the property, we observe that the definition of substitution of labels given in [25] can be extended in an obvious way – that need not to be pedantically detailed here – to all the formulas of our language and to neighbourhood labels. We'll have, for example, $x : \langle\ \rangle A(y/x) \equiv y : \langle\ \rangle A$, $a \Vdash^{\exists} A(b/a) \equiv b \Vdash^{\exists} A$, and $A \lhd a(b/a) \equiv A \lhd b$. Next, we prove that the calculus enjoys the property of hp-substitution both of world and neighbourhood labels:[19]

Proposition 4.3. *1. If* $\vdash_n \Gamma \Rightarrow \Delta$, *then* $\vdash_n \Gamma(y/x) \Rightarrow \Delta(y/x)$*;*

2. If $\vdash_n \Gamma \Rightarrow \Delta$, *then* $\vdash_n \Gamma(b/a) \Rightarrow \Delta(b/a)$*.*

Proof. Both statements are proved by induction on the height of the derivation.

If it is 0, then $\Gamma \Rightarrow \Delta$ is an initial sequent or a conclusion of $L\bot$. The same then holds for $\Gamma(y/x) \Rightarrow \Delta(y/x)$ and for $\Gamma(b/a) \Rightarrow \Delta(b/a)$.

If the derivation has height $n > 0$, we consider the last rule applied. If $\Gamma \Rightarrow \Delta$ has been derived by a rule without variable conditions, we apply the induction hypothesis and then the rule. Rules with variable conditions require that we avoid a clash of the substituted variable with the fresh variable in the premiss. This is the case for the logical rules $R \Vdash^{\forall}$, $L \Vdash^{\exists}$, $R[\ ]$, $L\langle\ ]$, $R[\ \rangle$, $L\langle\ \rangle$, $L\Box$, $L\Box'$ and for the neighbourhood rules $R \subseteq$, *Prebasic*/C', *Nondeg*. So, if $\Gamma \Rightarrow \Delta$ has been derived by any of these rules, we apply the inductive hypothesis twice to the premiss, first to replace the fresh variable with another fresh variable different, if necessary, from the one we want to substitute, then to make the substitution, and then apply the rule. QED

The rules of weakening for the language of a labelled system with internalized neighbourhood semantics such as **G3n*** have the following form, where ϕ is either a "relational" atom of the form $a \in I(x)$[20] or $x \in a$ or a labelled formula of the form $x : A$, $a \Vdash^{\forall} A$, $a \Vdash^{\exists} A$ or a formula of the form $A \lhd a$:

$$\frac{\Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta}\, L\text{-}Wkn \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \phi}\, R\text{-}Wkn$$

Proposition 4.4. *The rules of left and right weakening are hp-admissible in* **G3n****.*

Proof. Straightforward induction, with a similar proviso as in the above proof for rules with variable conditions. QED

Next, we prove *hp-invertibility* of the rules of **G3n***, i.e., for every rule of the form $\frac{\Gamma' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta}$, if $\vdash_n \Gamma \Rightarrow \Delta$ then $\vdash_n \Gamma' \Rightarrow \Delta'$, and for every rule of the form $\frac{\Gamma' \Rightarrow \Delta' \quad \Gamma'' \Rightarrow \Delta''}{\Gamma \Rightarrow \Delta}$ if $\vdash_n \Gamma \Rightarrow \Delta$ then $\vdash_n \Gamma' \Rightarrow \Delta'$ and $\vdash_n \Gamma'' \Rightarrow \Delta''$. Items $7'$ and $8'$ are the invertibility

[19] We remind that of the two possible notations for substitution we use the one in which $A(y/x)$ indicates the result of substituting y for x in A.

[20] Indeed, such formulas are not needed for right weakenening because they are never active on the right.

for the non-monotonic rules for $\langle\ ]$, $R\langle\ ]'$ and $L\langle\ ]'$, and $15'$ for the monotonic version of $L\Box$:

Lemma 4.5. *The following hold in* $\mathbf{G3n}^*$*:*

1. *If* $\vdash_n \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A$ *then* $\vdash_n x \in a, \Gamma \Rightarrow \Delta, x : A$.
2. *If* $\vdash_n x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta$ *then* $\vdash_n x \in a, x : A, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta$.
3. *If* $\vdash_n x \in a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A$ *then* $\vdash_n x \in a, \Gamma \Rightarrow \Delta, x : A, a \Vdash^{\exists} A$.
4. *If* $\vdash_n a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta$ *then* $\vdash_n x \in a, x : A, \Gamma \Rightarrow \Delta$.
5. *If* $\vdash_n \Gamma \Rightarrow \Delta, x : [\ ]A$ *then* $\vdash_n a \in I(x), \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A$.
6. *If* $\vdash_n a \in I(x), x : [\ ]A, \Gamma \Rightarrow \Delta$ *then* $\vdash_n a \in I(x), x : [\ ]A, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta$.
7. *If* $\vdash_n a \in I(x), \Gamma \Rightarrow \Delta, x : \langle\ ]A$ *then* $\vdash_n a \in I(x), \Gamma \Rightarrow \Delta, x : \langle\ ]A, a \Vdash^{\forall} A$.

7′. *If* $\vdash_n a \in I(x), \Gamma \Rightarrow \Delta, x : \langle\ ]A$ *then* $\vdash_n a \in I(x), \Gamma \Rightarrow \Delta, x : \langle\ ]A, a \Vdash^{\forall} A$ *and* $\vdash_n a \in I(x), \Gamma \Rightarrow \Delta, x : \langle\ ]A, A \lhd a$.

8. *If* $\vdash_n x : \langle\ ]A, \Gamma \Rightarrow \Delta$ *then* $\vdash_n a \in I(x), a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta$.

8′. *If* $\vdash_n x : \langle\ ]A, \Gamma \Rightarrow \Delta$ *then* $a \in I(x), a \Vdash^{\forall} A, A \lhd a, \Gamma \Rightarrow \Delta$.

9. *If* $\vdash_n \Gamma \Rightarrow \Delta, x : [\ \rangle A$ *then* $\vdash_n a \in I(x), \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A$.
10. *If* $\vdash_n a \in I(x), x : [\ \rangle A, \Gamma \Rightarrow \Delta$ *then* $\vdash_n a \in I(x), x : [\ \rangle A, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta$.
11. *If* $\vdash_n a \in I(x), \Gamma \Rightarrow \Delta, x : \langle\ \rangle A$ *then* $\vdash_n a \in I(x), \Gamma \Rightarrow \Delta, x : \langle\ \rangle A, a \Vdash^{\exists} A$.
12. *If* $\vdash_n x : \langle\ \rangle A, \Gamma \Rightarrow \Delta$ *then* $\vdash_n a \in I(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta$.
13. *If* $\vdash_n A \lhd a, y : A, \Gamma \Rightarrow \Delta$ *then* $\vdash_n y \in a, A \lhd a, y : A, \Gamma \Rightarrow \Delta$.
14. *If* $\vdash_n \Gamma \Rightarrow \Delta, A \lhd a$ *then* $\vdash_n y : A, \Gamma \Rightarrow \Delta, y \in a$.
15. *If* $\vdash_n x : \Box A, \Gamma \Rightarrow \Delta$ *then* $\vdash_n a \in I(x), a \Vdash^{\forall} A, A \lhd a, \Gamma \Rightarrow \Delta$.

15′. *If* $\vdash_n x : \Box A, \Gamma \Rightarrow \Delta$ *then* $\vdash_n a \in I(x), a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta$.

16. *If* $\vdash_n a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A$ *then* $\vdash_n a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A, a \Vdash^{\forall} A$ *and* $\vdash_n a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A, A \lhd a$.
17. *If* $\vdash_n \Gamma \Rightarrow \Delta, a \subseteq b$ *then* $\vdash_n x \in a, \Gamma \Rightarrow \Delta, x \in b$.

18. *If* $\vdash_n x \in a, a \subseteq b, \Gamma \Rightarrow \Delta$ *then* $\vdash_n x \in a, a \subseteq b, x \in b, \Gamma \Rightarrow \Delta$.

Proof. Observe first that all the cases (2, 3, 6, 7, 7′, 10, 11, 13, 16, 18) that are instances of hp-admissibility of weakening follow from Proposition 4.4 above. For the rest, the proof is by induction on n and we show in detail, by way of example item 5., the other cases being shown in a similar way.

Base case: Suppose that $\Gamma \Rightarrow \Delta, x : [\,]A$ is an initial sequent or conclusion of $L\bot$. Then, in the former case, $x : [\,]A$ not being atomic or of the form $x \in a$, $a \in I(x), \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A$ is an initial sequent, in the latter it is a conclusion of $L\bot$.

Inductive step: Assume hp-invertibility up to n, and let $\vdash_{n+1} \Gamma \Rightarrow \Delta, x : [\,]A$. If $x : [\,]A$ is principal, then the premiss $a \in I(x), \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A$ (possibly obtained through hp-substitution) has a derivation of height n. If $x : [\,]A$ is not principal in the last rule, we distinguish the case in which the last rule is not a rule with eigenvariable from the case in which it is. In the former case, the last rule has one or two premisses of the form $\Gamma' \Rightarrow \Delta', x : [\,]A$ of derivation height $\leq n$. By induction hypothesis we have $a \in I(x), \Gamma' \Rightarrow \Delta', a \Vdash^{\forall} A$ for each premiss, with derivation height at most n. Thus, $\vdash_{n+1} a \in I(x), \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A$. In the latter case, we proceed as in the previous case if the last rule has the eigenvariable for world labels, the critical case being (here) the one with the eigenvariable for neighbourhood labels. So, if the last rule is, say, $L\langle\,]$, then $\Gamma = \langle\,]B, \Gamma'$ and we have a premiss that we can assume to be of the form $b \in I(x), b \Vdash^{\forall} B, \Gamma' \Rightarrow \Delta, x : [\,]A$ with b different from a (this can be assumed without loss of generality because of hp-substitution). By inductive hypothesis we obtain a derivation of height n of $a \in I(x), b \in I(x), b \Vdash^{\forall} B, \Gamma' \Rightarrow \Delta, a \Vdash^{\forall} A$ and by a step of $L\langle\,]$ we conclude derivability of $a \in I(x), \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A$ with height $n+1$. Cases 8, 8′, 9, 12, 15, 15′, are proved with a similar analysis. There is a final group of cases (items 1, 4, 14, 17), those of rules with an eigenvariable condition for world labels. The treatment is similar to the case detailed above, with a similar distinction of cases as for the last rule applied in the derivation. A special proviso is needed for the case in which the last rule is not the rule with the principal formula in question and it is a rule with eigenvariable of the same type, namely a world label. The claim is obtained by inductive hypothesis after use, if needed, of hp-substitution on the premisses of such rules to avoid a clash of variables so that the last rule can be applied after the inductive step to restore the original contexts. QED

Lemma 4.6. *All the propositional rules of* **G3n*** *are hp-invertible.*

Proof. Similar to the proof for **G3c** (Theorem 3.1.1 in [31]). QED

Therefore, as a general result, we have:

Corollary 4.7. *All the rules of* **G3n*** *are hp-invertible.*

Proof. By Lemmas 4.5, 4.6, and 4.4 (the latter gives hp-invertibility of the neighbourhood rules). QED

The rules of contraction for the language of a labelled system with internalized neighbourhood semantics such as **G3n*** have the following form, where ϕ is either a "relational" atom of the form $a \in I(x)$ or $x \in a$ or a labelled formula of the form $x : A$, $a \Vdash^{\forall} A$, $a \Vdash^{\exists} A$ or a formula of the form $A \lhd a$:

$$\frac{\phi, \phi, \Gamma \Rightarrow \Delta}{\phi, \Gamma \Rightarrow \Delta}\,\textit{L-Ctr} \qquad \frac{\Gamma \Rightarrow \Delta, \phi, \phi}{\Gamma \Rightarrow \Delta, \phi}\,\textit{R-Ctr}$$

Theorem 4.8. *The rules of left and right contraction are hp-admissible in* **G3n****.*

Proof. By simultaneous induction on the height of derivation for left and right contraction.

If $n = 0$ the premiss is either an initial sequent or a conclusion of a zero-premiss rule. In each case, the contracted sequent is also an initial sequent or a conclusion of the same zero-premiss rule.

If $n > 0$, consider the last rule used to derive the premiss of contraction. There are two cases, depending on whether the contraction formula is principal or not in the rule.[21]

1. If the contraction formula is not principal in it, both occurrences are found in the premisses of the rule and they have a smaller derivation height. By the induction hypothesis, they can be contracted and the conclusion is obtained by applying the rule to the contracted premisses.

2. If the contraction formula is principal in it, we distinguish two sub-cases:

2.1. The last rule is one in which the principal formulas appear also in the premiss (such as $L \Vdash^{\forall}$, $R \Vdash^{\exists}$, $L[\ ]$, $R\langle\ ]$, $L[\ \rangle$, $R\langle\ \rangle$, $L \lhd$, $R\Box$, $R\Box'$, $L \subseteq$, and the neighbourhood rules). In all these cases we apply the induction hypothesis to the premiss(es) and then the rule. For example, if the last rule used to derive the premiss of contraction is $R\Box$ we have:

$$\frac{a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A, x : \Box A, a \Vdash^{\forall} A \quad a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A, x : \Box A, A \lhd a}{a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A, x : \Box A}\, R\Box$$

[21] We recall that the *principal formula* of a logical rule is the formula containing the constant named by the rule in question, which in this case can be a connective, a modality, or a local forcing relation ($\Vdash^{\exists}$, $\Vdash^{\forall}$), or the inclusion operator; the other formulas in the rule are active or side formulas. *Side formulas* are the formulas in the contexts and the other formulas, which are neither side not principal formulas are *active formulas*. In the case of labelled systems there can be active formulas in the conclusion of the rules. For example, the formula $a \in I(x)$ in the conclusion of $R\Box$ is regarded as an active formula.

By induction hypothesis applied to the premiss we obtain a one step shorter derivation of $a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A, a \Vdash^{\forall} A$ and $a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A, A \lhd a$ and thus by a step of $R\Box$ we obtain $a \in I(x), \Gamma \Rightarrow \Delta, x : \Box A$ with the same derivation height of the given premiss of contraction.

For the neighbourhood rules we follow the standard procedure as for added extra-logical rules and observe that in case the contraction formulas are both principal in the rule (as in the case of rule C) we apply the closure condition

2.2. The last rule is one in which the principal formula does not appear in the premiss(es) (such as the rules for &, $\vee$, $\supset$, $R \Vdash^{\forall}$, $L \Vdash^{\exists}$, $R[\ ]$, $L\langle\ ]$, $R[\ \rangle$, $L\langle\ \rangle$, $L\Box$, $R \subseteq$). In all such cases, we apply hp-invertibility to the premiss(es) of the rule so that we have a duplication of formulas at a smaller derivation height, then apply the induction hypothesis (as many times as needed) then apply the rule in question. For example, if the last rule is $L\Box$, we have:

$$\frac{a \in I(x), a \Vdash^{\forall} A, A \lhd a, x : \Box A, \Gamma \Rightarrow \Delta}{x : \Box A, x : \Box A, \Gamma \Rightarrow \Delta}\ L\Box,\ a\,\textit{fresh}$$

Using hp-invertibility of $L\Box$ we obtain from the premiss a derivation of height $n-1$ of

$$a \in I(x), a \in I(x), a \Vdash^{\forall} A, a \Vdash^{\forall} A, A \lhd a, A \lhd a, \Gamma \Rightarrow \Delta$$

By the induction hypothesis we get a derivation of the same height of the sequent $a \in I(x), a \Vdash^{\forall} A, A \lhd a, \Gamma \Rightarrow \Delta$ and application of $L\Box$ gives a derivation of height n of $x : \Box A, \Gamma \Rightarrow \Delta$. QED

Cut is a rule of the form

$$\frac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ Cut$$

where ϕ is any formula of the language of the labelled calculus $\mathbf{G3n}^*$. We have:

Theorem 4.9. *Cut is admissible in* $\mathbf{G3n}^*$.

Proof. By double induction, with primary induction on the weight of the cut formula and subinduction on the cut height, i.e., the sum of the heights of derivations of the premisses of cut. The cases in which the premisses of cut are either initial sequents or obtained through the rules for &, $\vee$, or $\supset$ follow the treatment of Theorem 3.2.3 of [31]. Among such cases, we just consider a significant one here, the case in which the initial sequent is $x \in a, \Gamma \Rightarrow \Delta, x \in a$ and the other premiss is conclusion of a rule for inclusion in which $x \in a$ is an active formula. The cut, with $\Gamma' = a \subseteq b, \Gamma''$, is as follows

$$\dfrac{x \in a, \Gamma \Rightarrow \Delta, x \in a \qquad \dfrac{x \in a, x \in b, a \subseteq b, \Gamma'' \Rightarrow \Delta'}{x \in a, \Gamma' \Rightarrow \Delta'}\; L\subseteq}{x \in a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\; Cut$$

and it is converted into a cut of reduced height as follows

$$\dfrac{\dfrac{x \in a, \Gamma \Rightarrow \Delta, x \in a \qquad x \in a, x \in b, a \subseteq b, \Gamma'' \Rightarrow \Delta'}{x \in a, x \in b, a \subseteq b, \Gamma, \Gamma'' \Rightarrow \Delta, \Delta'}\; Cut}{x \in a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\; L\subseteq$$

For the cases in which the cut formula is a side formula in at least one rule used to derive the premisses of cut, the cut reduction is dealt with in the usual way by permutation of cut, with possibly an application of hp-substitution to avoid a clash with the fresh variable in rules with variable condition. In all such cases the cut height is reduced. We give one example to give concreteness to this qualitative analysis:

$$\dfrac{\Gamma \Rightarrow \Delta, b \Vdash^{\forall} B \qquad \dfrac{a \in I(x), a \Vdash^{\forall} A, A \lhd a, b \Vdash^{\forall} B, \Gamma' \Rightarrow \Delta'}{x : \Box A, b \Vdash^{\forall} B, \Gamma' \Rightarrow \Delta'}\; L\Box}{x : \Box A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\; Cut$$

the neighbourhood label in the premiss of $L\Box$ is fresh, but nothing prevents it from appearing in the left premiss of cut; therefore, prior to the permutation of cut, we need to replace it with a neighbourhood label which is fresh not just with respect to the conclusion of $L\Box$ but also with respect to the left premiss of cut. Let the new fresh variable be c. The transformed derivation, with cut reduced to a cut of smaller height, is as follows:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, b \Vdash^{\forall} B \qquad c \in I(x), c \Vdash^{\forall} A, A \lhd c, b \Vdash^{\forall} B, \Gamma' \Rightarrow \Delta'}{c \in I(x), c \Vdash^{\forall} A, A \lhd a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\; Cut}{x : \Box A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\; L\Box$$

Next we consider in full detail the cases with cut formula principal in both premisses of cut and of the form $a \Vdash^{\forall} A$, $a \Vdash^{\exists} A$, $x : [\ ]A$, $x : \langle\ ]A$, $x : [\ \rangle A$, $x : \langle\ \rangle A$ or $A \lhd a$, $x : \Box A$.

1. The cut formula is $a \Vdash^{\forall} A$, principal in both premisses of cut. We have a derivation of the form

$$\dfrac{\dfrac{\overset{\mathcal{D}}{x \in a, \Gamma \Rightarrow \Delta, x : A}}{\Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}\, R \Vdash^{\forall} \quad \dfrac{y : A, y \in a, a \Vdash^{\forall} A, \Gamma' \Rightarrow \Delta'}{y \in a, a \Vdash^{\forall} A, \Gamma' \Rightarrow \Delta'}\, L \Vdash^{\forall}}{y \in a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, Cut$$

This is converted into the following derivation:

$$\dfrac{\dfrac{\overset{\mathcal{D}(y/x)}{y \in a, \Gamma \Rightarrow \Delta, y : A} \quad \dfrac{\Gamma \Rightarrow \Delta, a \Vdash^{\forall} A \quad y : A, y \in a, a \Vdash^{\forall} A, \Gamma' \Rightarrow \Delta'}{y \in a, y : A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, Cut_1}{y \in a, y \in a, \Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta'}\, Cut_2}{y \in a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, Ctr^*$$

Here $\mathcal{D}(y/x)$ denotes the result of application of hp-substitution to $\mathcal{D}$, using the fact that x is a fresh variable; compared to the original cut, Cut_1 is a cut of reduced height, Cut_2 is one of reduced weight of cut formula, because $\mathtt{w}(y : A) < \mathtt{w}(a \Vdash^{\forall} A)$, and Ctr^* denote repreated applications of (hp-)admissible contraction steps.

2. The cut formula is $a \Vdash^{\exists} A$, principal in both premisses of cut. The cut is reduced in a way similar to the one in the case above and the inequality to be used on formula weight is $\mathtt{w}(y : A) < \mathtt{w}(a \Vdash^{\exists} A)$.

3. The cut formula is $x : [\,]A$, principal in both premisses of cut.

We have a derivation of the form

$$\dfrac{\dfrac{\overset{\mathcal{D}}{a \in I(x), \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}}{\Gamma \Rightarrow \Delta, x : [\,]A}\, R[\,] \quad \dfrac{b \Vdash^{\forall} A, b \in I(x), x : [\,]A, \Gamma' \Rightarrow \Delta'}{b \in I(x), x : [\,]A, \Gamma' \Rightarrow \Delta'}\, L[\,]}{b \in I(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, Cut$$

The transformed derivation is obtained as follows:

$$\dfrac{\dfrac{\overset{\mathcal{D}(b/a)}{b \in I(x), \Gamma \Rightarrow \Delta, b \Vdash^{\forall} A} \quad \dfrac{\Gamma \Rightarrow \Delta, x : [\,]A \quad b \Vdash^{\forall} A, b \in I(x), x : [\,]A, \Gamma' \Rightarrow \Delta'}{b \Vdash^{\forall} A, b \in I(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, Cut}{b \in I(x), b \in I(x), \Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta'}\, Cut}{b \in I(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, Ctr^*$$

where the upper cut is of reduced height and the lower one of reduced weight because $\mathtt{w}(b \Vdash^{\forall} A) < \mathtt{w}(x : [\,]A)$.

The cases with cut formula of the form $x : \langle\,]A$, $x : [\,\rangle A$, and $x : \langle\,\rangle A$ are all treated in a similar way, using, respectively, the following inequalities that hold for the weight of the cut formulas, namely, $\mathtt{w}(b \Vdash^{\forall} A) < \mathtt{w}(x : \langle\,]A)$, $\mathtt{w}(b \Vdash^{\exists} A) < \mathtt{w}(x : [\,\rangle A)$, and $\mathtt{w}(b \Vdash^{\exists} A) < \mathtt{w}(x : \langle\,\rangle A)$.

We observe that it is essential here that the rules are in harmony in the sense that for each modality each pair of rules has either $\Vdash^{\forall}$ or $\Vdash^{\exists}$ in the premisses.

4. The cut formula is $A \lhd a$, principal in both premisses of cut. We have:

$$\cfrac{\cfrac{\overset{\mathcal{D}}{x : A, \Gamma \Rightarrow \Delta, x \in a}}{\Gamma \Rightarrow \Delta, A \lhd a}\,R\lhd \qquad \cfrac{y : A, y \in a, A \lhd a, \Gamma' \Rightarrow \Delta'}{y \in a, A \lhd a, \Gamma' \Rightarrow \Delta'}\,L\lhd}{y : A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\mathit{Cut}$$

The cut is converted as follows:

$$\cfrac{\cfrac{\overset{\mathcal{D}(y/x)}{y : A, \Gamma \Rightarrow \Delta, y \in a} \qquad \cfrac{\Gamma \Rightarrow \Delta, A \lhd a \quad y : A, y \in a, A \lhd a, \Gamma' \Rightarrow \Delta'}{y : A, y \in a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\mathit{Cut}}{y : A, y : A, \Gamma, \Gamma, \Gamma' \Rightarrow \Delta, \Delta, \Delta'}\,\mathit{Cut}}{y : A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\mathit{Ctr}^*$$

where the upper cut is of reduced cut height and the lower one of reduced weight of cut formula because $\mathtt{w}(y \in a) < \mathtt{w}(A \lhd a)$.

5. The cut formula is $x : \Box A$, principal in both premisses of cut. We have a cut of the form

$$\cfrac{\cfrac{b \in I(x), \Gamma \Rightarrow \Delta, x : \Box A, b \Vdash^{\forall} A \quad b \in I(x), \Gamma \Rightarrow \Delta, x : \Box A, A \lhd b}{b \in I(x), \Gamma \Rightarrow \Delta, x : \Box A}\,R\Box \qquad \cfrac{\overset{\mathcal{D}}{a \in I(x), a \Vdash^{\forall} A, A \lhd a, \Gamma' \Rightarrow \Delta'}}{x : \Box A, \Gamma' \Rightarrow \Delta'}\,L\Box}{b \in I(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\mathit{Cut}$$

This is transformed into derivation with four smaller cuts as follows. First we have

$$\cfrac{\cfrac{b \in I(x), \Gamma \Rightarrow \Delta, x : \Box A, b \Vdash^{\forall} A \quad x : \Box A, \Gamma' \Rightarrow \Delta'}{b \in I(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta', b \Vdash^{\forall} A}\,\mathit{Cut} \qquad \overset{\mathcal{D}(b/a)}{b \in I(x), b \Vdash^{\forall} A, A \lhd b, \Gamma' \Rightarrow \Delta'}}{(b \in I(x))^2, A \lhd b, \Gamma, (\Gamma')^2 \Rightarrow \Delta, (\Delta')^2}\,\mathit{Cut}$$

with two reduced cuts, the upper one with the original cut formula but smaller derivation height, and the lower one with a cut formula of reduced weight because $\mathtt{w}(b \Vdash^{\forall} A) < \mathtt{w}(x : \Box A)$.

We then continue with two more cuts as follows:

$$\cfrac{\cfrac{\cfrac{b \in I(x), \Gamma \Rightarrow \Delta, x : \Box A, A \lhd b \quad x : \Box A, \Gamma' \Rightarrow \Delta'}{b \in I(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta', A \lhd b}\,\mathit{Cut} \qquad (b \in I(x))^2, A \lhd b, \Gamma, (\Gamma')^2 \Rightarrow \Delta, (\Delta')^2}{(b \in I(x))^3, (\Gamma)^2, (\Gamma')^3 \Rightarrow (\Delta)^2, (\Delta')^3}\,\mathit{Cut}}{b \in I(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\mathit{Ctr}^*$$

where the upper cut is on the original cut formula, but of reduced height, and the lower one of reduced weight because $\mathtt{w}(A \lhd b) < \mathtt{w}(x : \Box A)$.

If instead the monotonic rules $R\Box'$, $L\Box'$ have been used, the conversion is simpler: We have a cut of the form

$$
\cfrac{\cfrac{b \in I(x), \Gamma \Rightarrow \Delta, x : \Box A, b \Vdash^{\forall} A}{b \in I(x), \Gamma \Rightarrow \Delta, x : \Box A}\, R\Box' \qquad \cfrac{\overset{\mathcal{D}}{a \in I(x), a \Vdash^{\forall} A, \Gamma' \Rightarrow \Delta'}}{x : \Box A, \Gamma' \Rightarrow \Delta'}\, L\Box'}{b \in I(x), \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, Cut
$$

This is converted into a derivation with two cuts, the upper one of reduced height and the lower one or reduced weight, followed by contractions, so that the inductive hypothesis applies. The details are easy and left to the reader.

For extensions of the basic system, we need to consider also the cases of cut with cut formula of the form $a \subseteq b$ or $x \in a \cap b$ principal in both premisses of cut. In the first case, we have a derivation of the form

$$
\cfrac{\cfrac{x \in a, \Gamma \Rightarrow \Delta, x \in b}{\Gamma \Rightarrow \Delta, a \subseteq b}\, R\subseteq \qquad \cfrac{y \in a, y \in b, a \subseteq b, \Gamma' \Rightarrow \Delta'}{y \in a, a \subseteq b, \Gamma' \Rightarrow \Delta'}\, L\subseteq}{y \in a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, Cut
$$

This is converted into a derivation with two cuts, the first of reduced height, the second of reduced weight, as follows:

$$
\cfrac{\cfrac{y \in a, \Gamma \Rightarrow \Delta, y \in b \qquad \cfrac{\Gamma \Rightarrow \Delta, a \subseteq b \quad y \in a, y \in b, a \subseteq b, \Gamma' \Rightarrow \Delta'}{y \in a, y \in b, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, Cut_1}{y \in a^2, \Gamma^2, \Gamma' \Rightarrow \Delta^2, \Delta'}\, Cut_2}{y \in a, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, Ctr^*
$$

Here the left premiss of the second cut is obtained by a hp-substitution on the premiss of $R\subseteq$.

In the second case, we have a derivation of the form

$$
\cfrac{\cfrac{\Gamma \Rightarrow \Delta, x \in a \cap b, x \in a \quad \Gamma \Rightarrow \Delta, x \in a \cap b, x \in b}{\Gamma \Rightarrow \Delta, x \in a \cap b}\, R\cap \qquad \cfrac{x \in a, x \in b, x \in a \cap b, \Gamma' \Rightarrow \Delta'}{x \in a \cap b, \Gamma' \Rightarrow \Delta'}\, L\cap}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, Cut
$$

This is converted into a derivation with five cuts, Cut_1, Cut_2 and Cut_4 of reduced height, and the remaining two of reduced weight of cut furmula:

$$
\cfrac{\cfrac{\cfrac{\Gamma \Rightarrow \Delta, x \in a \cap b, x \in b \quad x \in a \cap b, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x \in b}\, Cut_4 \qquad \overset{\vdots}{x \in b, \Gamma^2, \Gamma'^2 \Rightarrow \Delta^2, \Delta'^2}}{\Gamma^3, \Gamma'^3 \Rightarrow \Delta^3, \Delta'^3}\, Cut_5}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\, Ctr^*
$$

where the dotted part is continued as follows:

$$\cfrac{\cfrac{\Gamma \Rightarrow \Delta, x \in a \cap b, x \in a \quad x \in a \cap b, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x \in a}\,Cut_1 \quad \cfrac{\Gamma \Rightarrow \Delta, x \in a \cap b \quad x \in a, x \in b, x \in a \cap b, \Gamma' \Rightarrow \Delta'}{x \in a, x \in b, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,Cut_2}{x \in b, \Gamma^2, \Gamma'^2 \Rightarrow \Delta^2, \Delta'^2}\,Cut_3$$

QED

5 Soundness and completeness

Next, we give a proof of soundness and a direct proof of completeness of our calculus with respect to neighbourhood semantics. Specifically, we show that all the rules are sound, and show that proof search in the calculus either produces a proof, or provides us with a saturated branch which is used to define a countermodel. The countermodel will be defined *directly*, that is, using the syntactic elements (labels) and the forcing conditions in the saturated branch, without any need for additional constructions.

Soundness

We recall a definition from Chellas ([4], p. 215):

Definition 5.1. *Let $\mathcal{F} \equiv (W, I)$ be a neighbourhood frame.*

- *$\mathcal{F}$ is* **supplemented** *if for all subsets α, β of W and for all $x \in W$, if $\alpha \in I(x)$ and $\alpha \subseteq \beta$, we have $\beta \in I(x)$.*
- *$\mathcal{F}$ is* **closed under intersection** *if for all $x \in W$ for all α, β in $I(x)$, we have $\alpha \cap \beta \in I(x)$.*
- *$\mathcal{F}$ is* **contains the unit** *if for all $x \in W$, W is in $I(x)$.*

Definition 5.2. *Given a set S of world labels x and a set NL of neighbourhood labels a, and a neighbourhood model $\mathcal{M} = (W, I, \mathcal{V})$, an SN-realisation (ρ, σ) is a pair of functions mapping each $x \in S$ into $\rho(x) \in W$ and mapping each $a \in NL$ into $\sigma(a) \in I(w)$ for some $w \in W$. As SN-realisation (ρ, σ) has to respect formal intersection of the language, i.e., $\sigma(a \cap b) = \sigma(a) \cap \sigma(b)$[22]. We introduce the notion "$\mathcal{M}$ satisfies a formula F under an SN-realisation (ρ, σ)" and denote it by $\mathcal{M} \models_{\rho,\sigma}$*

[22] Observe that the symbol on the left denotes formal intersection, the one on the right set-theoretic intersection.

F, where we assume that the labels in F occur in S, NL. The definition extends the usual clauses for the propositional connectives by cases on the form of F:[23]

- $\mathcal{M} \models_{\rho,\sigma} x \in a$ *if* $\rho(x) \in \sigma(a)$
- $\mathcal{M} \models_{\rho,\sigma} a \in I(x)$ *if* $\sigma(a) \in I(\rho(x))$
- $\mathcal{M} \models_{\rho,\sigma} a \subseteq b$ *if* $\sigma(a) \subseteq \sigma(b)$
- $\mathcal{M} \models_{\rho,\sigma} x : A$ *if* $\rho(x) \Vdash A$
- $\mathcal{M} \models_{\rho,\sigma} a \Vdash^{\exists} A$ *if there exists* w *in* $\sigma(a)$ *such that* $w \Vdash A$
- $\mathcal{M} \models_{\rho,\sigma} a \Vdash^{\forall} A$ *if for all* w *in* $\sigma(a)$, $w \Vdash A$
- $\mathcal{M} \models_{\rho,\sigma} A \lhd a$ *if* $[A] \subseteq \sigma(a)$
- $\mathcal{M} \models_{\rho,\sigma} x : \Box A$ *if for some* a, $\sigma(a) \in I(\rho(x))$ *and* $\sigma(a) = [A]$

*Given a sequent $\Gamma \Rightarrow \Delta$, let S, NL be the sets of world and neighbourhood labels occurring in $\Gamma \cup \Delta$, and let (ρ, σ) be an SN-realisation; we define $\mathcal{M} \models_{\rho,\sigma} \Gamma \Rightarrow \Delta$ to hold if whenever $\mathcal{M} \models_{\rho,\sigma} F$ for all formulas $F \in \Gamma$ then $\mathcal{M} \models_{\rho,\sigma} G$ for some formula $G \in \Delta$. We further define $\mathcal{M}$-*validity *by*

$$\mathcal{M} \models \Gamma \Rightarrow \Delta \textit{ iff } \mathcal{M} \models_{\rho,\sigma} \Gamma \Rightarrow \Delta \textit{ for every } SN\textit{-realisation } (\rho, \sigma)$$

We finally say that a sequent $\Gamma \Rightarrow \Delta$ is valid *in a neighbourhood frame $\mathcal{F}$ if $\mathcal{M} \models \Gamma \Rightarrow \Delta$ for every neighbourhood model $\mathcal{M}$ based on $\mathcal{F}$.*

Below, we shall use the notation $\mathcal{M} \models_{\rho,\sigma} \Gamma$ for $\mathcal{M} \models_{\rho,\sigma} F$ for all $F \in \Gamma$. We shall denote with **G3nM***, **G3nC***, **G3nN*** the extensions of **G3n** which are monotonic, contain rule C, and rule N, respectively. Since extensions are obtained in a modular way, further extensions with rules that correspond to the frame properties * are indicated by the asterisk.

Theorem 5.3. *If $\Gamma \Rightarrow \Delta$ is derivable in* **G3n*** *(respectively* **G3nM***, **G3nC***, **G3nN****), then it is valid in the class of neighbourhood frames (respectively neighbourhood frames which are supplemented, closed under intersection, containing the unit) with the * properties.*

[23] Observe that hereafter we use the more compact notation $[A]$, in place of $ext(A)$, for the extension of A.

Proof. By induction on the height n of the derivation of $\Gamma \Rightarrow \Delta$ in $\mathbf{G3nE}^*$ (resp. $\mathbf{G3nM}^*$, $\mathbf{G3nC}^*$, $\mathbf{G3nN}^*$).

For $n = 0$, observe that initial sequents have the same labelled formula in the antecedent and in the succedent so the claim is obvious. Similarly if the antecedent contains $x : \bot$ because we assume that for no $w \in W$, $w \Vdash \bot$.

For the inductive step, consider the last rule in the derivation of $\Gamma \Rightarrow \Delta$. If it is a propositional rule, the claim is immediate by the definition of the forcing clauses for the propositional connectives.

If the last rule is $R \Vdash^{\forall}$, assume by induction hypothesis that $\mathcal{M} \models x \in a, \Gamma \Rightarrow \Delta, x : A$. Let (ρ, σ) be an arbitrary SN-realisation for the conclusion and assume that $\mathcal{M} \models_{\rho,\sigma} \Gamma$. Since x is fresh, it can be extended to ρ', an S-realization for the premiss with $\rho'(x) \in \sigma(a)$. Then (using the assumption that $x \notin \Gamma$) we have $\mathcal{M} \models_{\rho',\sigma} x \in a, \Gamma$. By the hypothesis $\mathcal{M} \models x \in a, \Gamma \Rightarrow \Delta, x : A$, we have that either (1) $\mathcal{M} \models_{\rho',\sigma} G$ for some G in Δ or (2) $\mathcal{M} \models_{\rho',\sigma} x : A$. In the former case we are done, so let us assume that $\mathcal{M} \models_{\rho',\sigma} G$ for no G in Δ. Since $x \notin \Delta$, this will be the case uniformly, independently of the choice of $\rho'(x)$, so we'll have $\mathcal{M} \models_{\rho',\sigma} x : A$ for all $\rho'(x) \in \sigma(a)$, and therefore $\mathcal{M} \models_{\rho,\sigma} a \Vdash^{\forall} A$.

If the last rule is $L \Vdash^{\forall}$, the claim holds because if $\mathcal{M} \models_{\rho,\sigma} x \in a$ and $\mathcal{M} \models_{\rho,\sigma} a \Vdash^{\forall} A$, then $\mathcal{M} \models_{\rho,\sigma} x : A$ by simply unfolding the definitions.

If the last rule is $R \Vdash^{\exists}$, consider an arbitrary SN-realisation (ρ, σ) and assume that (1) $\mathcal{M} \models_{\rho,\sigma} x \in a, \Gamma$. Then, by induction hypothesis, either (2) $\mathcal{M} \models_{\rho,\sigma} G$ for some $G \in \Delta$, or (3) $\mathcal{M} \models_{\rho,\sigma} x : A$, or (4) $\mathcal{M} \models_{\rho,\sigma} a \Vdash^{\exists} A$. If (2) or (4) hold, then the claim follows. If (3) holds, we have $\rho(x) \Vdash A$. Observe that (1) gives in particular $\rho(x) \in \sigma(a)$, so there is $w \in \sigma(a)$ such that $w \Vdash A$. It follows that the conclusion of the rule is $\mathcal{M}$-valid for the SN-realization (ρ, σ).

If the last rule is $L \Vdash^{\exists}$, assume that $\mathcal{M} \models_{\rho,\sigma} a \Vdash^{\exists} A, \Gamma$ for an arbitrary SN-realisation for the conclusion (ρ, σ). Then there is $w \in \sigma(a)$ such that $w \Vdash A$. Since x is fresh, we can extend ρ to and S-realization for the premiss by choosing $\rho'(x) = w$. Then we have $\mathcal{M} \models_{\rho',\sigma} x \in a, x : A$ by definition, and $\mathcal{M} \models_{\rho',\sigma} \Gamma$ because $x \notin \Gamma$. By induction hypothesis, the premiss of the rule is $\mathcal{M}$-valid, and therefore there is G in Δ such that $\mathcal{M} \models_{\rho',\sigma} G$. Since $x \notin \Delta$, this is the same as $\mathcal{M} \models_{\rho,\sigma} G$.

If the last rule is $R \lhd$, with premiss $y : A, \Gamma \Rightarrow \Delta, y \in a$, let (ρ, σ) be an arbitrary SN-realisation for the conclusion and assume that $\mathcal{M} \models_{\rho,\sigma} \Gamma$. The claim is that for some formula B in Δ, $\mathcal{M} \models_{\rho,\sigma} B$ or $\mathcal{M} \models_{\rho,\sigma} A \lhd a$. Since y is fresh, we can extend ρ to a S-realization for the premiss ρ' by choosing $\rho'(y) \in [A]$. Since $\mathcal{M} \models y : A, \Gamma \Rightarrow \Delta, y \in a$, we have that there exists $B \in \Delta$ such that $\mathcal{M} \models_{\rho',\sigma} B$ or $\mathcal{M} \models_{\rho',\sigma} y \in a$. In the first case, since y does not occur in B, we have also $\mathcal{M} \models_{\rho,\sigma} B$. In the second case, since $\rho'(y)$ was arbitrary in $[A]$, we have $\mathcal{M} \models_{\rho,\sigma} A \lhd a$.

If the last rule is $L \lhd$, assume that the premiss $y \in a, A \lhd a, y : A, \Gamma \Rightarrow \Delta$ is valid, and let (ρ, σ) be an arbitrary SN-realisation with $\mathcal{M} \models_{\rho,\sigma} A \lhd a, y : A, \Gamma$. Then we have $\rho(y) \in [A]$ and $[A] \subseteq \sigma(a)$, so that $\rho(y) \in \sigma(a)$, thus $\mathcal{M} \models_{\rho,\sigma} y \in a, A \lhd a, y : A, \Gamma$. By the assumption, there is B in Δ such that $\mathcal{M} \models_{\rho,\sigma} B$ and thus the claim follows.

If the last rule is $L\Box$, assume the premiss valid and let (ρ, σ) be an arbitrary SN-realisation with $\mathcal{M} \models_{\rho,\sigma} x : \Box A, \Gamma$. This means in particular that $\rho(x) \in [A]$, i.e., there is α in $I(\rho(x))$ with $\alpha = [A]$. Since a is fresh, we can extend σ to σ' by having $\sigma'(a) = \alpha$. We have $\mathcal{M} \models_{\rho,\sigma'} a \in I(x), A \lhd a, a \Vdash^{\forall} A$ by the definitions and also $\mathcal{M} \models_{\rho,\sigma'} \Gamma$ because $a \notin \Gamma$ and by hypothesis $\mathcal{M} \models_{\rho,\sigma} \Gamma$. Again by hypothesis, there is B in Δ with $\mathcal{M} \models_{\rho,\sigma'} B$ and thus by freshness of a (not in B) we have $\mathcal{M} \models_{\rho,\sigma} B$.

If the last rule is $R\Box$, assume the premisses valid and assume for an arbitrary SN-realisation (ρ, σ) that $\mathcal{M} \models_{\rho,\sigma} a \in I(x), \Gamma$. From the validity of the premisses we have that one of the following alternatives holds: 1: $\mathcal{M} \models_{\rho,\sigma} B$ for some B in Δ. 2. $\mathcal{M} \models_{\rho,\sigma} x : A$. 3. $\mathcal{M} \models_{\rho,\sigma} a \Vdash^{\forall} A, A \lhd a$. Observe that the latter gives, together with $\mathcal{M} \models_{\rho,\sigma} a \in I(x)$ that $\mathcal{M} \models_{\rho,\sigma} x : \Box A$ so in each of the three cases we have proved the claim.

Next, we consider the rules for inclusion. If the last rule is $R \subseteq$, consider an SN-realisation such that $\mathcal{M} \models_{\rho,\sigma} \Gamma$. Since x is fresh, we can extend ρ to ρ' by choosing $\rho'(x) \in \sigma(a)$. Since the premiss in $\mathcal{M}$-valid, by inductive hypothesis we have that $\mathcal{M} \models_{\rho',\sigma} G$ for some $G \in \Delta$ or $\mathcal{M} \models_{\rho',\sigma} x \in b$. Since x is not in Δ, the former gives $\mathcal{M} \models_{\rho,\sigma} G$ for some $G \in \Delta$, whereas the latter gives, by the choice in the range of $\rho'(x)$, $\mathcal{M} \models_{\rho,\sigma} a \subseteq b$.

The case with $L \subseteq$ as the last rule is immediate.

The preservation of validity in the case of rules $[\,]$, $\langle\,]$, $[\,\rangle$, and $\langle\,\rangle$ follows the same pattern of that for the $\Box$ rules. To conclude, it is immediate that rules M, C, N (and the monotonic variants C', *Nondeg*) are valid in frames frames which are supplemented, closed under intersection, containing the unit (and supplemented for the latter two with the monotonic variants) respectively. QED

Definition 5.4. *We say that a branch in a proof search from the endsequent up to a sequent $\Gamma \Rightarrow \Delta$ is* saturated *with respect to a rule R if condition (R) below holds, where we indicate with $\downarrow \Gamma$ ($\downarrow \Delta$) the union of the antecedents (succedents) in the branch from the end-sequent up to $\Gamma \Rightarrow \Delta$:*

($Init_0$) There is no $x \in a$ in $\Gamma \bigcap \Delta$.

(Init) There is no $x : P$ in $\Gamma \bigcap \Delta$.

($L\bot$) There is no $x : \bot$ in Γ.

(L&) If $x : A \& B$ *is in* $\downarrow\Gamma$*, then* $x : A$ *and* $x : B$ *are in* $\downarrow\Gamma$*.*

(R&) If $x : A \& B$ *is in* $\downarrow\Delta$*, then either* $x : A$ *or* $x : B$ *is in* $\downarrow\Delta$*.*

(L$\vee$) If $x : A \vee B$ *is in* $\downarrow\Gamma$*, then either* $x : A$ *or* $x : B$ *is in* $\downarrow\Gamma$*.*

(R$\vee$) If $x : A \vee B$ *is in* $\downarrow\Delta$*, then* $x : A$ *and* $x : B$ *are in* $\downarrow\Delta$*.*

(L$\supset$) If $x : A \supset B$ *is in* $\downarrow\Gamma$*, then either* $x : A$ *is in* $\downarrow\Delta$ *or* $x : B$ *is in* $\downarrow\Gamma$*.*

(R$\supset$) If $x : A \supset B$ *is in* $\downarrow\Delta$*, then* $x : A$ *is in* $\downarrow\Gamma$ *and* $x : B$ *is in* $\downarrow\Delta$

(R $\Vdash^{\forall}$) If $a \Vdash^{\forall} A$ *is in* $\downarrow\Delta$*, then for some* x *there is* $x \in a$ *in* Γ *and* $x : A$ *in* $\downarrow\Delta$

(L $\Vdash^{\forall}$) If $x \in a$ *and* $a \Vdash^{\forall} A$ *and are in* Γ*, then* $x : A$ *is in* $\downarrow\Gamma$*.*

(R $\Vdash^{\exists}$) If $x \in a$ *is in* Γ *and* $a \Vdash^{\exists} A$ *is in* Δ*, then* $x : A$ *is in* $\downarrow\Delta$*.*

(L $\Vdash^{\exists}$) If $a \Vdash^{\exists} A$ *is in* $\downarrow\Gamma$*, then for some* x *there is* $x \in a$ *in* Γ *and* $x : A$ *is in* $\downarrow\Gamma$

(L $\lhd$) If $A \lhd a$ *and* $y : A$ *are in* $\downarrow\Gamma$*, then* $y \in a$ *is in* Γ*.*

(R $\lhd$) If $A \lhd a$ *is in* $\downarrow\Delta$*, then for some* y*,* $y : A$ *is in* $\downarrow\Gamma$ *and* $y \in a$ *is in* Δ*.*

(L$\Box$) If $x : \Box A$ *is in* $\downarrow\Gamma$*, then for some* a*,* $a \in I(x)$*,* $a \Vdash^{\forall} A$*,* $A \lhd a$ *are in* $\downarrow\Gamma$*.*

(L$\Box'$) If $x : \Box A$ *is in* $\downarrow\Gamma$*, then for some* a*,* $a \in I(x)$*,* $a \Vdash^{\forall} A$ *are in* $\downarrow\Gamma$*.*

(R$\Box$) If $a \in I(x)$ *is in* Γ *and* $x : \Box A$ *is in* $\downarrow\Delta$*, then either* $a \Vdash^{\forall} A$ *or* $A \lhd a$ *is in* $\downarrow\Delta$*.*

(R$\Box'$) If $a \in I(x)$ *is in* Γ *and* $x : \Box A$ *is in* $\downarrow\Delta$*, then* $a \Vdash^{\forall} A$ *is in* $\downarrow\Delta$*.*

(L $\subseteq$) If $x \in a$ *and* $a \subseteq b$ *are in* $\downarrow\Gamma$*, then* $x \in b$ *is in* Γ*.*

(R $\subseteq$) If $a \subseteq b$ *is in* $\downarrow\Delta$*, then for some* x *there is* $x \in a$ *in* Γ *and* $x \in b$ *in* Δ*.*

(L$\cap$) If $x \in a \cap b$ *is in* Γ*, then* $x \in a$ *and* $x \in b$ *are in* Γ*.*

(R$\cap$) If $x \in a \cap b$ *is in* Δ*, then either* $x \in a$ *or* $x \in b$ *are in* Δ*.*

(M) If $a \in I(x), a \subseteq b$ *are in* Γ*, then* $b \in I(x)$ *is in* Γ*.*

(C) If $a \in I(x), b \in I(x)$ *are in* Γ*, then* $a \cap b$ *is in* Γ*.*

(C$'$) If $a \in I(x), b \in I(x)$ *are in* Γ*, then for some* c*,* $c \in I(x)$*,* $c \subseteq a$*,* $c \subseteq b$ *are in* Γ*.*

(N) For some a, $a \in I(x), \top \lhd a$ are in Γ.

(Nondeg) For some a, $a \in I(x)$ is in Γ.

A branch is saturated relative to a systems $\mathcal{S}$ of rules if it is saturated with respect each rule of $\mathcal{S}$.

The definition of saturation with respect to the rules for the modalities $[\,]$, $\langle\,]$, $[\,\rangle$, and $\langle\,\rangle$ has been left out as it involves eight more clauses and it should be by now clear from the meaning of saturation with respect a rule and the pattern of the other cases. The definition of saturated branch is extended to infinite branches $\mathcal{B} \equiv \{\Gamma_i \Rightarrow \Delta_i\}_{i\geq 0}$ by replacing, in the definition above, Γ (or $\downarrow\Gamma$) by $\mathbf{\Gamma}$, the union of the Γ_i, and Δ (or $\downarrow\Delta$) by $\mathbf{\Delta}$, the union of the Δ_i. The first and second clause (*Init*$_0$, *Init*) are modified to requiring that for all i, there is no $x \in a$ in $\Gamma_i \cap \Delta_i$ and for all i, there is no $x : P$ in $\Gamma_i \cap \Delta_i$.

Given a sequent $\Gamma \Rightarrow \Delta$ we apply root-first all the available rules. Observe that by invertibility of the rules, there is no prescribed order in which they need to be applied. We want to avoid the possibility that the search produces an infinite branch which is not saturated, something that would result, e.g., from applying the same rule infinitely many times in consecutive steps. This is achieved as usual in such proofs through a counter: if there are m rules, apply at step 1 rule R_1 to all formulas that match its conclusion, at step 2 rule R_2, and in general for all $n \geq 0$ apply at step $n \times m + j$ rule R_j. In this way we'll obtain a *proof-search tree* that can be a derivation, or a non-derivation; the latter can either be a finite search tree that contains finite saturated branches, or an infinite search that, by König's lemma contains an infinite, saturated branch. We shall now prove that a saturated branch (either finite or infinite) for a sequent $\Gamma \Rightarrow \Delta$ gives a countermodel.

Lemma 5.5. *Let $\mathcal{B} \equiv \{\Gamma_i \Rightarrow \Delta_i\}$ be a saturated branch in a proof-search tree for $\Gamma \Rightarrow \Delta$. Then there exists a countermodel $\mathcal{M}$ to $\Gamma \Rightarrow \Delta$, which makes all the formulas in $\mathbf{\Gamma}$ true, and all the formulas in $\mathbf{\Delta}$ false.*

Proof. Consider a saturated branch and define the countermodel $\mathcal{M} \equiv (W, I, \mathcal{V})$ as follows:

1. The set W of worlds consists of all the world labels in $\mathbf{\Gamma}$;

2. For each neighbourhood label a in $\mathbf{\Gamma}$, we associate α_a, the set that consists of all the y in W such that $y \in a$ is in $\mathbf{\Gamma}$;

3. For each x in W, the set of neighbourhoods of x consists of all the α_a such that $a \in I(x)$ is $\mathbf{\Gamma}$;

4. The valuation is defined by $x \in \mathcal{V}(P)$ if $x : P$ is in $\mathbf{\Gamma}$.

We then define a realization (ρ, σ) by $\rho(x) \equiv x$ and $\sigma(a) \equiv \alpha_a$. Next we prove the following:

1. If A is in $\mathbf{\Gamma}$, then $\mathcal{M} \models_{\rho,\sigma} A$.

2. If A is in $\mathbf{\Delta}$, then $\mathcal{M} \not\models_{\rho,\sigma} A$.

The two claims are proved simultaneously by cases/induction on the weight of A (cf. Definition 4.1).

(a) If A is a formula of the form $a \in I(x)$, $x \in a$, $a \subseteq b$, claim 1. holds by definition of $\mathcal{M}$; if A is $x \in a \cap b$, by saturation we have that $x \in a$ and $x \in b$ are in $\mathbf{\Gamma}$. These are lighter formulas, so the inductive hypothesis applies and we have $\rho(x) \in \sigma(a)$ and $\rho(x) \in \sigma(b)$, so $\rho(x) \in \sigma(a) \cap \sigma(b)$. The conclusion $\rho(x) \in \sigma(a \cap b)$ follows from the fact that σ respects intersection. Claim 2. is empty for $a \in I(x)$ because such formulas never occur on the right-hand side of sequents. If $x \in a$ is in $\mathbf{\Delta}$, then $x \in a$ is not in $\mathbf{\Gamma}$ and thus $\rho(x) \notin \alpha_a$, so $\mathcal{M} \not\models_{\rho,\sigma} x \in a$. If $a \subseteq b$ is in $\mathbf{\Delta}$, then for some x, $x \in a$ is in $\mathbf{\Gamma}$ and $x \in b$ is in $\mathbf{\Delta}$, so by inductive hypothesis $\mathcal{M} \models_{\rho,\sigma} x \in a$ and $\mathcal{M} \not\models_{\rho,\sigma} x \in b$, and therefore $\mathcal{M} \not\models_{\rho,\sigma} a \subseteq b$.

(b) If A is a labelled atomic formula $x : P$, the claims hold by definition of $\mathcal{V}$ and by the saturation clause *Init* no inconsistency arises. If A is $\perp$, it holds by definition of the forcing relation that it is never forced, and therefore 2. holds, whereas 1. holds by the saturation clause for $L\perp$. If A is a conjunction, or a disjunction, or an implication, the claim holds by the corresponding saturation clauses and inductive hypothesis on smaller formulas.

(c) If $a \Vdash^{\exists} A$ is in $\mathbf{\Gamma}$, by the saturation clause $(L \Vdash^{\exists})$, for some x there is $x \in a$ in $\mathbf{\Gamma}$ and $x : A$ is in $\mathbf{\Gamma}$. Then $\mathcal{M} \models_{\rho,\sigma} x \in a$ by (a) and by induction hypothesis $\mathcal{M} \models_{\rho,\sigma} x : A$, therefore $\mathcal{M} \models_{\rho,\sigma} a \Vdash^{\exists} A$. If $a \Vdash^{\exists} A$ is in $\mathbf{\Delta}$, consider an arbitrary world x in α_a. Then by definition of $\mathcal{M}$ we have $x \in a$ in $\mathbf{\Gamma}$ and thus by the saturation clause $(R \Vdash^{\exists})$ we also have $x : A$ is in $\mathbf{\Delta}$. By induction hypothesis we have $\mathcal{M} \not\models_{\rho,\sigma} x : A$ and therefore $\mathcal{M} \not\models_{\rho,\sigma} a \Vdash^{\exists} A$. The proof for formulas of the form $a \Vdash^{\forall} A$ is similar.

(d) If $A \lhd a$ is in $\mathbf{\Gamma}$, let y be an arbitrary label such that $\mathcal{M} \models_{\rho,\sigma} y : A$. Then by definition of $\mathcal{M}$ we have $y : A$ in $\mathbf{\Gamma}$ and then by saturation $y \in a$ is in $\mathbf{\Gamma}$ thus by inductive hypothesis and by definition of $\mathcal{M}$ we obtain $\mathcal{M} \models_{\rho,\sigma} A \lhd a$.

If $A \lhd a$ is in $\mathbf{\Delta}$, by the corresponding saturation clause we have that for some y, $y : A$ is in $\mathbf{\Gamma}$ and $y : a$ is in Δ , so by induction hypothesis we have that there is y such that $\mathcal{M} \models_{\rho,\sigma} y : A$ and $\mathcal{M} \not\models_{\rho,\sigma} y \in a$. Overall, this means that $\mathcal{M} \not\models_{\rho,\sigma} A \lhd a$.

(e) If $x : \Box A$ is in $\mathbf{\Gamma}$, then for some a, $a \in I(x)$, $a \Vdash^{\forall} A$, $A \lhd a$ are in $\mathbf{\Gamma}$. By induction hypothesis we obtain $\mathcal{M} \models_{\rho,\sigma} a \Vdash^{\forall} A$ and $\mathcal{M} \models_{\rho,\sigma} A \lhd a$, and therefore $\mathcal{M} \models_{\rho,\sigma} x : \Box A$.

If $x : \Box A$ is in $\mathbf{\Delta}$, let α_a be a neighbourhood in $I(x)$ in the model. By the saturation clause, we have that either $a \Vdash^{\forall} A$ or $A \lhd a$ is in $\mathbf{\Delta}$. By induction hypothesis we obtain $\mathcal{M} \not\models_{\rho,\sigma} a \Vdash^{\forall} A$ or $\mathcal{M} \not\models_{\rho,\sigma} A \lhd a$, and therefore $\mathcal{M} \not\models_{\rho,\sigma} x : \Box A$.

In order to prove completeness for extensions of $\mathbf{E}$ we need to prove that the countermodel $\mathcal{M}$ is in the intended class. For M (1), we shall consider the version of the $\Box$ rules with monotonicity built-in and modify the model to impose monotonicity; for C (2) and N (3) instead we shall extend in a consistent way the saturated branch.

(1) Let $\mathcal{M}$ be defined as $\mathcal{M}$ above, but taking for $I(x)$ *supersets* of the α_a. In this way $\mathcal{M}$ is supplemented. We need to verify that if $x : \Box A$ is in $\mathbf{\Gamma}$ then $\mathcal{M} \models_{\rho,\sigma} \Box A$: by the saturation clause for $L\Box$ we have that for some a such that $a \in I(x)$ is in $\mathbf{\Gamma}$, $a \Vdash^{\forall} A$ is in $\mathbf{\Gamma}$. By inductive hypothesis, $\mathcal{M} \models_{\rho,\sigma} a \Vdash^{\forall} A$ and therefore, since $\mathcal{M}$ is supplemented, $\mathcal{M} \models_{\rho,\sigma} \Box A$. If $x : \Box A$ is in $\mathbf{\Delta}$, let α_a be a neighbourhood of x in the model. This means that $a \in I(x)$ is in $\mathbf{\Gamma}$. By the $R\Box$ saturation clause, $a \Vdash^{\forall} A$ is in $\mathbf{\Delta}$, so by inductive hypothesis $\mathcal{M} \not\models_{\rho,\sigma} a \Vdash^{\forall} A$, and therefore it is not the case that for all w in α_a, $w \Vdash A$. Since α_a was an arbitrary neighbourhood of x, we have $\mathcal{M} \not\models_{\rho,\sigma} \Box A$.

(2) The saturated branch is extended as follows: whenever $\mathbf{\Gamma}$ contains $x \in a$ and $x \in b$, we add $x \in a \cap b$ to $\mathbf{\Gamma}$ (observe that this move doesn't collapse the saturated branch into an initial sequent since if $x \in a \cap b$ was in $\mathbf{\Delta}$, then by saturation either $x \in a$ or $x \in b$ would be in $\mathbf{\Delta}$, against the assumption that we started with a saturated branch). We call the branch thus obtained a C-extended saturated branch. Next we prove that the model $\mathcal{M}$ built on the C-extended saturated branch is closed under intersection. Let α_a and α_b be in $I(x)$. This means that $a \in I(x)$ and $b \in I(x)$ are in $\mathbf{\Gamma}$. We show that $\alpha_a \cap \alpha_b = \alpha_{a \cap b}$ and therefore conclude that $\alpha_a \cap \alpha_b$ is also in $I(x)$. Clearly, if $y \in \alpha_{a\cap b}$, i.e., $y \in a \cap b$ in $\mathbf{\Gamma}$, then by saturation $y \in a$ and $y \in b$ are in $\mathbf{\Gamma}$, therefore $y \in \alpha_a \cap \alpha_b$. The converse inclusion is guaranteed by the C-extension of the saturated branch. Observe that the equality just proved also shows that the added formulas $x \in a \cap b$ are true in the model.

(3) The saturated branch is extended as follows: for every label y in the branch, we add the formula $y : \top$ to $\mathbf{\Gamma}$. The branch thus obtained is an N-extended saturated branch. By the saturation condition for N we have that for some $a \in I(x)$, $\top \lhd a$ is in $\mathbf{\Gamma}$. By the N-extension and the saturation with respect to $L\lhd$, we have that α_a coincides with W, and therefore W is a neighbourhood of x, so the model contains the unit. QED

We are ready to prove the completeness of the calculus.

Theorem 5.6. *If A is valid then there is a derivation of $\Rightarrow x : A$, for any label x.*

Proof. For every A we either find a derivation or a saturated branch. By the above lemma a saturated branch gives a countermodel to A. It follows that if A is valid it has to be derivable. QED

The above completeness proof gives a method to construct countermodels for unprovable sequents. It is also possible to give a simple completeness proof as a direct consequence of the structural properties of the calculus and the derivability of the characteristic axiom of each of the non-normal systems considered:

Theorem 5.7. *Let A be a formula in the language of the modal propositional logic, and let* $\mathbf{E}^*$ *be any extension of* $\mathbf{E}$ *with axioms M, C, N (and combinations thereof) and* $\mathbf{G3n}^*$ *the corresponding labelled sequent calculus. Then if* $\mathbf{E}^* \vdash A$*, we have* $\mathbf{G3n}^* \vdash \Rightarrow x : A$ *where x is an arbitrary world label.*

Proof. By induction on the derivation in the axiomatic system. Observe that the result holds for classical propositional axioms and has been proved for each specific modal axiom in Section 3, so it is enough to prove the inductive step for the only rule of the axiomatic system, i.e., that if $\mathbf{E}^* \vdash A$ is obtained by *modus ponens*, then $\mathbf{G3n}^* \vdash \Rightarrow x : A$. Consider derivable premisses B and $B \supset A$. By inductive hypothesis we have $\mathbf{G3n}^* \vdash \Rightarrow x : B$ and $\mathbf{G3n}^* \vdash \Rightarrow x : B \supset A$. The latter gives by (hp-) invertibility of $R\supset$, $\mathbf{G3n}^* \vdash x : B \Rightarrow x : A$. An admissible step of cut gives the desired conclusion. QED

Computational issues about the calculi are not in the scope of the present paper, and we shall deal with termination and complexity of our calculi in further work. However, following the line of our [30] and [13], we can outline the recipe to obtain a terminating proof search in the calculi here presented. First of all, it is useful to make the distinction between *static* and *dynamic* rules. The former do not introduce new labels in moving from conclusions to premisses, whereas the latter do.[24] The main difficulty in obtaining termination is that a proof branch may potentially introduce infinitely many world and neighbourhood labels by unconstrained application of the dynamic rules. The termination of proof search requires to adopt a suitable strategy of rule application which on the one hand preserves the completeness and on the other ensures that in any proof branch only a finite number of labels will be introduced. The strategy will be specific to each calculus, but it contains at least the following constraints:

[24] For example, in **G3n** the rule $L\Box$ is dynamic and $R\Box$ static.

1. Do not apply a rule R to a sequent $\Gamma \Rightarrow \Delta$ if $\downarrow \Gamma$ and/or $\downarrow \Delta$ satisfy the saturation condition associated to R.

2. Apply static rules before dynamic rules.

The strategy may specify further constraints on the order of applications of rules (e.g. rule R_1 must always be applied before rule R_2) or on the temporal order in which the labels must be treated (e.g. apply all rules to a label x before applying any rule to y if x is "older" than y, that is, introduced earlier in the branch).

There is also an additional difficulty for systems where intersection of neighbourhood labels is allowed, as neighbourhood labels become complex terms so that infinitely many terms can be generated from a finite number of labels. To handle this case we shall need to identify term labels which are equivalent modulo commutativity and associativity of intersection.

We shall carry on a detailed analysis of all computation issues along the above lines in further work.

6 Concluding remarks

We have presented a systematic development of labelled sequent calculi for logical systems based on neighbourhood semantics, with focus on classical modal systems. Other approaches to the proof theory of classical modal logics besides the ones mentioned in the introduction include the nested sequent calculi of [21].[25] Additionally, in [16] standard sequent systems (most of them cut free) are provided for extensions of the monotonic system M by all combinations of the modal axioms D, T, 4, B, and 5. Similar results are obtained for *congruent modal logics* (another name for extensions of E) in [17]. Standard sequents are also considered in [19] via an approach based on a treatment of "sequents as sets" that makes contraction implicit, rather than admissible as in the G3-calculi. When such approach to sequent calculus proof theory is followed, all the rules become context-dependent and the proof of cut elimination presents some difficulties that one does not have with the usual approach to sequents as lists or multisets (cf. [33]). Standard, contraction-free sequent calculi for non-normal systems of deontic logic are presented in [37]. A different approach to the proof-theoretic study of non-normal modal logics, with focus on conditional logics, is pursued in [39]: here a criterion is developed for guaranteeing absorption of the structural rules into a system of sequent rules. The conditions the system has to satisfy are closure conditions and typically generate a large number of rules.

[25] We remark that nested sequent calculi have been developed also in other venues in non-classical logics (e.g. for modal logic [3] and bi-intuitionistic logic [14]); complexity has been studied in [2].

Labelled calculi for monotonic and regular modal logics have already been considered in [15]. As in our work the labelling originates from neighbourhood semantics, but there are important differences: first, the proof system is a tableau with signed formulas, rather than a sequent style proof system. Second, the calculus has labels with a path structures and no relations, whereas in our approach we have two sorts of labels and the explicit relation of formal membership. Correspondingly, in one systems there are rules that operate on the structured labels through an unification algorithm, whereas in our system there are rules for the neighbourhood semantics counterpart of the accessibility relation of Kripke semantics.

Labelled systems, on the whole, have several advantages over other formalisms for modal logic. First, the systems originate from a uniform methodology which has reached a wide range of applications; the transparent semantic motivation behind the rules makes them intuitive and allows a direct completeness proof. As we have seen, we can use a ground basic system to find, through proof search by invertible rules, which additional rules are needed to obtain complete systems for extensions; this can be useful especially in the absence of known correspondence results.

This extension of the labelled approach inherits the flexibility and far reach of neighbourhood semantics. Here we have focused on the most basic classical systems but it is possible extend the approach to systems with further requirements on the neighbourhood frames, as those listed in section 7.4 of [4]. A property such as $a \in I(x) \to x \in a$ is straightforward to handle and corresponds to an added rule of the form $\frac{x \in a, a \in I(x), \Gamma \Rightarrow \Delta}{a \in I(x), \Gamma \Rightarrow \Delta}$. Other properties, such as $a \in I(x) \to a^c \notin I(x)$ can also be treated by the method of conversion into rules, but one also needs rules for the formal complement of a neighbourhood.

Rather than dwelling on abstract generality, we stress that alongside with the completion of this ground work, labelled calculi based on neighbourhood semantics have been developed for other logics that cannot be studied *simpliciter* through possible world semantics, such as *preferential conditional logic* [30] and *conditional doxastic logic* [13]. Classical modal logics are decidable. The finite model property and finitary proof search can be established in parallel for labelled calculi; we expect that no special difficulties should arise in the case of classical logics, but a detailed proof, along the lines sketched at the end of Section 5, is left to further work.

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