Meaning in Use

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Abstract The historical origins of provability semantics are illustrated by so far unexplored manuscript passages of Gentzen and Gödel. Next the determination of elimination rules in natural deduction through a generalized inversion principle is treated, proposed earlier by the authors as a pedagogical device.
The notion of validity in intuitionistic logic is related to the notion of formal provability through a direct translation. Finally, it is shown how the correspondence between rules and meaning can be used for setting up complete labelled sequent calculi, first for intuitionistic logic with the remarkable property of invertibility of all the logical rules, and then for modal and related logics.

1 Meaning explanations and provability conditions

The discussions about proof theory and meaning in the past few decades date back to the early years of intuitionistic logic. The very name "BHK-explanations" reminds us of this fact. The *locus classicus*, however, was not penned down by Brouwer, Heyting, or Kolmogorov, but by Gentzen. He writes in his published thesis that the introduction rules of natural deduction are definitions of sorts of the logical connectives, and that the elimination rules are consequences of these definitions (Gentzen, 1934-1935, III.5.12). He suggests further that it should be possible to actually determine the elimination rules, "as unique functions of the *I*-rules." Unfortunately the topic is not pursued further.

It turned out in 2005, when the second author found a handwritten manuscript version of the thesis, that the passage in the printed thesis was taken directly from a longer discourse that begins with:

> The "introductions" present, so to say, the "definitions" of the signs in question, and the "eliminations" are actually just consequences thereof, expressed more or

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> less as follows: In the elimination of a sign, the proposition the outermost sign of which is in question, must "be used only as what it means on the basis of the introduction of this sign." An example will clarify what is meant hereby: The proposition $A \to B$ could be introduced when a derivation of B from the assumption A was at hand. If one now wants to use $A \to B$ further with the elimination of the sign $\to$ (uses for the construction of *longer* propositions, such as $A \to B. \vee C$ (*OI*), are naturally also possible), one can do it straightaway so that one concludes at once B from A that has been proved (*FE*). For $A \to B$ documents the existence of a derivation of B from A. Note well: It is not necessary to rely on a "contentful sense" of the sign ! $\to$.
>
> I think one could show, by making precise this idea, that the *E*-inferences are, through certain conditions, *unique* consequences of the respective *I*-inferences.

Save for a few stylistic changes and the horseshoe implication in the published version in place of the arrow, the two texts are the same to this point, but the manuscript version has the following continuation:

> I shall limit myself to the indication of a consequence of this connection, one that can be established purely formally. It will form the basis of later investigations into decidability and consistency. It goes as follows:
>
> If in an *N1I*-proof an introduction (*I*) of a sign is followed immediately by its elimination (*E*), the proposition with the sign in question (as its outermost sign) can be eliminated through a simple "reduction" of the proof.
>
> These reductions proceed after the following schemes: ($\alpha, \beta, \ldots, \varepsilon, \zeta$ denote the further lines of the proof, in a way that can be easily seen. Square brackets mean that the respective part of the proof is to be written as many times as there occurred the respective assumption before the reduction.)

$$\&\qquad \dfrac{\dfrac{\begin{matrix}\alpha & \beta\\ A & B\end{matrix}}{A\&B}\,AI}{A}\,AE \atop \varepsilon \quad \text{becomes:}\quad \begin{matrix}\alpha\\ A\\ \varepsilon\end{matrix}$$

$$\vee\qquad \dfrac{\dfrac{\begin{matrix}\alpha\\ A\end{matrix}}{A\vee B}\,OI \quad \begin{matrix}[A]\\ \gamma\\ C\end{matrix} \quad \begin{matrix}[B]\\ \delta\\ C\end{matrix}}{C}\,OE \atop \varepsilon \quad \text{into:}\quad \begin{matrix}\begin{bmatrix}\alpha\\ A\end{bmatrix}\\ \gamma\\ C\\ \varepsilon\end{matrix}$$

> (it is quite analogous with the other form of *AE* resp. *OI*.)

These "simple reductions" of a proof are nothing but steps of conversion to normal form. At the time of writing the above, Gentzen had not yet proved the normalization theorem of intuitionistic natural deduction, but just conjectured it. The passage continues with the conversion schemes for the quantifiers and implication and ends with:

> It requires some considerations to see to it that a correct proof is in fact produced in each case. I shall refrain from the exact realization, because I will not make any use of these facts, but rather present them for the sake of intuitiveness. —

The manuscript does contain an "exact realization," though, for it has a chapter added later with a detailed proof of the normalization theorem, published for the first time seventy-five years after it was written (cf. von Plato, 2008). Gentzen's idea at this time was to extend the normal form of derivations from pure logic to arithmetic, in a proof of the consistency of arithmetic; These are the "later investigations into decidability and consistency" that he mentions.

Gentzen had finalized his set of logical principles of proof, the system of natural deduction, by September 1932. His analysis of "actual proofs" in mathematics led to intuitionistic logic, a topic well-defined after Arend Heyting's article of 1930 that was based on standard axiomatic logic in the tradition of Frege, Peano, Russell, Hilbert, and Bernays. There is in Heyting's subsequent article (1931)a brief explanation of negation through "a proof procedure that leads to a contradiction." Next, it is stated that a proof of a disjunction consists in a proof of one of the disjuncts.

Heyting's explanations evolved later into the well-known proof conditions: $A \,\&\, B$ is proved whenever A and B have been proved separately, $A \vee B$ is proved whenever one of A and B has been proved, $A \supset B$ is proved whenever any proof of A turns into some proof of B. For the quantifiers, $\forall x A(x)$ is proved whenever $A(y)$ is proved for an arbitrary y, and $\exists x A(x)$ is proved whenever $A(a)$ is proved for some object a. It was realized soon that the explanation of implication need not reduce a proof of $A \supset B$ into something simpler, for A could have been obtained by any proof. The difficulty is mentioned by Gentzen in a manuscript from the fall of 1932 with no reference to Heyting, and by Gödel repeatedly in the late 1930s. Heyting's short article of 1931 suggestive of the BHK-explanations was in a volume that contained some of the proceedings of a conference held in Königsberg in September 1930, the very occasion in which Gödel! made his incompleteness result public. So we can trust that Heyting's paper had been read by those involved.

Gentzen's stenographic notes contain an item from the fall of 1932, some twenty-five dense pages, with a few pages added in the next spring and ten more in October 1934. The title is "Formal conception of the notion of contentful correctness in pure number theory, relation to proof of consistency" (*Die formale Erfassung des Begriffs der inhaltlichen Richtigkeit in der reinen Zahlentheorie, Verhältnis zum Widerspuchsfreiheitsbeweis*). Most of it was written within a month in October-November, and it was meant to be a groundwork for systematic formal studies, after the basic structure of mathematical reasoning had been cleared in September. We abbreviate the manuscript in the same way he did, as **INH**. The first task in it is to explain the notion of *correctness* for intuitionistic logic and arithmetic, quite similarly to Heyting's explanations:

| 14.X. | *Contentful correctness in intuitionistic proofs*

One defines contentful correctness like this: The mathematical axioms are correct. $A\&B$ is correct when A is correct and B is correct, $A \vee B$ when at least one of them is correct, Ax when for each number substitution for x this correct, the same with [the universally quantified] $x\,Ax$, Aa when a number can be given so that Aa holds, the same for $Ex\,Ax$, $A \rightarrow B$ when from the correctness of A that of B can be concluded, $\neg A$ when from A a contradiction can be concluded.

It is to be shown now that the result of a proof is correct.

In the case of $A \& B$ and $A \vee B$, a well-founded notion is achieved, but $A \supset B$ remained problematic. A few weeks later, Gentzen writes:

> 3.XI *The* $\rightarrow$ plays a special role in the *definition of correctness*, because correctness is always reduced with the other signs to the correctness of smaller propositions. This does not happen with $\rightarrow$. The correctness of $A \rightarrow B$ can be conceived as the existence of a proof of B from A. However, there is a circle in this conception once the proof operates in its turn with $\rightarrow$. Maybe one has to do a recursion of a theory to one closest below (of which the former is the metatheory).

As can be seen, Gentzen is requesting that if $A \supset B$ is provable, it should have a proof that is somehow made up from the components of $A \supset B$. The correctness of a notion of proof with this property would not be circular.

Doubts about the explanation of implication through hypothetical proof were raised from early on also by others. Here is a passage from Bernays (*Grundlagen der Mathematik*, vol. 1, p. 43):

> The methodological point of "intuitionism" that is at the basis of Brouwer, is formed by a certain *extension of the finitary position*, namely, an extension in so far as Brouwer allows the introduction of an assumption about the presence of a consequence, resp. of a proof, even if such a consequence, resp. proof, is not determined in respect of its visualizable constitution. For example, from Brouwer's point of view, propositions of the following forms are permitted: "If proposition B holds under assumption A, also C holds," and also: "The assumption that A is refutable leads to a contradiction," or in Brouwer's mode of expression, "the absurdity of A is absurd."

It is hard to believe that the idea of a hypothetical proof, so common today, was taken to be *the* new methodological idea of intuitionism. The passage calls for a revision of the view of the tradition of axiomatic logic, from Frege to Hilbert, to the effect that it was exclusively based on a categorical notion of truth as in the logicist thesis.

Here is another discussion of the intuitionistic meaning of implication:

> By far the most important and interesting of these notions here is $P \rightarrow Q$. Now to explain the meaning of a proposition in a constructive system means to state under which circumstances one is entitled to assert it. And the answer in this case is: If one is able to deduce Q from the assumption P. But one has to be careful: the assumption P in a constructive logic means the assumption, that a proof for P is given, since truth in itself without proof makes no sense in a constructive logic. So $P \rightarrow Q$ means: Given a proof for P one can construct a proof for Q or in other words: One has a method to continue any given proof of P to a proof of Q. It is quite essential that $\rightarrow$ is not interpreted as meaning Q is deducible from the assumption that P is true, because certain theorems of intuitionistic logic don't hold for it.

This is not Dummett or Prawitz, but Gödel himself in the lectures on intuitionism he gave in Princeton in 1941. The influence of Gentzen in the passage seems clear.

Hypothetical reasoning has its pitfalls, as indicated by Gödel. His warning in the passage goes equally well for classical logic: If from the truth of P the truth of Q follows, $P \rightarrow Q$ need not be derivable. By the completeness of propositional logic, substitute truth by derivability and you have:

If from $\vdash P$ it follows that $\vdash Q$, it need not follow that $\vdash P \to Q$. Thus, the former condition is that Q is derivable whenever P is, the latter the stronger condition that Q be derivable from P. After eighty years, the erroneous conclusion can still be found in books and articles written by otherwise competent logicians, even dubbed "failure of the deduction theorem" by those who commit the error of mixing an assumption about provability with an assumption (see Hakli and Negri, 2012, for a detailed treatment).

It was a real pity that Gentzen did not present his normalization theorem for natural deduction in the published thesis or explain it to Heyting and Gödel in correspondence. Bernays seems not to have realized that Gentzen had the result (see von Plato, 2012, p. 330)). The normalization theorem would have cut short the talk about the possible circularity of Heyting's explanation of implication, at least in first-order logic: First assertions without open assumptions are covered by the fact that their normal derivations end with an introduction rule, as in the BHK-explanations. Gentzen calls these "direct proofs" in his 1936 paper on the consistency of arithmetic (end of §10.3). Then hypothetical assertions are covered in the sense that whenever their hypotheses receive direct proofs, a direct proof of the assertion can be obtained through normalization. This explanation is found very clearly stated in Gentzen's 1936 paper. The central point of that work was to extend such a meaning explanation to cover also the rule of induction: The inductive step consists in a derivation of $A(x+1)$ from the hypothesis $A(x)$ that may be "transfinite," and the conclusion is $\forall x A(x)$. Whenever a numerical instance $A(n)$ is concluded, the hypothesis can be made disappear through a composition of the sequence $A(0), A(0) \supset A(1), A(1) \supset A(2), \ldots, A(n-1) \supset A(n)$ (ibid., §10.5).

2 Determination of the elimination rules

A minimum condition for the "unique determination" Gentzen is calling for is given by the Gentzen-Prawitz *inversion principle*:[1] The elimination rule of a connective or quantifier should bring back that which is *included in the sufficient conditions* for introducing that connective or quantifier. The detour conversion schemes, as in the above quote from Gentzen, have been seen as a formal manifestation of this idea: They justify the elimination rules in terms of the introduction rules, by showing how the immediate grounds for introducing a formula are recovered in the conversions.

The Gentzen-Prawitz inversion principle does not meet Gentzen's requirement of actually determining the elimination rules from the introduction rules, instead of only justifying them. Thus, the possibility remains that the elimination rules are in some way too weak. The principle can be gener-

[1] We do not enter into the discussion of the background of this principle beyond Prawitz, but refer to (Moriconi and Tesconi, 2008) for that.

alized, as in (Negri and von Plato, 2001, p. 6), into one in which one looks at the *arbitrary consequences of the sufficient grounds* for introducing a formula, instead of just those grounds. For conjunction, the grounds are A and B separately, and their arbitrary consequences give the following *general* elimination rule:

$$\dfrac{A\&B \quad \begin{matrix}[A,B] \\ \vdots \\ C\end{matrix}}{C}\ \&E$$

For implication, the sufficient ground for introducing $A \supset B$ is, in Gentzen's words, "the existence of a derivation of B from A" (1934-1935, II.5.23). First-order logic is not able to represent formally within its language the existence of a derivation. Therefore (Schroeder-Heister, 1984b) considered a system of higher-order rules. In 1980, with publication in his (1984), Martin-Löf formulated a scheme for elimination rules in his constructive type theory in which the existence of a derivation can be expressed. The general lesson from his discourse is that introduction rules correspond to "constructor" functions in an inductive definition, and a general elimination scheme for any such functions is a principle of recursive definition of functions over the inductively defined class.

In this light, the Gentzen-Prawitz inversion principle covers the base case of the recursive definition of functions over proofs of a compound formula, the one in which the arbitrary consequences of the sufficient grounds for introducing the formula are just these sufficient grounds. In the case of conjunction elimination, the way the elimination scheme computes a proof of the consequence C from a proof of $A\&B$ and a proof of C from assumed proofs of A and B separately has the base case that the proof of C is the proof of A, and the second base case that the proof of C is the proof of B. Thus, Martin-Löf's general elimination scheme gives us for these base cases the two rule instances:

$$\dfrac{A\&B \quad [A]}{C}\ \&E \qquad \dfrac{A\&B \quad [B]}{C}\ \&E$$

To recover the Gentzen-Prawitz elimination rules, it is sufficient to leave unwritten the degenerate derivations of the minor premisses in these two rule instances.

For implication, the sufficient ground for concluding $A \supset B$ is that there is a derivation of B from the assumption A. Such an existence can be indicated only schematically, and no way has been found to express in first-order logic the idea that C is the consequence of the existence of a derivation. In (Negri and von Plato, 2001, p. 8), the following is suggested: If there is a derivation of B from A, then, whatever follows from B follows already from A. Thus, the rule scheme becomes:

$$\dfrac{A \supset B \quad A \quad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C}\;\supset E$$

As with conjunction elimination, the standard rule comes out as the base case when the derivation of the minor premiss C is degenerate.

What has been said of implication goes also for universal quantification: The sufficient ground for concluding $\forall x A$ is the existence of a derivation of $A(y/x)$ for an *arbitrary* y. Type theory can hypothesize the existence of a higher-order function that produces, for any value of y, a proof of $A(y/x)$, and express that a formula C follows from the existence of such a function. In first-order logic, an elimination rule can be written, with t an arbitrary term, as:

$$\dfrac{\forall x A \qquad \begin{matrix}[A(t/y)] \\ \vdots \\ C\end{matrix}}{C}\;\forall E$$

The type-theoretical version of this rule is presented in (Martin Löf, 1984, preface), and the first-order rule in (Schroeder-Heister, 1984a). The full set of general elimination rules is found in (Dyckhoff, 1988), then in (Tennant 1992), (Lopez Escobar, 1999), and (von Plato, 2000; von Plato 2001).

Natural deduction with general elimination rules can be brought into a direct correspondence with the left rules of sequent calculus, with the following result, as established in von Plato (2001):

Isomorphism between natural deduction and sequent calculus. *A cut-free derivation in sequent calculus translates isomorphically into a derivation in natural deduction with general elimination rules with the property that all major premisses of elimination rules are assumptions.*

The correspondence between left rules and elimination rules and right rules and introduction rules, as well as the order of the logical rules, is maintained by the translation.

The translation goes also in the other direction, from natural deduction to sequent calculus, and the property singled out by the isomorphism gives a simple notion of normal derivability:

Normal derivations. *A derivation in natural deduction with general elimination rules is normal if all major premisses of elimination rules are assumptions.*

Further results include that instances of the structural rules of weakening and contraction in sequent calculus correspond to vacuous and multiple discharges, respectively, of assumptions in natural deduction. These results come out directly from the isomorphic translation between derivations in natural deduction and sequent calculus. The normalization of derivations is carried through so that cases with major premisses of elimination rules derived by other elimination rules are first removed in what are known as *permutative conversions.* Such conversions for disjunction and existence elimination were first published in (Prawitz 1965) but actually known and used already by Gentzen (von Plato, 2008). With general elimination rules, there are permutative conversions for all the elimination rules. After permutative conversions have been exhausted, there come the cases of major premisses of elimination rules that are derived by the corresponding introduction rules, i.e., the *detour convertibilities.* A direct proof of normalization for natural deduction with general elimination rules was given in *Structural Proof Theory* (pp. 199–201, see also *Proof Analysis*, pp. 27–28). The related result of strong normalization was proved in (Joachimski and Matthes, 2003).

The last rule in a normal derivation of a theorem, i.e., a derivation without open assumptions, must be an introduction rule, because an elimination rule would leave its major premiss as an open assumption. Results that were earlier proved through sequent calculus, such as the disjunction and existence properties of intuitionistic logic, can now be carried through in natural deduction. There are many later applications of the very strong property of normal derivability that is made possible by general elimination rules, such as the existence property of Heyting arithmetic (von Plato, 2006).

The point with the inversion principle of *Structural Proof Theory* was mainly a pedagogical one in three steps: 1. To motivate the rules of natural deduction through the standard meaning explanations of the connectives and quantifiers that give rise to the introduction rules. 2. To determine the elimination rules by the general inversion principle. 3. To arrive at the rules of sequent calculus by the translation of 1 and normal instances in 2. Somewhat surprisingly, the inversion principle turned out to be more than a pedagogical device, namely a very useful tool in research, as we shall point out below.

3 From semantical explanations to rules of proof

One half of natural deduction, the introduction rules, is a formalization of the BHK provability conditions. Thus, we can say that Gentzen was the one who took the step of extracting a rule system from semantical explanations.[2] These developments led by 1970 or so to the remarkable *computational semantics* of intuitionistic logic, an idea developed further in intuitionistic type theory. Formal proofs are coded as functions and steps of normalization become interpreted as steps of computation of these functions. Strong normalization was also established around 1970 (see Prawitz, 1971), and becomes interpreted as the termination of the computations, and the uniqueness of normal form as the uniqueness of values of the functions.

[2] If that was his way, which is by no means certain as discussed in (von Plato, 2012).

Thirty years after Gentzen, and well before the computational semantics was understood in detail, Saul Kripke gave another semantics for intuitionistic logic in terms of *possible worlds.* In classical propositional logic, there is a situation at hand in which to the atomic formulas are assigned truth values that determine the truth values of compound formulas. In Kripke's semantics, these situations are indexed by the worlds, denoted $w, o, r, \dots$ with $\mathcal{W}$ standing for the collection of all possible worlds, and the notation $w \Vdash P$ standing for the "forcing relation": *atom P holds in world w*. This machinery gains strength when the idea of possible worlds is put into use, with the intuition that there is an initial world w_0 in which some thing or other, possibly nothing at all, is known about the atomic facts $P, Q, R, \dots$, and that information comes in in the form of added atomic facts, in new worlds $o, r, \dots$ related to the present one through an *accessibility relation* $w \leqslant o$. The accessibility relation is assumed to have the following properties:

1. There is an *initial world* w_0 such that $w_0 \leqslant w$ for any w in $\mathcal{W}$.
2. The accessibility relation is *reflexive*: $w \leqslant w$ for any w in $\mathcal{W}$.
3. The accessibility relation is *transitive*: If $w \leqslant o$ and $o \leqslant r$, then $w \leqslant r$ for any w, o, r in $\mathcal{W}$.

It is further required that no information be lost, i.e., that the forcing relation be *monotonic*: If $w \Vdash A$ and $w \leqslant o$, then $o \Vdash A$. For compound formulas, forcing is defined inductively, as in the semantical clauses for the connectives:

1. $w \Vdash A \& B$ whenever $w \Vdash A$ and $w \Vdash B$.
2. $w \Vdash A \vee B$ whenever $w \Vdash A$ or $w \Vdash B$.
3. $w \Vdash A \supset B$ whenever from $w \leqslant o$ and $o \Vdash A$ follows $o \Vdash B$.
4. $w \Vdash \neg A$ whenever from $w \leqslant o$ and $o \Vdash A$ follows $o \Vdash C$ for any C.
5. $w \Vdash C$ for any C if $w \Vdash A$ and $w \Vdash \neg A$ for some A.

This definition will work for intuitionistic logic with a primitive notion of negation. With a defined notion of negation, clause 4 is left out and clause 5 can be put as: no world forces $\perp$. It then happens that proofs of the properties of the forcing relation have to rely on somewhat awkward meta-level reasonings. For example, for $w \Vdash \perp \supset C$, one needs: From $w \leqslant o$ and $o \Vdash \perp$ follows $o \Vdash C$. This is the case because $o \Vdash \perp$ is false.

Under the above clauses 1–5 for compound formulas, the forcing relation for a world w becomes *trivial*, in the sense that w forces all formulas, whenever $w \Vdash A$ and $w \Vdash \neg A$ for some A. It is natural to pose the requirement of *nontriviality: No world must force all formulas. Validity* of a formula A can now be defined as "truth in all possible worlds," or more formally, as $w \Vdash A$ for an arbitrary w.

The correspondence between the inductive clauses of forcing and the provability conditions of natural deduction is straightforward, as a couple of examples show:

For conjunction, one direction of the semantical clause is: If $w \Vdash A$ and $w \Vdash B$, also $w \Vdash A \& B$. Therefore, if w is arbitrary and the premisses A and B of rule $\&I$ are assumed valid, also its conclusion $A \& B$ is. In the other direction, the clause is that if $w \Vdash A \& B$, then $w \Vdash A$ and $w \Vdash B$. Therefore, if the premiss $A \& B$ of rules $\&E_1$ and $\&E_2$ is valid, also the conclusions A and B are.

For the rule of implication introduction, the definition of validity has to be extended: B is *forced under assumptions* Γ *in world* w whenever from $w \leqslant o$ and $o \Vdash A$ for each A in Γ follows $o \Vdash B$. If w is arbitrary, B is *valid under assumptions* Γ.

The clause for implication is in one direction: If from $w \leqslant o$ and $o \Vdash A$ follows $o \Vdash B$, also $w \Vdash A \supset B$. Therefore, if the premiss B of rule $\supset I$ is valid under the assumption A, i.e., if from $o \Vdash A$ follows $o \Vdash B$, also the conclusion $A \supset B$ of the rule is valid by the clause. In the other direction, assume $w \Vdash A \supset B$ and $w \Vdash A$. By the semantical clause, $o \Vdash B$ whenever $w \leqslant o$ and $o \Vdash A$. In particular, setting w for o, we have that if $w \leqslant w$ and $w \Vdash A$, also $w \Vdash B$. The first condition holds by the reflexivity of the accessibility relation, the second by assumption. Therefore, if the premisses of rule $\supset E$ are valid, also the conclusion is.

The lesson from the above correspondence between syntax and semantics is that one direction of a semantical clause corresponds to an introduction rule, the other direction to an elimination rule.

In perfect analogy to the proof terms of typed lambda-calculus that lead to the computational semantics of intuitionistic logic, we can make the semantics of possible worlds for intuitionistic logic formal, by including these worlds and the forcing relation as parts of a system of rules: Formulas come with *labels* $w, o, r, \ldots$ with the forcing relation written compactly as $w : A$, and the accessibility relation $w \leqslant o$ is a new type of atomic formula. The rules for conjunction and implication are, directly from the semantical clauses:

$$\frac{w : A \quad w : B}{w : A \& B}\,\&I \qquad \frac{w : A \& B}{w : A}\,\&E_1 \qquad \frac{w : A \& B}{w : B}\,\&E_2$$

$$\frac{\begin{matrix} w \overset{1}{\leqslant} o, o \overset{1}{:} A \\ \vdots \\ o : B \end{matrix}}{w : A \supset B}\,{\supset}I,1 \qquad \frac{w \leqslant o \quad w : A \supset B \quad o : A}{o : B}\,{\supset}E$$

In rule $\supset I$, the label o has to be arbitrary, i.e., an eigenvariable of the rule.

Accessibility relations are now a part of the formal calculus and their properties have to be represented. To this end, we use the well-developed machinery of *proof analysis*, i.e., of the extension of logical calculi by rules that represent mathematical axioms. The rules can be written in the style of natural deduction as:

$$\frac{w : A \quad w \leqslant o}{o : A}\,Mon \qquad \frac{}{w_0 \leqslant w}\,Init \qquad \frac{}{w \leqslant w}\,Ref \qquad \frac{w \leqslant o \quad o \leqslant r}{w \leqslant r}\,Tr$$

If a semantics is going to be more than just suggestive words, the notion of *proof of validity* has to be considered instead of mere validity. An example from the Kripke semantics for intuitionistic logic shows that proofs of validity can turn out to be essentially the same as formal proofs by syntactic rules:

An example of a semantical proof of validity. $\Vdash A \supset (B \supset A \& B)$. Let w be arbitrary and assume $w \leqslant o$ and $o \Vdash A$. To show $o \Vdash B \supset A \& B$, assume $o \leqslant r$ and $r \Vdash B$. By monotonicity, $r \Vdash A$, so by definition, $r \Vdash A \& B$. Therefore $o \Vdash B \supset A \& B$, and finally $w \Vdash A \supset (B \supset A \& B)$.

Reproduction by the rules of formal semantics.

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{2}{o : A} \quad \overset{1}{o \leqslant r}}{r : A}\,Mon \qquad \overset{1}{r : B}}{r : A \& B}\,\&I}{o : B \supset A \& B}\,{\supset}I,1}{w : A \supset (B \supset A \& B)}\,{\supset}I,2$$

In the upper instance of rule $\supset I$, the accessibility relation $o \leqslant r$ is closed together with the assumption $r : B$. In the lower instance of rule $\supset I$, the assumption $o : A$ is closed, but the associated accessibility relation $w \leqslant o$ is not used in the derivation. It is closed vacuously.

Translation to a formal derivation in natural deduction. Given a formal proof of validity, it can be translated by an easy algorithm into a formal derivation in natural deduction: First delete all labels and accessibility relations. Now instances of rules *Init, Ref*, and *Tr* have disappeared. Next delete the repetitions that rule *Mon* has left. The result for the above example is:

$$\dfrac{\dfrac{\dfrac{\overset{2}{A} \quad \overset{1}{B}}{A \& B}\,\&I}{B \supset A \& B}\,{\supset}I,1}{A \supset (B \supset A \& B)}\,{\supset}I,2$$

The approach to labelled deduction with the internalization of the Kripke semantics has been developed in the literature in several forms, based on either natural deduction, sequent calculi, or tableau systems. Closest to the approach illustrated here are the works of (Simpson, 1994) that uses natural deduction and (Viganò, 2000), based on sequent calculus but with frame rules that correspond to frame properties that do not contain disjunctions in positive parts.

4 An intuitionistic sequent calculus with invertible rules

Kripke's most fundamental discovery was perhaps the correspondence between conditions on the accessibility relation and axioms of systems of logic. For example, if to the conditions of reflexivity and transitivity of intuitionistic logic the condition of symmetry is added, the possible worlds collapse into one equivalence class and the logic becomes classical. By the correspondence, logical systems between the intuitionistic and classical ones can be captured either by suitable axioms, such as Dummett's axiom $(A \supset B) \vee (B \supset A)$, or by a suitable "frame condition" on the accessibility relation, the linearity condition $w \leqslant o \vee o \leqslant w$ in this case. However, as is seen, the condition employs the same connective $\vee$ as the axiom. A similar problem was met when Gentzen wanted to reason about provability in natural deduction, and his solution was to distinguish between an internal implication $A \supset B$ and an external derivability relation $A \vdash B$ ($A \rightarrow !B$ in Gentzen's notation). A similar method is possible here: With frame property *Tr*, a two-premiss "logic-free rule" was used that acts on the atomic premisses $w \leqslant o$ and $o \leqslant r$, to give the atomic conclusion $w \leqslant r$, with no interference with the logical operations of conjunction and implication that would otherwise be used in the expression of the axiom of transitivity. Thus, we have the correspondence between a "logical" and a "logic-free" derivation of $w \leqslant r$ from the assumptions $w \leqslant o$ and $o \leqslant r$, the former with an instance of the transitivity axiom:

$$\dfrac{w \leqslant o \,\&\, o \leqslant r \supset w \leqslant r \qquad \dfrac{w \leqslant o \quad o \leqslant r}{w \leqslant o \,\&\, o \leqslant r}\,{\scriptstyle \&I}}{w \leqslant r}\,{\scriptstyle \supset E} \qquad\qquad \dfrac{w \leqslant o \quad o \leqslant r}{w \leqslant r}\,{\scriptstyle Tr}$$

More generally, those quantifier-free frame properties that do not contain essential disjunctions, i.e., disjunctions in positive parts, can be converted to additional rules of natural deduction of the type of *Tr*. No mixing of logical properties is produced. The method of conversion of axioms into "logic-free" additional rules had been already developed successively in (Negri, 1999), (Negri and von Plato, 1998), and (Negri and von Plato, 2001, ch. 8), when the first author realized the possibility of converting frame properties of modal logic into rules. This earlier work covered those cases in which frame properties are expressed by universal formulas. The much wider class of *geometric implications*, including typical existence axioms, was covered in (Negri, 2003).

The limitation on disjunction inherent to additional rules in the style of natural deduction is surpassed if a multisuccedent sequent calculus is used. The logical rules of the labelled sequent calculus are written with shared contexts as in the *G3*-calculi, to support root-first proof search:

$$\dfrac{w : A, w : B, \Gamma \rightarrow \Delta}{w : A\&B, \Gamma \rightarrow \Delta}\,{\scriptstyle L\&} \qquad\qquad \dfrac{\Gamma \rightarrow \Delta, w : A \quad \Gamma \rightarrow \Delta, w : B}{\Gamma \rightarrow \Delta, w : A\&B}\,{\scriptstyle R\&}$$

$$\frac{w : A, \Gamma \to \Delta \quad w : B, \Gamma \to \Delta}{w : A \vee B, \Gamma \to \Delta} {\scriptstyle L\vee} \qquad \frac{\Gamma \to \Delta, w : A, w : B}{\Gamma \to \Delta, w : A \vee B} {\scriptstyle R\vee}$$

$$\frac{w : A \supset B, \Gamma \to \Delta, w : A \quad w : B, \Gamma \to \Delta}{w : A \supset B, \Gamma \to \Delta} {\scriptstyle L\supset} \qquad \frac{w \leqslant o, o : A, \Gamma \to \Delta, o : B}{\Gamma \to \Delta, w : A \supset B} {\scriptstyle R\supset}$$

$$\frac{}{w : \bot, \Gamma \to \Delta} {\scriptstyle L\bot}$$

Contrary to unlabelled sequent calculus, rule $R\supset$ has the context Δ also in the premiss. The label o in the rule has to be arbitrary, i.e., an eigenvariable of the rule.

The frame rules of intuitionistic logic in the notation of labelled sequent calculus are:

$$\frac{o : A, w : A, w \leqslant o, \Gamma \to \Delta}{w : A, w \leqslant o, \Gamma \to \Delta} {\scriptstyle Mon}$$

$$\frac{w \leqslant w, \Gamma \to \Delta}{\Gamma \to \Delta} {\scriptstyle Ref} \qquad \frac{w \leqslant r, w \leqslant o, o \leqslant r, \Gamma \to \Delta}{w \leqslant o, o \leqslant r, \Gamma \to \Delta} {\scriptstyle Tr}$$

The calculus has initial sequents of the form $w : P, \Gamma \to \Delta, w : P$ with P an atomic formula.

To obtain a calculus with strong structural properties, the rule of monotonicity is left out in favour of initial sequents with in-built monotonicity, of the form

$$w \leqslant o, w : P, \Gamma \to \Delta, o : P$$

The sequent calculus thus obtained and called *G3I* has all structural rules admissible and, moreover, contraction is admissible with the property that a step of contraction preserves the height of derivation. (This is the reason for the repetition of the atoms from the conclusion in the premiss.) As a result, steps of root-first proof search that would produce a duplication are not permitted. By completeness, a formula A is provable in intuitionistic logic if and only if for a label w, the sequent $\to w : A$ is derivable in the calculus. It follows that the sequent calculus version of the rule *Init* that produces an initial label is not needed as an explicit rule of *G3I*.

Even if *G3I* does not have the restriction of a single-succedent premiss in rule $R\supset$, as if by miracle the calculus does not become classical: An attempt at a root-first derivation of the law of excluded middle gives

$$\dfrac{\dfrac{x \leqslant y, y : P \to x : P, y : \bot}{\to x : P, x : \neg P} {\scriptstyle R\supset,\, y\ fresh}}{\to x : P \vee \neg P} {\scriptstyle R\vee}$$

The eigenvariable condition is $y \neq x$ by which no initial sequent is reached.

The most remarkable feature of the labelled sequent calculus for intuitionistic logic is the *invertibility of all of its rules*, a property encountered earlier only with unlabelled classical sequent calculi. By this invertibility, the rules preserve countermodels and a terminal node in a failed proof search defines a Kripke countermodel which is automaticaly a countermodel to the conclusion. In the example above the countermodel is given by a the following

$$\begin{array}{l} \bullet\ y \Vdash P \\ \uparrow \\ \bullet\ x \nVdash P \end{array}$$

The parallel proof search/countermodel construction works in full generality for the *G3K*-based modal labelled calculi (Negri, 2009).

To obtain a classical calculus, a rule of symmetry is added to the frame rules of *G3I*. Alternatively, no accessibility relation is considered and all the rules, including those for implication, are obtained by labelling in a "flat" way all the rules of *G3c*, as in the propositional part of the calculus *G3K*.

5 Labelled modal calculi

Turning now to modal logic, the inductive definition of forcing of a modal formula in a possible world w follows from the basic idea of Kripke semantics, which is to define necessity relative to a world w simply as that which happens to hold in all worlds accessible from w, as in:

$w \Vdash \Box A$ *whenever for all* o, *from* $w \leqslant o$ *follows* $o \Vdash A$.

The definition gives:

If $o : A$ *can be derived for an arbitrary* o *accessible from* w, *then* $w : \Box A$ *can be derived.*

Formally, we have the rule in natural deduction style:

$$\cfrac{\begin{array}{c} w \overset{1}{\leqslant} o, \Gamma \\ \vdots \\ o : A \end{array}}{w : \Box A}\ {\scriptstyle \Box I,1}$$

The condition is that o does not occur in Γ. By generalizing the rule to an arbitrary conclusion, that is one in which $o : A$ comes together with an arbitrary succedent Δ, it becomes the sequent calculus rule

$$\frac{w \leqslant o, \Gamma \to \Delta, o : A}{\Gamma \to \Delta, w : \Box A}\ {\scriptstyle R\Box}$$

In the rule, the arbitrariness of o becomes the variable condition that o must not occur in Γ, Δ.

The inversion principle is stated in (Negri and von Plato, 2001, p. 6) in the form *"Whatever follows from the direct grounds for deriving a proposition must follow from that proposition."* Through this principle, one has that consequences of the derivability of $o : A$ from an arbitrary o accessible from w are consequences of $w : \Box A$. Then, similarly to the determination of the "lower level" rule of general implication elimination, we find that whatever follows from $o : A$ must already follow from $w : \Box A$ and $w \leqslant o$, that is, we have the general elimination rule

$$\frac{w : \Box A \quad wRo \quad \begin{array}{c} \overset{1}{o : A} \\ \vdots \\ w : C \end{array}}{w : C}\ {\scriptstyle \Box E,1}$$

If the major premiss is an assumption, the rule can be written in sequent calculus notation, as:

$$\frac{\Gamma \to \Delta, wRo \quad o : A, \Gamma \to \Delta}{w : \Box A, \Gamma \to \Delta}\ {\scriptstyle L\Box'}$$

The rule can be equivalently given as a one-premiss rule in the following form

$$\frac{o : A, w : \Box A, wRo, \Gamma \to \Delta}{w : \Box A, wRo, \Gamma \to \Delta}\ {\scriptstyle L\Box}$$

The recipe for "meaning in use" is: Meaning-conferring introduction rules are scrutinized under the inversion principle, to obtain general forms of elimination rules. The normal instances of these rules have direct translations to sequent calculus.

The inductive clause for the possibility operator $\Diamond$ is:

$w : \Diamond A$ *whenever for some* o, $w \leqslant o$ *and* $o : A$.

The rules for $\Diamond$ are obtained from the semantic explanation analogously to those of $\Box$. They are:

$$\frac{w \leqslant o, o : A, \Gamma \to \Delta}{w : \Diamond A, \Gamma \to \Delta}\ {\scriptstyle L\Diamond} \qquad \frac{w \leqslant o, \Gamma \to \Delta, w : \Diamond A, o : A}{w \leqslant o, \Gamma \to \Delta, w : \Diamond A}\ {\scriptstyle R\Diamond}$$

In rule $L\Diamond$, o is an eigenvariable that corresponds to the existential quantifier in the inductive clause.

In all, the labelled calculi are constructed so that they are equivalent to corresponding axiomatic calculi. More precisely, because the language includes the accessibility and forcing relations, they are conservative extensions of the axiomatic calculi (cf. Negri, 2005, for details).

Properties of the accessibility relation such as reflexivity and transitivity correspond to modal axioms, as in the table:

	Axiom	Frame property
T	$\Box A \supset A$	$\forall w\, w \leqslant w$ reflexivity
4	$\Box A \supset \Box\Box A$	$\forall wor(w \leqslant o \,\&\, o \leqslant r \supset w \leqslant r)$ transitivity
E	$\Diamond A \supset \Box\Diamond A$	$\forall wor(w \leqslant o \,\&\, w \leqslant r \supset o \leqslant r)$ Euclideanness
B	$A \supset \Box\Diamond A$	$\forall wo(w \leqslant o \supset o \leqslant w)$ symmetry
D	$\Box A \supset \Diamond A$	$\forall w \exists o\, w \leqslant o$ seriality
W	$\Box(\Box A \supset A) \supset \Box A$	no infinite R-chains + transitivity

Let us take as another example the determination of the rules for the "actuality operator" @ from the semantic explanation. The formula $@A$, read *actually A true at world w*, expresses that A is true at the actual world w_a. The forcing notation is:

$w \Vdash @A$ *whenever* $w_a \Vdash A$.

Now we can read out from the semantical explanation, similarly to the modalities of necessity and possibility, the introduction and elimination rules:

$$\frac{w_a : A}{w : @A}\,{\scriptstyle @I} \qquad \frac{w : @A}{w_a : A}\,{\scriptstyle @E}$$

The formulation in terms of labelled sequent calculus is:

$$\frac{w_a : A, \Gamma \to \Delta}{w : @A, \Gamma \to \Delta}\,{\scriptstyle L@} \qquad \frac{\Gamma \to \Delta, w_a : A}{\Gamma \to \Delta, w : @A}\,{\scriptstyle R@}$$

An axiomatization of modal systems augmented by the actuality operator has been provided by Hodes (1984), as an extension of first-order S5 and shall not be recalled it here. It is straightforward to verify that the axioms are all derivable in the labelled sequent calculus for actuality here obtained as an extension of the basic modal system with reflexivity, transitivity and symmetry of the accessibility relation plus the rules for actuality. For example, axiom $@(A \supset B) \supset (@A \supset @B)$ is derived as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{w_a : A \to w_a : A \quad w_a : B \to w_a : B}{w_a : A, w_a : A \supset B \to w_a : B}\,{\scriptstyle L\supset}}{w_a : A, w : @(A \supset B) \to w_a : B}\,{\scriptstyle L@}}{w : @A, w : @(A \supset B) \to w : @B}\,{\scriptstyle R@}}{w : @(A \supset B) \to w : @A \supset @B}\,{\scriptstyle R\supset}}{\to w : @(A \supset B) \supset (@A \supset @B)}\,{\scriptstyle R\supset}$$

The labelled approach allows for a fine distinction between various notions of logical consequence that can be adopted: *actualistic* logical consequence is logical consequence relative to the actual world, whereas *universal* (or strong) consequence is relative to an arbitrary world.

The contraction-free labelled sequent calculi were first developed for modal and related logics (Negri, 2005), but are not limited to them. They can be applied equally well to create proof systems for pure predicate logic, for example, and for the intermediate logics that were mentioned above. Such logical systems are typically characterized by frame conditions that are added to those of intuitionistic logic, until the conditions of classical logic are reached. This idea is carried through in (Dyckhoff and Negri, 2012) in which intermediate logical systems are obtained by adding to the labelled calculus for intuitionistic logic rules that correspond to frame conditions. The uniformity provided by the labelled calculi leads to a simple syntactic proof of soundness and faithfulness of the embedding of a wide class of intermediate logics into their modal companions.

6 Completeness and decidability

The connection between derivations in natural deduction and proofs of validity in Kripke semantics is close and suggestive of a completeness theorem. The unification of the semantic and syntactic dimension in labelled sequent calculi leads to such uniform, simple, and direct proofs of completeness for modal logics. Strangely enough, the style of completeness proof that was favored in the literature on modal logic since the late 1960s has been the Henkin-style completeness proof, even if Kripke's (1963) proof of completeness was based on a direct construction of countermodels for failed tableau proofs. Apparently, as documented in (Negri, 2009), the reasons behind this turn are to be found in negative reviews of Kripke's paper. The review by Kaplan (1966) contained also a sketch of an alternative, Henkin-style, completeness proof, which became the standard until the present days. Labelled sequent calculi, however, allow to recover the original explicit character of Kripke's c! ompleteness proof, without the insufficiency in formalization that was lamented by the early reviewers.

The direct proof for labelled sequent calculi is obtained through a Schütte-style argument: all the rules (logical rules and frame rules) of the calculus for a given modal logic are applied, root-first, from a given logical sequent $\Gamma_0 \to \Delta_0$ labelled uniformly with an arbitrary label w. In this way a big tree is built. If all the branches lead to initial sequents or instances of $L\bot$, then the sequent is derivable. Otherwise it may happen that at some stage no rule is applicable and the sequent is neither initial nor an instance of $L\bot$, or that the construction goes on forever. In the two latter cases, a countermodel is built. If the search stops at a sequent $\Gamma \to \Delta$ because no rule is any longer applicable, a countermodel is built by considering all the worlds in Γ, related to each other through the accessibility relations in Γ, and the valuation that forces in w all the atomic formulas for which $w : P$ is in Γ and th! at does not force in w atomic formulas for which $w : Q$ is in Δ. In the case of an infinite process, by König's lemma, the tree has an infinite branch of sequents $\Gamma_i \to \Delta_i$. Again, the countermodel is built directly by taking as possible worlds all the labels and all the relations in the antecedents Γ_i, with a valuation that forces on the world w the formulas for which $w : P$ is in some of the Γ_i and does not force on w those for which $w : Q$ is in some Δ_i. An inductive arguments then shows that the valuation has the property of forcing on w all the formulas A for which $w : A$ is in one of the antecents and no formula B for which $w : B$ is in some of the succedents. A countermodel to $\Gamma_0 \to \Delta_0$ is thus found.

The completeness proof can be turned into a constructive proof of decidability whenever the potentially infinite growth of the search tree can be truncated. The finite countermodel is not extracted from an infinite one, but is built directly from a proof search which has at least a truncated branch. Rather than describing the procedure in general, we illustrate it with an example. First, observe that a check of derivability for a formula A is equivalent to a check of validity. We can thus start with applying root-first the rules of the labelled calculus for intuitionistic logic for the the sequent $\to w : \neg\neg A \supset A$, where w is an arbitrary label, as follows (in applications of $L\supset$ the derivable right premiss is omitted):

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\vdots}{w \leqslant o, o \leqslant o, o \leqslant r, r \leqslant l, l : A, r : A, o : \neg\neg A \to o : A, o : \neg A, r : \bot, l : \bot}\,R\supset}{w \leqslant o, o \leqslant o, o \leqslant r, r : A, o : \neg\neg A \to o : A, o : \neg A, r : \bot, r : \neg A}\,L\supset}{w \leqslant o, o \leqslant o, o \leqslant r, r : A, o : \neg\neg A \to o : A, o : \neg A, r : \bot}\,R\supset}{w \leqslant o, o \leqslant o, o : \neg\neg A \to o : A, o : \neg A}\,L\supset}{w \leqslant o, o \leqslant o, o : \neg\neg A \to o : A}\,Ref}{w \leqslant o, o : \neg\neg A \to o : A}\,R\supset}{\to w : \neg\neg A \supset A}$$

Clearly, the proof search goes on indefinitely, but there are two ways to see already at this point that it does not lead to a derivation. The first is strictly proof-theoretic and consists in appealing to structural properties of the labelled calculus, namely height-preserving admissibility of substitution for labels (here r/l) and height preserving admissibility of contraction. By these two properties, the above search would yield, together with reflexivity, a shortening of the purported derivation, which contradicts the quest for a minimal one. Alternativelly, and probably more convincingly, we observe that l is a *looping label*, i.e., a label of a formula that already appeared earlier in the search. We obtain a finite countermodel already from this segment of the infinite branch by taking as worlds the labels w, o, r, l with the accessibility relations $w \leqslant o$, $o \leqslant r$, $r \leqslant l$ which are in the search tree, plus the accessibility that witnesses the loop! , namely $l \leqslant r$, and their transitive closures plus reflexivities. The valuation is defined by the forcing of A in r and l but not in o. It is clear that $x \nVdash \neg\neg A \supset A$, so a finite countermodel has been found. Adding the extra accessibility relations is not strictly needed in the case of intuitionistic logic, but becomes necessary for example for systems which have a frame property of seriality in place of reflexivity.

Countermodel constructions inspired by the above methodology are used to obtain decision procedures for modal logics with transitive and serial accessibility relations such such as the logic of Priorean linear time (Boretti and Negri, 2009) and several classes of intuitionistic multi-modal logics (Garg, Genovese and Negri, 2012). The general results guarantee that the frame that arises from the truncated failed proof search gives indeed a countermodel to the conclusion of the failed proof-search, with no need to check that the endformula is not valid in it.

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