Journal of Logic and Computation Advance Access published July 18, 2013

A cut-free sequent system for Grzegorczyk logic, with an application to the Gödel–McKinsey–Tarski embedding

ROY DYCKHOFF, *School of Computer Science, St Andrews University, St Andrews, Fife KY16 9SX, Scotland*
E-mail: rd@st-andrews.ac.uk

SARA NEGRI, *Department of Philosophy, University of Helsinki, P.O. Box 24, 00014 Finland*
E-mail: sara.negri@helsinki.fi

Abstract

It is well-known that intuitionistic propositional logic **Int** may be faithfully embedded not just into the modal logic **S4** but also into the provability logics **GL** and **Grz** of Gödel-Löb and Grzegorczyk, and also that there is a similar embedding of **Grz** into **GL**. Known proofs of these faithfulness results are short but model-theoretic and thus non-constructive. Here a labelled sequent system **Grz** for Grzegorczyk logic is presented and shown to be complete and therefore closed with respect to *Cut*. The completeness proof, being constructive, yields a constructive decision procedure, i.e. both a proof procedure for derivable sequents and a countermodel construction for underivable sequents. As an application, a constructive proof of the faithfulness of the embedding of **Int** into **Grz** and hence a constructive decision procedure for **Int** are obtained.

Keywords: Sequent calculus, modal logic, provability logic, Grzegorczyk logic, intuitionistic logic, decision procedures, labelled deduction.

1 Introduction

Motivated by the idea that intuitionism expresses a modal notion of provability, Gödel [13] defined in 1933 a translation of intuitionistic logic **Int** into the modal logic **S4**, stated without proof the soundness of the translation and remarked [14][1] that 'presumably a formula holds in Heyting's calculus if and only if its translation is provable in **S4**'.[2] It took some years before McKinsey and Tarski [23] proved that which Gödel had merely conjectured, faithfulness of the embedding of intuitionistic logic in **S4** by means of a suitable translation: a translated formula $A^{\Box}$ is provable in **S4** iff A is provable in **Int**. The result was proved indirectly, using algebraic semantics and completeness of **S4** with respect to closure algebras and of **Int** with respect to Heyting algebras. Various slightly different translations (modifying that of [13]), proof methods and results in this area are summarized in [39] (p. 314). The translation $\cdot^{\Box}$ that we use will be defined in Section 6.

The result was later extended in various ways: first, into embedding results for intermediate logics in modal logics between **S4** and **S5** by Dummett and Lemmon [9], and second, into the embeddings

[1]Here in translation, including use of the name **S4** in place of the name he used.

[2]As a referee remarked, Gödel's translation, taken literally, is seen not to be faithful by considering the intuitionistically unprovable formula $(P{\supset}P){\supset}(P\vee\neg P)$ which is translated to the **S4** provable formula $\Box(P{\supset}P){\supset}\Box(P\vee\neg P)$; however one can assume that in his note the recursive part of the definition was left implicit.

doi:10.1093/logcom/ext036

of **Int** into what were later called the provability logics **GL** and **Grz** of Gödel-Löb and (resp.) of Grzegorczyk. Dummett and Lemmon proved that $\mathbf{Int}+\Gamma\vdash A$ if and only if $\mathbf{S4}+\Gamma^{\Box}\vdash A^{\Box}$. A similar result, with **Grz** in place of **S4**, was stated (without proof) in [12]. See also [24] for a survey.

The provability logic **GL** extends the normal modal logic **K** by Löb's axiom schema $\Box(\Box A\supset A)\supset\Box A$ (from which the 4 schema $\Box A\supset\Box\Box A$ follows, and the T schema $\Box A\supset A$ does not follow); its semantics is given by irreflexive transitive Noetherian frames. In 1976, Solovay [37] emphasized the importance of **GL** by presenting it as the logic that characterizes arithmetic provability, i.e. he showed that, for any modal formula A, $\mathbf{GL}\vdash A$ if and only if, for every realization r of the atoms in A as sentences of **PA** (Peano Arithmetic), one has $\mathbf{PA}\vdash P_r(A)$, where, for formula X, the sentence $P_r(X)$ of **PA** is defined on atoms X as $r(X)$, routinely on conjunctions, disjunctions, negations, implications and absurdity, but for modal formulae $\Box B$ one has $P_r(\Box B)\equiv Bew(\ulcorner P_r(B)\urcorner)$, where $\ulcorner\cdot\urcorner$ is a fixed Gödel numbering of sentences of **PA** as numerals and *Bew* (short for *Beweisbar*) is a defined unary predicate with $Bew(\ulcorner\cdot\urcorner)$ capturing arithmetic provability of sentences. The Löb axiom schema $\Box(\Box A\supset A)\supset\Box A$ is then interpreted as saying, for any sentence S of **PA**, that, if **PA** $\vdash Bew(\ulcorner S\urcorner)\supset S$, then already $\mathbf{PA}\vdash S$, i.e. Löb's Theorem.

It had been observed already by Gödel in 1933 that a naive 'provability' interpretation of **Int** would clash with the alethic interpretation of necessity: **S4** proves $\Box(\Box A\supset A)$, but the instance of this with $\perp$ in place of A, translated by a provability interpretation into **PA**, expresses the provability of consistency, $Bew(\ulcorner\neg Bew(\ulcorner\perp\urcorner)\urcorner)$, which by the second incompleteness theorem fails in any system containing arithmetic.

Grzegorczyk defined in [16], along the lines of the semantic topological method of McKinsey and Tarski, a special class of topological (point-free) spaces associated to finite reflexive transitive and antisymmetric frames (i.e. finite partial orders) and showed that Heyting algebras can be embedded in these frames. He also provided an axiomatization of the logic (now called **Grz**) characterized by such frames, as the extension of **S4** by the axiom $((A\sqsupset\Box B)\sqsupset\Box B)\wedge((\neg A\sqsupset\Box B)\sqsupset\Box B)\sqsupset\Box B)$, where $C\sqsupset D$ abbreviates $\Box(C\supset D)$, and proved semantically that it is a proper extension of **S4**, not contained in **S5** but in which **Int** is faithfully embedded by means of the translation $\cdot^{\Box}$. Segerberg later [36] gave a simpler axiomatization over **S4** using the schema $\Box(G(A)\supset A)\supset A$, where $G(A)\equiv\Box(A\supset\Box A)$.

Several authors independently proposed [2–4, 6, 15, 18, 19] a modified interpretation (the *provability-truth interpretation*) of modality in terms of arithmetic provability; this uses a translation $\cdot^{+}$ from **Grz** to **GL**, in which $(\Box A)^{+}$ is defined as $A^{+}\wedge\Box(A^{+})$; for its motivation see [20, 21]. Provability of A in **Grz** is then equivalent to provability in **GL** of its translation A^{+} and therefore to provability in **PA** of every $P_r(A^{+})$. That **Int** can be embedded into **GL** then follows from a modification $\cdot^{\boxdot}$ of the translation $\cdot^{\Box}$ used for the embedding into **S4** (and **Grz**); this modification $\cdot^{\boxdot}$ interprets atoms P as $P\wedge\Box P$ and implicational formulae $A\supset B$ as $(A^{\boxdot}\supset B^{\boxdot})\wedge\Box(A^{\boxdot}\supset B^{\boxdot})$. The modal interpretation $\cdot^{\Box}$ of **Int**, together with the translation $\cdot^{+}$ of **Grz** into **GL**, thus gives the embedding $\cdot^{\boxdot}$ of **Int** into **GL**.

However, unlike the proofs of soundness, the syntactic proofs of faithfulness of these embeddings are not entirely straightforward, as witnessed in section 9.2 of [39] for the relatively simple case of the embedding of **Int** into **S4**. Semantic arguments can be found in, e.g., [8, 15]. In the case of the syntactic proofs, careful invention of a strengthened induction hypothesis, stronger than that which has to be proved, is required. This is simplified in that simple case by the use of labelled systems [11]; we show in Section 6 how to attack the problem for the faithfulness of the embedding $\cdot^{\Box}$ of **Int** into **Grz**.

A first step to establishing such a faithfulness result consists in the formulation of a cut-free sequent system for the logic (in this case, **Grz**) that is the target of the embedding. A (traditional) sequent calculus for **Grz** (and for **GL**) was presented by Avron in [1] and shown to be complete with respect to Kripke semantics by the method of saturated sequents and canonical models constructions. Borga and Gentilini [7] prove cut-admissibility for an unlabelled calculus. A similar approach, but with tableaux rather than sequent systems, was pursued in [33]. As argued in [28], a more direct completeness proof than the one based on canonical model constructions is obtained in labelled systems: a failed proof search explicitly contains a Kripke countermodel. In addition, uniformity of syntax is a strong desideratum in view of syntactic embedding results.

In our previous work [11], to which we refer for a short account of the background, based on [25, 26, 29], on labelled sequent calculi for modal systems (especially the labelled calculus **G3K** for the modal logic **K** and the reflexivity and transitivity rules *Ref* and *Trans* needed for **G3S4** (for **S4**)) and for the labelled calculus for **Int** used here, we have given a simple and uniform embedding result for a wide class of intermediate logics and their corresponding modal companions. In particular, the proof of faithfulness of the embeddings is achieved in a syntactic way and is as straightforward as the proof of soundness and the proof for the **Int** into **S4** embedding considerably simplifies the earlier syntactic proof of [39] for this simple case.

Our goals in this article are thus, after some preliminaries including the setting up of a labelled sequent calculus, to give a constructive proof of completeness of the calculus and to give a simple syntactic proof of the faithfulness of this embedding.

The method of labelled sequent calculus we build upon covers in a uniform way all logics characterized by universal or geometric conditions on their Kripke frames. Grzegorczyk logic is characterized by reflexive, transitive and Noetherian frames: the last condition is not first-order, but it can nevertheless be internalized in the syntax of the calculus by a suitable characterization of the forcing relation for boxed formulas in such frames. This results in a modification of the rules for the necessity operator, analogous to the one used in [26] for **GL**.

We present in Section 2 a labelled sequent calculus **G3Grz** for **Grz** and give in Section 3 a constructive completeness proof for it, which both establishes the finite model property and gives a decision procedure for **Grz**: we show in fact that, for any given labelled sequent in the modal language, either a derivation in the calculus or a finite countermodel can be constructed. (It is this last feature that merits the epithet 'constructive' for the decision procedure.)

The complete sequent calculus permits a proof that the standard modal translation of intuitionistic logic is a faithful embedding into **Grz**. The proof of faithfulness is obtained in a constructive syntactic fashion through an induction on height of derivations and so, in contrast with semantic proofs of the same result (cf. [8]), one can recover a derivation of $\Rightarrow A$ in **G3I** from a derivation of $\Rightarrow A^{\Box}$ in **G3Grz**.

Thus, one may obtain a constructive decision procedure for **Int** as a consequence of the faithfulness of the embedding and the constructive decision procedure for **Grz**.

2 A labelled sequent system for Grzegorczyk logic

The provability logic **Grz** is characterized by reflexive, transitive and Noetherian frames. The latter property is not first-order, so the general methods used in our earlier work [11, 26, 27] for internalizing the frame properties into the syntax of sequent calculus cannot be applied in a straightforward way. Instead, analogous to the special method followed for **GL** in Negri [26], a suitable characterization of the forcing relation for modal formulas, and a consequent modification of the rules for $\Box$, permits the formulation of an appropriate labelled sequent calculus. However, **Grz** is characterized by *reflexive* (rather than irreflexive), transitive and Noetherian frames; this tiny detail makes important differences in the proof-theoretical analysis of these logics, even if the guiding idea for internalizing Noetherianity is similar.

We start with recalling the well-known definition of a Noetherian relation.

Definition 2.1
Let R be a relation on a set S.

1. An R-sequence[3] is a (finite or infinite) sequence (x_i) of elements of S such that, for any two successive elements x_i and x_{i+1} of the sequence, $x_i R x_{i+1}$ holds.
2. An R-sequence (x_i) is *convergent* iff, for some i, for all $j > i$ for which x_j is defined, $x_j = x_i$.
3. R is *Noetherian* iff every R-sequence is convergent.

'Is stationary', 'is eventually stationary', 'becomes stationary' and 'stabilises' are sometimes used in place of 'is convergent'. Jeřábek [17] shows that the axiom DC of Dependent Choice is equivalent in Zermelo–Fraenkel set theory to the proposition that every poset that is 'upwards well-founded', i.e., such that every non-empty subset has a maximal element, is Noetherian in the above sense; and he isolates an interesting condition which, in the absence of DC, lies strictly between this converse version of well-foundedness and Noetherian (as in the definition we have given) and which helps exactly characterize the frames that are models of **Grz**. Note that we phrase the definition so that the relation can be reflexive. If the relation is irreflexive, then 'convergent' is equivalent to 'finite'.

Since every finite sequence is convergent, we have the first part of:

Lemma 2.2
The Noetherian condition on R is equivalent to the convergence of every infinite R-sequence. In the presence of irreflexivity, it is equivalent to the finiteness of every R-sequence.

Proof. It remains to prove the second part: this is routine. ■

We recall the standard definition of Kripke semantics (w.r.t. a set S, a relation R on S and a 'world' $x \in S$) for normal modal logics:

$$x \Vdash \Box A \iff \text{for all } y \in S,\ xRy \text{ implies } y \Vdash A.$$

It is convenient to use the abbreviation $G(A)$ for the formula $\Box(A \supset \Box A)$.

Next, we prove a characterization of forcing of boxed formulas in Noetherian models.[4]

Lemma 2.3
If R is a transitive Noetherian relation on S, then, for all $x \in S$ and formula A,

$$x \Vdash \Box A \iff \text{for all } y \in S,\ xRy \text{ and } y \Vdash G(A) \text{ imply } y \Vdash A.$$

Proof. In one direction (left to right) this is immediate. In the other direction, suppose $x \nVdash \Box A$, but that the right-hand side (the RHS) holds. Let $x_0 = x$. Then, for some $x_1 \in S$ with $x_0 R x_1$, $x_1 \nVdash A$. By the RHS, $x_1 \nVdash \Box(A \supset \Box A)$, so for some $x_2 \in S$ with $x_1 R x_2$, $x_2 \nVdash A \supset \Box A$, i.e. $x_2 \Vdash A$ but $x_2 \nVdash \Box A$. In particular, $x_2 \neq x_1$, since $x_1 \nVdash A$. Since $x_2 \nVdash \Box A$, there is some $x_3 \in S$ with $x_2 R x_3$, $x_3 \nVdash A$. By transitivity, $x R x_3$. By the RHS, $x_3 \nVdash \Box(A \supset \Box A)$; so for some $x_4 \in S$ with $x_3 R x_4$, $x_4 \nVdash (A \supset \Box A)$, i.e. $x_4 \Vdash A$ but $x_4 \nVdash \Box A$.

[3] Also known as an *R-chain*. The word 'sequence' is used in the sense of being indexed either by the ordered set of all natural numbers less than some fixed natural number or by the ordered set of all natural numbers.

[4] We observe that the same result can be obtained via the equivalence, valid in **Grz**-models $\Box A \equiv \Box(G(A) \supset A)$ (cf., e.g., [8]), but since the purpose here is to motivate the definition of the proof system we prefer to give an independent proof.

In particular, $x_4 \neq x_3$, since $x_3 \not\Vdash A$. The argument can be repeated (using *DC*) to obtain x_5, x_6 with $x_4 R x_5$, $x_5 R x_6$, $x_5 \not\Vdash A$, $x_6 \Vdash A$ but $x_6 \not\Vdash \Box A$ and so on, giving an infinite sequence that cannot converge, since $\forall i \geq 0.\ x_{2i+2} \neq x_{2i+1}$. This contradicts the assumption that R is Noetherian. ■

Different notions of Noetherianity have been considered in the literature on constructive mathematics and a useful survey on their interrelations, in view of applications in algebra, is provided in [32]. Among these variants is the constructively weaker notion due to Richman and Seidenberg stating that for every chain $x_0 R x_1 R x_2 \ldots$ there exists n such that $x_n = x_{n+1}$. The proof above shows (since one has not just $\forall i \geq 0.\ x_{2i+2} \neq x_{2i+1}$ but also $\forall i \geq 0.\ x_{2i} \neq x_{2i+1}$) that Lemma 2.3 can indeed be proved (using *DC*) also with this weaker assumption on the relation R.

The following rules (in the context of labelled sequent calculi such as **G3K** for the modal logic **K**, as presented in [26], in which Γ and Δ are multisets of labelled formulae $x:A$, with possibly also some 'relational atoms' xRy in Γ) are justified by the characterization in Lemma 2.3 of the forcing relation in frames with a transitive Noetherian relation:

$$\frac{xRy, x:\Box A, \Gamma \Rightarrow \Delta, y:G(A) \qquad xRy, x:\Box A, y:A, \Gamma \Rightarrow \Delta}{xRy, x:\Box A, \Gamma \Rightarrow \Delta}\ L\Box Z$$

$$\frac{xRy, y:G(A), \Gamma \Rightarrow \Delta, y:A}{\Gamma \Rightarrow \Delta, x:\Box A}\ R\Box Z \text{ (with } y \text{ fresh)}\ .$$

Rule $R\Box Z$ has the condition that y is *fresh*, i.e. is not in the conclusion. The fresh label in $R\Box Z$ will be called an 'eigenlabel' (also known as an 'eigenvariable' [31]) by analogy with the usage in first-order logic.

There are several alternative possibilities for presentation of **Grz** as a labelled sequent calculus; we defer to Section 4 discussion thereof. Our choice here is to use the $R\Box Z$ just given but the standard $L\Box$ rule; this allows hp-invertibility of all inference rules and hp-admissibility of *Weakening* and of *Contraction*. (For this terminology, see, e.g., [30] (pp. 31, 34) for 'height-preserving' (hp-), or [39] (pp. 76–77), where the equivalent variant 'depth-preserving' (dp-) is used.) Syntactic cut elimination is not proved for the system; on the other hand this formulation of the calculus permits a completeness proof that yields at the same time a semantic proof of admissibility of *Cut*, the finite model property and a constructive decision procedure.

In the following, we shall denote by **G3Grz** the system obtained from **G3K** by replacing rule $R\Box$ by $R\Box Z$, leaving $L\Box$ (from **G3K**) unchanged, and adding the rules *Ref* and *Trans* for reflexivity and transitivity:

$$\frac{xRy, x:\Box A, y:A, \Gamma \Rightarrow \Delta}{xRy, x:\Box A, \Gamma \Rightarrow \Delta}\ L\Box \quad \frac{xRx, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ Ref \quad \frac{xRz, xRy, yRz, \Gamma \Rightarrow \Delta}{xRy, yRz, \Gamma \Rightarrow \Delta}\ Trans\ .$$

The *initial sequents* are those of **G3K**, i.e. those of the form $x:P, \Gamma \Rightarrow \Delta, x:P$ where P is atomic. The notation *size*(A) denotes the size of the formula A, i.e. the number of logical connectives.

3 Structural properties of G3Grz

It is unproblematic to verify that all the preliminary properties proved for **G3K** and its extensions in [26] hold also for **G3Grz**. In particular, we have:

LEMMA 3.1
All sequents of the form $x:A, \Gamma \Rightarrow \Delta, x:A$ are derivable in **G3Grz** with derivation height at most $2 * size(A)$.

LEMMA 3.2
The *Substitution* rule

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma(y/x) \Rightarrow \Delta(y/x)}\ (y/x)$$

is hp-admissible in **G3Grz**.

PROPOSITION 3.3
The rules of *Weakening*

$$\frac{\Gamma \Rightarrow \Delta}{x{:}A, \Gamma \Rightarrow \Delta}\ LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x{:}A}\ RW \qquad \frac{\Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}\ LW_R$$

are hp-admissible in **G3Grz**.

As a consequence of admissibility of *Weakening*, rule $R\Box$ of **G3K** is admissible in **G3Grz**, which is thus an extension of **G3S4**, and admissibility of *Necessitation* is established in a way similar to that in [26]:

COROLLARY 3.4
The rule, where y does not occur in the conclusion,

$$\frac{xRy, \Gamma \Rightarrow \Delta, y{:}A}{\Gamma \Rightarrow \Delta, x{:}\Box A}\ R\Box$$

is admissible in **G3Grz**.

PROPOSITION 3.5
The *Necessitation* rule

$$\frac{\Rightarrow x{:}A}{\Rightarrow x{:}\Box A}\ Nec$$

is admissible in **G3Grz**.

PROPOSITION 3.6
Every sequent of the form

$$\Rightarrow x{:}\Box A \supset A$$

is derivable in **G3Grz**.

PROOF. Routine use of $R\supset$, *Ref*, $L\Box$ and Lemma 3.1. ■

PROPOSITION 3.7
Every sequent of the form

$$\Rightarrow x{:}\Box A \supset \Box\Box A$$

is derivable in **G3Grz**.

PROOF. Routine use of $R\supset$, $R\Box$ (twice), *Trans*, $L\Box$ and Lemma 3.1. ■

Axiomatic presentations of Grzegorczyk logic (such as those in [5, 8, 36]) use the '*Grzegorczyk formula*', namely

$$\Box(\Box(A \supset \Box A) \supset A) \supset A.$$

With the notation we have introduced, it can be rewritten as $\Box(G(A) \supset A) \supset A$. The following results shows the derivability of the corresponding sequents in **G3Grz**.

LEMMA 3.8
All sequents of the form

$$x{:}\Box(G(A)\supset A),\Gamma\Rightarrow\Delta,x{:}G(A)$$

are derivable in **G3Grz**.

PROOF. We have the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{xRz,xRy,yRz,x{:}\Box(G(A)\supset A),y{:}A,z{:}G(A),z{:}G(A)\supset A,\Gamma\Rightarrow\Delta,z{:}A}{xRz,xRy,yRz,x{:}\Box(G(A)\supset A),y{:}A,z{:}G(A),\Gamma\Rightarrow\Delta,z{:}A}\,L\Box}{xRy,yRz,x{:}\Box(G(A)\supset A),y{:}A,z{:}G(A),\Gamma\Rightarrow\Delta,z{:}A}\,Trans}{xRy,x{:}\Box(G(A)\supset A),y{:}A,\Gamma\Rightarrow\Delta,y{:}\Box A}\,R\Box Z}{xRy,x{:}\Box(G(A)\supset A),\Gamma\Rightarrow\Delta,y{:}A\supset\Box A}\,R\supset}{x{:}\Box(G(A)\supset A),\Gamma\Rightarrow\Delta,x{:}G(A)}\,R\Box \quad .$$

with $R\Box$ justified by Corollary 3.4 and the top-sequent derivable by $L\supset$ and Lemma 3.1. ■

PROPOSITION 3.9
The '*Grzegorczyk sequents*', i.e. those of the form

$$\Rightarrow x{:}\Box(G(A)\supset A)\supset A,$$

are derivable in **G3Grz**.

PROOF. We have the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{xRx,x{:}\Box(G(A)\supset A)\Rightarrow x{:}G(A)\qquad xRx,x{:}\Box(G(A)\supset A),x{:}A\Rightarrow x{:}A}{xRx,x{:}\Box(G(A)\supset A),x{:}G(A)\supset A\Rightarrow x{:}A}\,L\supset}{xRx,x{:}\Box(G(A)\supset A)\Rightarrow x{:}A}\,L\Box}{x{:}\Box(G(A)\supset A)\Rightarrow x{:}A}\,Ref}{\Rightarrow x{:}\Box(G(A)\supset A)\supset A}\,R\supset \quad .$$

with top-sequents derivable by Lemmas 3.8 and 3.1, respectively. ■

More simply, we may derive sequents with a variant form of the **Grz** formula, which (in the absence of reflexivity) plays a role in the weak Grzegorczyk logic **wGrz** studied by, e.g., Litak [22]:

PROPOSITION 3.10
All sequents of the form

$$\Rightarrow x{:}\Box(G(A)\supset A)\supset\Box A$$

are derivable in **G3Grz**.

PROOF. We have the derivation

$$\dfrac{\dfrac{\dfrac{xRy,x{:}\Box(G(A)\supset A),y{:}G(A)\supset A,y{:}G(A)\Rightarrow y{:}A}{xRy,x{:}\Box(G(A)\supset A),y{:}G(A)\Rightarrow y{:}A}\,L\Box}{x{:}\Box(G(A)\supset A)\Rightarrow x{:}\Box A}\,R\Box Z}{\Rightarrow x{:}\Box(G(A)\supset A)\supset\Box A}\,R\supset$$

with top-sequent derivable by $L\supset$ and Lemma 3.1. ■

Without loss of generality we now assume that derivations are *pure*, i.e., that each eigenlabel used at a step of $R\Box Z$ appears only in the subtree above that step. Clearly, by hp-admissibility of substitution, such a condition can always be satisfied.

LEMMA 3.11
The following rule is hp-admissible for all labels y:

$$\frac{\Gamma \Rightarrow \Delta, x:\Box A}{xRy, y:G(A), \Gamma \Rightarrow \Delta, y:A}\ .$$

PROOF. By induction on the height n of the derivation of $\Gamma \Rightarrow \Delta, x:\Box A$.

First, suppose $n=0$. Since $x:\Box A$ is not principal in initial sequents, then also $\Gamma \Rightarrow \Delta$ and hence $xRy, y:G(A), \Gamma \Rightarrow \Delta, y:A$ are initial sequents.

Second, suppose $n>0$. If $x:\Box A$ is not principal in the last step, then the conclusion follows by the induction hypothesis, application of the last rule and possibly an hp-admissible substitution. Otherwise, with $x:\Box A$ principal, the final step has as premiss, for some fresh y', the sequent $xRy', y':G(A), \Gamma \Rightarrow \Delta, y':A$. By substituting y for y' we obtain the desired conclusion, using Lemma 3.2.

■

THEOREM 3.12
All the rules of the system **G3Grz** are hp-invertible.

PROOF. For hp-invertibility of the rules for $\wedge$, $\vee$, $\supset$ and $L\Box$ see the argument for Proposition 4.11 of [26]. The hp-invertibility of $R\Box Z$ is the content of Lemma 3.11. ■

We then have:

THEOREM 3.13
The *Contraction* rules

$$\frac{x:A, x:A, \Gamma \Rightarrow \Delta}{x:A, \Gamma \Rightarrow \Delta}\ LC \qquad \frac{\Gamma \Rightarrow \Delta, x:A, x:A}{\Gamma \Rightarrow \Delta, x:A}\ RC \qquad \frac{xRy, xRy, \Gamma \Rightarrow \Delta}{xRy, \Gamma \Rightarrow \Delta}\ LC_R$$

are hp-admissible in **G3Grz**.

PROOF. The hp-admissibility of contraction LC_R on relational atoms xRy is routine.

The other two rules are dealt with by simultaneous induction on the height of the derivation, by case analysis on the last rule applied. For example, when the contracted labelled formula $x:\Box B$ is principal for an application of $R\Box Z$ we have

$$\frac{xRy, y:G(B), \Gamma \Rightarrow \Delta, x:\Box B, y:B}{\Gamma \Rightarrow \Delta, x:\Box B, x:\Box B}\ R\Box Z.$$

By Lemma 3.11, we have a derivation of

$$xRy, xRy, y:G(B), y:G(B), \Gamma \Rightarrow \Delta, y:B, y:B$$

with lower height than the original derivation. By the induction hypothesis we can contract the two occurrences of $y:G(B)$ and, as already shown, relational atoms xRy can be contracted. An application of $R\Box Z$ gives the contracted conclusion $\Gamma \Rightarrow \Delta, x:\Box B$. ■

The following two easy consequences will be useful later:

LEMMA 3.14
The following rule is admissible:

$$\frac{xRy, x:\Box A, y:\Box A, \Gamma \Rightarrow \Delta}{xRy, x:\Box A, \Gamma \Rightarrow \Delta}\ .$$

PROOF. By induction on the derivation of the premiss. If the last step has $y:\Box A$ principal, using some atom yRz in Γ, the premiss is

$$xRy, x:\Box A, y:\Box A, z:A, \Gamma \Rightarrow \Delta$$

to which we apply the induction hypothesis to obtain

$$xRy, x:\Box A, z:A, \Gamma \Rightarrow \Delta,$$

which we weaken with xRz, apply $L\Box$ on $x:\Box A$ using xRz to obtain

$$xRy, xRz, x:\Box A, \Gamma \Rightarrow \Delta$$

from which we remove xRz by *Trans*. Otherwise we just use the induction hypothesis. ∎

LEMMA 3.15
The following rule is admissible:

$$\frac{xRy, x:G(A), \Gamma \Rightarrow \Delta, y:A, y:\Box A}{xRy, x:G(A), \Gamma \Rightarrow \Delta, y:A}\ .$$

PROOF. By Theorem 3.12, from the premiss we can obtain

$$xRy, yRy, x:G(A), y:G(A), \Gamma \Rightarrow \Delta, y:A, y:A.$$

We remove yRy by *Ref*, use *Contraction* on $y:A$ and obtain

$$xRy, x:G(A), y:G(A), \Gamma \Rightarrow \Delta, y:A$$

from which we remove $y:G(A)$ by appeal to Lemma 3.14 (since $G(A)$ is of the form $\Box B$). ∎

4 Other formulations

We discuss here other possible formulations of Grz as a labelled sequent calculus. Each of the various possibilities is suitable for a different purpose.

First, there is the possibility of using, along with $R\Box Z$, the rule $L\Box Z$ presented earlier; this gives a harmonious system, i.e. one with, for each connective, a left and a right rule justified by the same semantic explanation, of which the importance has often been stressed in the literature on Gentzen systems. One of the reasons for this choice is that, when different explanations are used for the left and the right rules, syntactic cut elimination can be lost, or at least the standard reductions fail. This was the second author's approach in [26] for **GL**, which included a syntactic cut-elimination proof. However, one then has to include as initial sequents those where the principal formulae are of the form $\Box A$ (which ensures the derivability of the first premiss of the $L\Box Z$ rule), and the hp-invertibility of rules is lost.

A second variation is to use not a reflexive relation R but an non-reflexive one (which we write as $<$), with two premisses for the right rule for $\Box$ and a corresponding split of the $L\Box$ (or $L\Box Z$) rule into two rules. Thus, with the rule *Trans* as before but the rule *Ref* discarded, the changed logical rules would be:

$$\frac{x<y,\Gamma,y:G(A)\Rightarrow\Delta,y:A \quad \Gamma,x:G(A)\Rightarrow\Delta,x:A}{\Gamma\Rightarrow\Delta,x:\Box A}\; R\Box' \text{ (with } y \text{ fresh)}$$

$$\frac{x<y,x:\Box A,y:A,\Gamma\Rightarrow\Delta}{x<y,x:\Box A,\Gamma\Rightarrow\Delta}\; L\Box_1 \qquad \frac{x:\Box A,x:A,\Gamma\Rightarrow\Delta}{x:\Box A,\Gamma\Rightarrow\Delta}\; L\Box_2.$$

This approach appears to be promising for a proof of the faithfulness of the embedding of **Grz** into **GL**, but with a more complicated meta-theory.

5 Soundness and completeness

Instead of proving admissibility of *Cut* syntactically, we proceed by showing that the calculus **G3Grz** is sound and complete; we shall prove that derivable sequents are valid in reflexive and transitive Noetherian frames and that for any sequent in the language of **Grz** either a proof in the calculus or a countermodel on a reflexive and transitive Noetherian frame can be found.

We start with the definitions of interpretation in a frame, of truth and of validity (from [31]) adapted to the case of **Grz**:

Definition 5.1
Let K be a frame with a reflexive, transitive and Noetherian accessibility relation $\mathcal{R}$. Let W be the set of labels used in derivations in **G3Grz**. An *interpretation* of the labels in the frame K is a function $[[\cdot]]:W\to K$. A *valuation* of atomic formulas in the frame K is a function $\mathcal{V}:AtFrm\to\mathcal{P}(K)$ that assigns to each atom P a set of nodes of K, i.e. 'the set of nodes at which P holds'; the standard notation for $k\in\mathcal{V}(P)$ is $k\Vdash P$, read as 'P holds at k'.

Valuations are extended to arbitrary formulas by the following inductive clauses:

$k\Vdash\bot$ for no k,
$k\Vdash A\wedge B$ if $k\Vdash A$ and $k\Vdash B$,
$k\Vdash A\vee B$ if $k\Vdash A$ or $k\Vdash B$,
$k\Vdash A\supset B$ if from $k\Vdash A$ follows $k\Vdash B$,
$k\Vdash\Box A$ if, for all k', from $k\mathcal{R}k'$ follows $k'\Vdash A$.

Definition 5.2
A sequent $\Gamma\Rightarrow\Delta$ is *true* for an interpretation $[[\cdot]]$ and a valuation $\mathcal{V}$ in the frame $(K,\mathcal{R})$ if, whenever for all labelled formulas $x:A$ and relational atoms yRz in Γ it is the case that $[[x]]\Vdash A$ and $[[y]]\mathcal{R}[[z]]$, then, for some $w:B$ in Δ, $[[w]]\Vdash B$. A sequent is *valid* in a frame if it is true for every interpretation and every valuation in the frame.

Theorem 5.3
If the sequent $\Gamma\Rightarrow\Delta$ is derivable in **G3Grz**, then it is valid in every reflexive, transitive and Noetherian frame.

Proof. Let $(K,\mathcal{R})$ be such a frame. We argue by induction on the derivation of $\Gamma\Rightarrow\Delta$ in **G3Grz**. All the cases are similar to those for extensions of the basic modal system **G3K** considered in [31] except for the rule $R\Box Z$ specific to the system.

If $\Gamma \Rightarrow \Delta$ is the conclusion of $R\Box Z$ from the premiss $xRy, y:G(A), \Gamma \Rightarrow \Delta', y:A$, where y is fresh, assume as induction hypothesis that the premiss is true for every interpretation and some valuation $\mathcal{V}$ in K. Consider an arbitrary interpretation $[[\cdot]]$; suppose that $[[\cdot]]$ and $\mathcal{V}$ make true all members of Γ but no members of Δ'. Let k be an arbitrary member of K and $[[\cdot]]'$ the interpretation that is like $[[\cdot]]$ but with $[[y]]=k$. By the induction hypothesis, specialized to $[[\cdot]]'$, and the freshness of y, one sees that $[[x]]\mathcal{R}k$ and $k \Vdash G(A)$ imply $k \Vdash A$; since that holds for all $k \in K$, by Lemma 2.3 one concludes that $x:\Box A$ is true for $[[\cdot]]$ and $\mathcal{V}$. The validity of the premiss therefore implies that of the conclusion. ■

THEOREM 5.4
Let $\Gamma \Rightarrow \Delta$ be a sequent in the language of **G3Grz**. Then it is decidable whether the sequent is derivable in **G3Grz**. In the negative case, the failed proof search gives a countermodel to the sequent on a reflexive, transitive and Noetherian frame.

PROOF. We use an adaptation to labelled sequents of the method of *reduction trees* detailed for Gentzen's LK by Takeuti (cf. [38] ch. 1, pf. 8) and in turn due to Schütte [35]. For an arbitrary sequent $\Gamma \Rightarrow \Delta$ in the language of **G3Grz** we apply, whenever possible, root-first the rules of **G3Grz**, in a given order. The procedure will construct either a derivation in **G3Grz** or a countermodel. The proof is similar to the proof of Theorem 11.28 in [31] and therefore some common details will be omitted. We stress however one difference: rather than constructing a countermodel on an infinite branch, we shall construct it on an appropriately pruned branch.

1. Construction of the reduction tree: The reduction tree is defined inductively in stages as follows: Stage 0 has $\Gamma \Rightarrow \Delta$ at the root of the tree. For each branch, stage $n>0$ has two cases:

Case I: If the top-sequent is either an initial sequent or has some $x:A$, not necessarily atomic, on both left and right, or is a conclusion of $L\bot$, the construction of the branch ends.

Case II: Otherwise we continue the construction of the branch by writing, above its top-sequent, other sequents that are obtained by applying root-first the rules of **G3Grz** whenever possible, in a given order and under suitable conditions.

There are 10 different stages: 8 for the rules of the basic modal systems, 2 for *Ref* and *Trans*. At stage $n=10+1$ we repeat stage 1, at stage $n=10+2$ we repeat stage 2, and so on for every n until an initial sequent, or a conclusion of $L\bot$, or a saturated branch (defined below) is found.

The stages for the propositional rules and for $L\Box$ are similar to those in the cited Theorem 11.28 of [31]. Note that the propositional rules discard the principal formula but $L\Box$ retains it; all such formulae however are available somewhere on the branch for when we need to discuss the countermodel construction.

For the stage relative to $R\Box Z$, we consider all labelled formulas of the form $x:\Box B$ in the succedent. If the succedent of the top-sequent contains both $x:\Box B$ and $x:B$, and the antecedent contains, for some x_0, both x_0Rx and $x_0:G(B)$, we do not further analyse $x:\Box B$; this is justified by Lemma 3.15. More generally, if $x:B$ is in the succedent of any sequent on the branch, we do the same. For each of the remaining labelled boxed formulas $x_i:\Box B_i$, $i=1,\dots,m$, we apply several times (indicated by the superfix $*$) the rule $R\Box Z$, that is, we construct the step

$$\frac{x_1Ry_1,\dots,x_mRy_m, y_1:G(B_1),\dots,y_m:G(B_m),\Gamma \Rightarrow \Delta, y_1:B_1,\dots,y_m:B_m}{\Gamma \Rightarrow \Delta, x_1:\Box B_1,\dots,x_m:\Box B_m}\ R\Box Z^*,$$

where $y_1,\dots,y_m$ are fresh variables.

Finally, for $n=9,10$, we consider the cases of the frame rules *Ref* and *Trans*. By an easy adaptation of the argument detailed in Section 8 of [11], it is enough to instantiate *Ref* only on terms in the top-sequent.

Observe also that, because of height-preserving admissibility of contraction, once a rule has been considered, it need not be instantiated again on the same principal formulas (for $L\Box$ such principal formulas are pairs of the form $xRy, x:\Box B$) and it need not be applied whenever its application produces a duplication of labelled formulas or relational atoms.

To show that the procedure terminates, it is enough to show that every branch in the reduction tree for a sequent $\Gamma \Rightarrow \Delta$ is finite. Every branch contains one or more chains of labels $x_1Ry_1, \ldots, x_mRy_m \ldots$; each label that was not already in the endsequent is introduced by a step of $R\Box Z$. By inspection of the rules of **G3Grz**, it is clear that all the formulas that occur in the branch are subformulas of Γ, Δ or formulas of the form $\Box(A \supset \Box A)$ or of the form $A \supset \Box A$ for some subformula $\Box A$ of Γ, Δ. To ensure that all proper chains of labels in the reduction tree are finite, it is therefore enough to prove the following statement

Rule $R\Box Z$ cannot be applied twice to the same formula along a chain of labels.

This done, we can conclude that all the chains of labels in the tree are finite. To conclude that the branch is finite, it is enough to observe that it contains only a finite number of such chains (the number of chains is bounded by a function of the number of disjunctions or commas in the positive part of the endsequent; observe that this argument would break down in the labelled calculus for intuitionistic logic because here we rely on the fact that propositional rules have premisses in which the active formulas are strictly simpler than the principal formula).

To prove the above statement, suppose, e.g., that we have a derivation that contains the following steps (in which recall that $G(A) \equiv \Box(A \supset \Box A)$):

$$\dfrac{\begin{array}{c} xRy, yRz, xRz, y:G(A), \Gamma'' \Rightarrow \Delta'', z:\Box A \\ \vdots \\ xRy, y:G(A), \Gamma' \Rightarrow \Delta', y:A \end{array}}{\Gamma' \Rightarrow \Delta', x:\Box A}\ R\Box Z$$

and is closed under all the available rules (excluding $R\Box Z$) of the reduction procedure. Then, by the closure properties for $L\Box$ (operating on yRz and $y:\Box(A \supset \Box A)$) and $L\supset$, we have that either $z:\Box A$ is in Γ'' (in which case the top-sequent is initial) or $z:A$ is in Δ'' (or in the succedent somewhere below), in which case (since also yRz and $y:G(A)$ are in the antecedent) extension by $R\Box Z$ is blocked. Therefore the application of $R\Box Z$ to $z:\Box A$ is blocked by definition of the reduction tree. The general case, where the chain is longer than just xRy, yRz, is similar.

A branch which either ends in an initial sequent or in a sequent with the same labelled formula, even compound, in both the antecedent and succedent, or at the conclusion of $L\bot$, or has a top-sequent amenable to any of the reduction steps, is called *unsaturated*. Every other branch is said to be *saturated*.

2. Construction of the countermodel: If the reduction tree for $\Gamma \Rightarrow \Delta$ is not a derivation, it has at least one saturated branch. Let $\Gamma^* \Rightarrow \Delta^*$ be the union (respectively, of the antecedents and succedents) of all the sequents $\Gamma_i \Rightarrow \Delta_i$ of the branch up to its top-sequent. We define a Kripke model that forces all the formulas in Γ^* and no formula in Δ^* and is therefore a countermodel to the sequent $\Gamma \Rightarrow \Delta$.

Consider the frame K, the nodes of which are the labels that appear in the relational atoms in Γ^* and the order on which is given by these relational atoms. Clearly, the construction of the reduction tree imposes the frame properties on the countermodel: *Ref* and *Trans* hold because the branch is saturated. Morever, any label that appears in the sequent will appear in a relational atom (and thus in the frame K), because the rule *Ref* has been applied. Noetherianity clearly holds because all the strictly ascending chains in the countermodel are finite by construction.

The model is defined as follows. First, the interpretation $[[x]]$ of each label x is just x itself. As for the valuation, for each labelled atomic formula $x:P$ in Γ^* we stipulate that $x \Vdash P$. Since the top-sequent is not initial, for all labelled atomic formulas $y:Q$ in Δ^* we infer that $y \nVdash Q$.

We then show by induction on *size*(A) that $x \Vdash A$ if $x:A$ is in Γ^* and that $x \nVdash A$ if $x:A$ is in Δ^*. Therefore we have a countermodel to the endsequent $\Gamma \Rightarrow \Delta$.

If A is $\bot$, it cannot be in Γ^*, by definition of saturated branch: so $x \nVdash A$.

If A is atomic, the claim holds by the definition of the model.

If $x:A \equiv x:B \wedge C$ is in Γ^*, then by the saturation of the branch we also have $x:B$ and $x:C$ in Γ^*. By the induction hypothesis, $x \Vdash B$ and $x \Vdash C$, and therefore $x \Vdash B \wedge C$.

If $x:A \equiv x:B \wedge C$ is in Δ^*, then by the saturation of the branch either $x:B$ or $x:C$ is in Δ^*, and therefore by the induction hypothesis, $x \nVdash B$ or $x \nVdash C$, and therefore $x \nVdash B \wedge C$.

If $x:A \equiv x:B \vee C$ is in Γ^*, we argue as with $x:A \equiv x:B \wedge C$ in Δ^*.

If $x:A \equiv x:B \vee C$ is in Δ^*, we argue as with $x:A \equiv x:B \vee C$ in Γ^*.

If $x:A \equiv x:B \supset C$ is in Γ^*, then, by saturation, either $x:B$ is in Δ^* or $x:C$ is in Γ^*. By the induction hypothesis, in the former case $x \nVdash B$, and in the latter $x \Vdash C$, so in both cases $x \Vdash B \supset C$.

If $x:A \equiv x:B \supset C$ is in Δ^*, then $x:B$ is in Γ^* and $x:C$ is in Δ^*. By the induction hypothesis $x \Vdash B$ and $x \nVdash C$, so $x \nVdash B \supset C$.

If $x:A \equiv x:\Box B$ is in Γ^*, for any occurrence of xRy in Γ^* we find, by the construction of the reduction tree, an occurrence of $y:B$ in Γ^*. By the induction hypothesis, $y \Vdash B$, and therefore $x \Vdash \Box B$.

If $x:A \equiv x:\Box B$ is in Δ^*, we consider the step where it is analysed. If $x:B$ is in the succedent of that step (or any succedent below it), then by the induction hypothesis $x \nVdash B$. Since xRx is also in Γ^* by construction of the reduction tree, it follows that $x \nVdash A$. Otherwise there is xRy in Γ^* and $y:B$ in Δ^*. By the induction hypothesis $y \nVdash B$, and therefore $x \nVdash A$. ■

COROLLARY 5.5
If a sequent $\Gamma \Rightarrow \Delta$ is valid in every reflexive, transitive and Noetherian frame, then it is derivable in **G3Grz**.

COROLLARY 5.6
A formula A is provable in **Grz** iff, for some (or any) label x, the sequent $\Rightarrow x:A$ is derivable in **G3Grz**.

PROOF. By 'provable in **Grz**' we mean 'provable in an axiomatic Hilbert-style system for **K** with the **Grz** formulae as axioms', and we take it as well-known [5] that this is equivalent to validity in all reflexive, transitive and Noetherian frames. So Theorems 5.3 and 5.4 extend this well-known result to relate **Grz**-provability to **G3Grz**-derivability. ■

We observe that completeness implies in particular closure of our sequent calculus with respect to *Cut*, so we have an indirect proof of admissibility of the *Cut* rule. The proof of Theorem 5.4 is also of interest because it establishes the finite model property for **Grz** and gives a constructive decision procedure for it, i.e. an algorithm that, given a sequent, constructs either a derivation or a countermodel.

6 Embedding of intuitionistic logic into Grzegorczyk logic

To make this article more self-contained, we take from [11] the main points of the labelled sequent calculus **G3I** for **Int**. We use $\leqslant$ rather than the relation symbol R, both in **G3I** and in **G3Grz**. The rules *Ref* and *Trans* given above for **G3Grz** are used (with this change) in **G3I** as well.

Initial sequents are those of the form $x \leqslant y,\, x{:}P,\, \Gamma \Rightarrow \Delta,\, y{:}P$; recall that P is a metavariable ranging over atomic formulae. The rules for $\wedge$ and $\vee$ are exactly the same for **G3I** as for **G3Grz**—but we give them here again for completeness. The logical rules are as follows (Table 1):

TABLE 1. Logical rules of the system **G3I**

$$\frac{}{x{:}\bot,\Gamma \Rightarrow \Delta}\; L\bot$$

$$\frac{x{:}A, x{:}B, \Gamma \Rightarrow \Delta}{x{:}A\wedge B, \Gamma \Rightarrow \Delta}\; L\wedge \qquad \frac{\Gamma \Rightarrow \Delta, x{:}A \quad \Gamma \Rightarrow \Delta, x{:}B}{\Gamma \Rightarrow \Delta, x{:}A\wedge B}\; R\wedge$$

$$\frac{x{:}A, \Gamma \Rightarrow \Delta \quad x{:}B, \Gamma \Rightarrow \Delta}{x{:}A\vee B, \Gamma \Rightarrow \Delta}\; L\vee \qquad \frac{\Gamma \Rightarrow \Delta, x{:}A, x{:}B}{\Gamma \Rightarrow \Delta, x{:}A\vee B}\; R\vee$$

$$\frac{x \leqslant y, x{:}A\supset B, \Gamma \Rightarrow \Delta, y{:}A \qquad x \leqslant y, x{:}A\supset B, y{:}B, \Gamma \Rightarrow \Delta}{x \leqslant y, x{:}A\supset B, \Gamma \Rightarrow \Delta}\; L\supset$$

$$\frac{x \leqslant y, y{:}A, \Gamma \Rightarrow \Delta, y{:}B}{\Gamma \Rightarrow \Delta, x{:}A\supset B}\; R\supset \text{ (with } y \text{ fresh)}$$

The main results [11] about **G3I** are that a formula A is provable in **Int** (e.g. in Heyting's calculus) iff for some (or any) x the sequent $\Rightarrow x{:}A$ is derivable in **G3I**; that *Weakening*, *Contraction* and *Cut* are admissible in **G3I**, and that all the logical rules are invertible.

The translation $\cdot^{\Box}$ from formulae of **Int** to **Grz** (as to **S4**) has $P^{\Box} = \Box P$, $\bot^{\Box} = \bot$, $(A\wedge B)^{\Box} = A^{\Box} \wedge B^{\Box}$, $(A\vee B)^{\Box} = A^{\Box} \vee B^{\Box}$ and $(A\supset B)^{\Box} = \Box(A^{\Box} \supset B^{\Box})$. Routinely, it determines a translation from sequents of **G3I** to **G3Grz**, as also [11] to **G3S4**: formulae are translated, labels are unchanged, and relational atoms $x \leqslant y$ are unchanged. So, if Ψ is a multiset of labelled formulae, then $\Psi^{\Box}$ is the result of applying the translation to all formulae in Ψ.

The translation is *sound*, i.e. if a sequent is derivable in **G3I** then its translation is derivable in **G3Grz**. This follows routinely from soundness of the translation of **G3I** into **G3S4** ([11]) because, by Corollary 3.4, the $R\Box$ rule of **G3S4** is admissible in **G3Grz** and therefore **G3S4** is a subsystem of **G3Grz** .

The main content of this section is that a faithfulness result can be proved for **G3Grz** just as for **G3S4**. The proof is complicated by the fact that in the calculus **G3Grz** the $R\Box$ rule of **G3S4** is replaced by $R\Box Z$, with extra antecedent formulae $G(A)$. The analogue for **G3Grz** of Lemma 4 of [11] is the following:

LEMMA 6.1
Suppose

1. Γ, Δ are multisets of labelled formulas from **Int**, possibly with relational atoms also in Γ;
2. Γ', Δ' are multisets of labelled atomic formulas;
3. Θ is a multiset of labelled formulae of the form $G(P)$ or $G(A^{\Box} \supset B^{\Box})$;

and that

$$\mathbf{G3Grz} \vdash \Gamma^{\Box}, \Gamma', \Theta \Rightarrow \Delta^{\Box}, \Delta'.$$

Then it follows that, if Θ^* is obtained from Θ by discarding every labelled $G(P)$ and (while retaining the label) replacing each $G(A^\Box \supset B^\Box)$ by A, then

$$\mathbf{G3I} \vdash \Gamma, \Gamma', \Theta^* \Rightarrow \Delta, \Delta'.$$

PROOF. By induction on the derivation of $\Gamma^\Box, \Gamma', \Theta \Rightarrow \Delta^\Box, \Delta'$.

If it is an initial sequent, then some labelled formula $x{:}P$ is both in Γ' and in Δ'; the conclusion then follows by *Ref* from the **G3I** initial sequent

$$x \leqslant x, \Gamma, \Gamma', \Theta^* \Rightarrow \Delta, \Delta'.$$

If it is a conclusion of $L\bot$, so also is $\Gamma, \Gamma', \Theta^* \Rightarrow \Delta, \Delta'$. If it is derived by a **G3Grz** rule for $\wedge$ or for $\vee$, the induction hypothesis applies to the premisses and then the corresponding rule in **G3I** gives the conclusion.

If it is derived by a modal rule, and the principal formula occurrence is in $\Gamma^\Box$ or in $\Delta^\Box$, then the principal formula, being a translated formula, can only be of the form $\Box P$ or of the form $\Box(A^\Box \supset B^\Box)$. There are thus four cases:

1. If $\Box P$ is principal on the left we have (with $\Gamma = x \leqslant y, x{:}P, \Gamma''$)

$$\frac{x \leqslant y, y{:}P, x{:}\Box P, \Gamma''^\Box, \Gamma', \Theta \Rightarrow \Delta^\Box, \Delta'}{x \leqslant y, x{:}\Box P, \Gamma''^\Box, \Gamma', \Theta \Rightarrow \Delta^\Box, \Delta'}\ L\Box,$$

which is (by the induction hypothesis) translated to the admissible **G3I** step

$$\frac{x \leqslant y, y{:}P, x{:}P, \Gamma'', \Gamma', \Theta^* \Rightarrow \Delta, \Delta'}{x \leqslant y, x{:}P, \Gamma'', \Gamma', \Theta^* \Rightarrow \Delta, \Delta'}\ .$$

2. If $\Box P$ is principal on the right, we have the step (with y fresh and $\Delta = \Delta'', x{:}P$)

$$\frac{x \leqslant y, \Gamma^\Box, \Gamma', \Theta, y{:}G(P) \Rightarrow \Delta''^\Box, \Delta', y{:}P}{\Gamma^\Box, \Gamma', \Theta \Rightarrow \Delta''^\Box, \Delta', x{:}\Box P}\ R\Box Z$$

from the premiss of which, by the induction hypothesis, we obtain a **G3I** derivation of

$$x \leqslant y, \Gamma, \Gamma', \Theta^* \Rightarrow \Delta'', \Delta', y{:}P;$$

since y is fresh, we can substitute x for y and then use *Ref* to remove $x \leqslant x$ and obtain

$$\Gamma, \Gamma', \Theta^* \Rightarrow \Delta'', \Delta', x{:}P.$$

3. If $\Box(A^\Box \supset B^\Box)$ is principal on the left, we have (with $\Gamma = x \leqslant y, x{:}A \supset B, \Gamma''$) the step

$$\frac{x \leqslant y, x{:}\Box(A^\Box \supset B^\Box), y{:}A^\Box \supset B^\Box, \Gamma''^\Box, \Gamma', \Theta \Rightarrow \Delta^\Box, \Delta'}{x \leqslant y, x{:}\Box(A^\Box \supset B^\Box), \Gamma''^\Box, \Gamma', \Theta \Rightarrow \Delta^\Box, \Delta'}\ L\Box$$

from which, by hp-invertibility of $L\supset$ in **G3Grz**, we get derivations in **G3Grz** of the sequents

$$x \leqslant y, x{:}\Box(A^\Box \supset B^\Box), \Gamma''^\Box, \Gamma', \Theta \Rightarrow \Delta^\Box, \Delta', y{:}A^\Box$$

and

$$x \leqslant y, x{:}\Box(A^{\Box} \supset B^{\Box}), y{:}B^{\Box}, \Gamma''^{\Box}, \Gamma', \Theta \Rightarrow \Delta^{\Box}, \Delta'.$$

to which the induction hypothesis applies. This gives derivations in **G3I** of the sequents

$$x \leqslant y, x{:}A \supset B, \Gamma'', \Gamma', \Theta^* \Rightarrow \Delta, \Delta', y{:}A$$

and

$$x \leqslant y, x{:}A \supset B, y{:}B, \Gamma'', \Gamma', \Theta^* \Rightarrow \Delta, \Delta'$$

from which the conclusion follows by a step of $L\supset$ in **G3I**.

4. If the formula $\Box(A^{\Box} \supset B^{\Box})$ is principal on the right, then the step is (with y fresh, and with $\Delta = \Delta'', x{:}A \supset B$):

$$\frac{x \leqslant y, \Gamma^{\Box}, \Gamma', \Theta, y{:}G(A^{\Box} \supset B^{\Box}) \Rightarrow \Delta''^{\Box}, \Delta', y{:}A^{\Box} \supset B^{\Box}}{\Gamma^{\Box}, \Gamma', \Theta \Rightarrow \Delta''^{\Box}, \Delta', x{:}\Box(A^{\Box} \supset B^{\Box})}\ R\Box Z$$

from the premiss of which, by hp-invertibility of $R\supset$ in **G3Grz**, we have a derivation of

$$x \leqslant y, \Gamma^{\Box}, \Gamma', \Theta, y{:}A^{\Box}, y{:}G(A^{\Box} \supset B^{\Box}) \Rightarrow \Delta''^{\Box}, \Delta', y{:}B^{\Box}$$

to which the induction hypothesis applies. This gives us a derivation in **G3I** of

$$x \leqslant y, \Gamma, \Gamma', \Theta^*, y{:}A, y{:}A \Rightarrow \Delta'', \Delta', y{:}B$$

and thus also (using a contraction) of

$$x \leqslant y, \Gamma, \Gamma', \Theta^*, y{:}A \Rightarrow \Delta'', \Delta', y{:}B;$$

an $R\supset$ step (using freshness of y) in **G3I** gives us the desired conclusion.

Finally, where the principal formula is in Θ, we have a case not encountered in the **G3S4** proof. Such a formula is either of the form $G(P)$ or of the form $G(A^{\Box} \supset B^{\Box})$, so there are two cases:

1. The labelled formula $x{:}G(P)$ in Θ is principal, and the last step is thus

$$\frac{x \leqslant y, \Gamma''^{\Box}, \Gamma', \Theta,\ y{:}P \supset \Box P \Rightarrow \Delta^{\Box}, \Delta'}{x \leqslant y, \Gamma''^{\Box}, \Gamma', \Theta \Rightarrow \Delta^{\Box}, \Delta'}\ L\Box$$

from the premiss of which, by hp-invertibility of $L\supset$ in **G3Grz**, we obtain derivations of

$$x \leqslant y, \Gamma''^{\Box}, \Gamma', \Theta \Rightarrow \Delta^{\Box}, \Delta', y{:}P$$

and

$$x \leqslant y, \Gamma''^{\Box}, \Gamma', \Theta, y{:}\Box P \Rightarrow \Delta^{\Box}, \Delta'$$

to which the induction hypothesis applies. This gives us **G3I** derivations of

$$x \leqslant y, \Gamma'', \Gamma', \Theta^* \Rightarrow \Delta, \Delta', y{:}P$$

and

$$x \leqslant y, \Gamma'', \Gamma', \Theta^*, y{:}P \Rightarrow \Delta, \Delta'$$

from which the desired conclusion $x \leqslant y, \Gamma'', \Gamma', \Theta^* \Rightarrow \Delta, \Delta'$ is obtained by *Cut*.

2. The labelled formula $x{:}G(A^{\Box}\supset B^{\Box})$ in Θ is principal, and the last step is thus

$$\frac{x\leqslant y,\Gamma''^{\Box},\Gamma',\Theta,\ y{:}(A^{\Box}\supset B^{\Box})\supset(\Box(A^{\Box}\supset B^{\Box}))\Rightarrow\Delta^{\Box},\Delta'}{x\leqslant y,\Gamma''^{\Box},\Gamma',\Theta\Rightarrow\Delta^{\Box},\Delta'}\ L\Box .$$

From this, two steps are required. First, using the hp-invertibility of $L\supset$ in **G3Grz**, we obtain a derivation of

$$x\leqslant y,\Gamma''^{\Box},\Gamma',\Theta,\ y{:}\Box(A^{\Box}\supset B^{\Box})\Rightarrow\Delta^{\Box},\Delta'$$

to which the induction hypothesis applies: we thus obtain a **G3I**-derivation of

$$\boxed{x\leqslant y,\Gamma'',\Gamma',\Theta^{*},\ y{:}A\supset B\Rightarrow\Delta,\Delta'}\ .$$

Second, using the hp-invertibility of $L\supset$ again and then that of $R\supset$, we obtain a derivation of

$$x\leqslant y,\Gamma''^{\Box},\Gamma',\Theta,y{:}A^{\Box}\Rightarrow\Delta^{\Box},\Delta',y{:}B^{\Box}$$

to which the induction hypothesis applies. We thus obtain a **G3I**-derivation of

$$\boxed{x\leqslant y,\Gamma'',\Gamma',\Theta^{*},y{:}A\Rightarrow\Delta,\Delta',y{:}B}\ .$$

Since $x{:}G(A^{\Box}\supset B^{\Box})$ is in Θ, $x{:}A$ is in Θ^{*}, so monotonicity in **G3I** gives us a **G3I**-derivation of

$$\boxed{x\leqslant y,\Gamma'',\Gamma',\Theta^{*}\Rightarrow y{:}A}$$

and we also have easily a **G3I**-derivation of

$$\boxed{y{:}B\Rightarrow y{:}A\supset B}\ .$$

Using the four sequents displayed in boxes, cuts on $y{:}A$, $y{:}B$ and $y{:}A\supset B$ and some contractions now give us, as required, a **G3I**-derivation of

$$x\leqslant y,\Gamma'',\Gamma',\Theta^{*}\Rightarrow\Delta,\Delta'.$$

■

THEOREM 6.2
If a sequent $\Gamma\Rightarrow\Delta$ of **G3I** has its translation $\Gamma^{\Box}\Rightarrow\Delta^{\Box}$ derivable in **G3Grz**, then the sequent is derivable in **G3I**.

PROOF. By application of Lemma 6.1, with Γ', Δ' and Θ empty. ■

We have thus proved faithfulness of the translation $\cdot^{\Box}$ (now to be called an 'embedding') of **G3I** into **G3Grz**, and thus of **Int** into **Grz**.

COROLLARY 6.3
The embedding of **G3I** into **G3Grz** gives a constructive decision procedure for **G3I** and thus for **Int**.

PROOF. Take a sequent $\Gamma\Rightarrow\Delta$ to be proved or refuted in **G3I**, translate it to $\Gamma^{\Box}\Rightarrow\Delta^{\Box}$, and construct its reduction tree for the rules of **G3Grz**, that is, apply the constructive decision procedure established by Theorem 5.4 for **G3Grz**. If a derivation is obtained, then by Lemma 6.1 this can be translated back to a proof in **G3I**, else the sequent is not provable and a finite countermodel is found. It is routine to show that a countermodel to a translated sequent of **Int** is a counter-model to the original sequent. ■

7 Conclusion

After a talk in Padova by the second author, Giovanni Sambin asked whether a labelled sequent calculus can be used to obtain a syntactic proof of the faithfulness of the embedding of intuitionistic logic **Int** into the Gödel-Löb provability logic **GL**. The motivation for looking into the embedding was to exploit the good meta-theoretic properties of **GL** for obtaining a constructive decision procedure for **Int**. Here we have answered a related question by giving a simple syntactic proof of the faithfulness of the embedding of **Int** into the provability logic **Grz**. Our answer leads to the same desired consequence. A similar syntactic proof of the faithfulness of the embedding into **GL** seems harder to establish because the characterizing frames for **GL** are, unlike those for **Int** and **Grz**, irreflexive. One may however observe that the decision procedure for **GL** in [34] (using an unlabelled calculus) is terminating without any loop-checking, and this gives a decision procedure for **Int** with a similar property, albeit less efficient than that of Vorob'ev (for details and variations of which see [10]).

References

[1] A. Avron. On modal systems having arithmetical interpretations. *Journal of Symbolic Logic*, **49**, 935–942, 1984.

[2] G. Boolos. On systems of modal logic with provability interpretations. *Theoria*, **46**, 7–18, 1980.

[3] G. Boolos. Provability in arithmetic and a schema of Grzegorczyk. *Fundamenta Mathematicae*, **106**, 41–45, 1980.

[4] G. Boolos. Provability, truth, and modal logic. *Journal of Philosophical Logic*, **9**, 1–7, 1980.

[5] G. Boolos. *The Logic of Provability*. Cambridge University Press, 1993.

[6] G. Boolos. *The Unprovability of Consistency*. Cambridge University Press, 1979; 2nd edn., 2008.

[7] M. Borga and P. Gentilini. On the proof theory of the modal logic **Grz**. *Zeitschrift f. math, Logik und Grundlagen d. Math*, **32**, 145–148, 1986.

[8] A. Chagrov and M. Zakharyaschev. *Modal Logic*. Oxford University Press, 1997.

[9] M. A. E. Dummett. and E. J. Lemmon. Modal logics between S4 and S5. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, **5**, 250–264, 1959.

[10] R. Dyckhoff. Contraction-free sequent calculi for intuitionistic logic. *Journal of Symbolic Logic*, **57**, 795–807, 1992.

[11] R. Dyckhoff, and S. Negri. Proof analysis in intermediate propositional logics. *Archive for Mathematical Logic*, **51**, 71–92, 2012.

[12] L. Esakia. On modal counterparts of superintuitionistic logics. In *The Seventh All-Union Symposium on Logic and Methodology of Science*, Abstracts (in Russian), Kiev, pp. 135–136, 1976.

[13] K. Gödel. Eine Interpretation des intuitionistichen Aussagenkalküls. *Ergebnisse eines mathematischen Kolloquiums*, **4**, 39–40, (English translation in [14]), 1933.

[14] K. Gödel. *Collected Works*, vol. 1, pp. 301–303. Oxford University Press, 1986.

[15] R. Goldblatt. Arithmetical necessity, provability and intuitionistic logic. *Theoria*, **44**, 38–46, 1978.

[16] A. Grzegorczyk. Relational systems and the associated topological spaces. *Fundamenta Mathematicae*, **60**, 223–231, 1967.

[17] E. Jeřábek. A note on Grzegorczyk's logic. *Mathematical Logic Quarterly*, **50**, 295–296, 2004.

[18] A. V. Kuznetsov and A. Yu. Muravitsky. The logic of provability. In *The 4th All-Union Conference on Mathematical Logic*, Abstracts (in Russian), p. 73, 1976.
[19] A. V. Kuznetsov and A. Yu. Muravitsky. Magari algebras. In *The 14th All-Union Conference on Algebra*, Abstracts (in Russian), Part 2, pp. 105–106, 1977.
[20] A. V. Kuznetsov and A. Yu. Muravitsky. Provability as modality. In *Current Problems in Logic and Methodology of Sciences*, (in Russian), Kiev, pp. 193–230, 1980.
[21] A. V. Kuznetsov and A. Yu. Muravitsky. On superintuitionistic logics as fragments of proof logic extensions. *Studia Logica*, **45**, 77–99, 1986.
[22] T. Litak. The non-reflexive counterpart of Grz. *Bulletin of the Section of Logic*, Lodz, **36**, 195–208, 2007.
[23] J. C. C. McKinsey and A. Tarski. Some theorems about the sentential calculus of Lewis and Heyting. *The Journal of Symbolic Logic*, **13**, 1–15, 1948.
[24] A. Y. Muravitsky. The embedding theorem: its further development and consequences. I. *Notre Dame J. Formal Logic*, **47**, 525–540, 2006.
[25] S. Negri. Contraction-free sequent calculi for geometric theories with an application to Barr's theorem. *Archive for Mathematical Logic*, **42**, 389–401, 2003.
[26] S. Negri. Proof analysis in modal logic. *Journal of Philosophical Logic*, **34**, 507–544, 2005.
[27] S. Negri. Proof analysis in non-classical logics. In *Logic Colloquium 2005*, pp. 107–128. Vol. 28 of *Lecture Notes in Logic*, Assoc. Symbol. Logic, Urbana, IL, 2008.
[28] S. Negri. Kripke completeness revisited. In *Acts of Knowledge – History, Philosophy and Logic*, G. Primiero and S. Rahman, eds, pp. 247–282. College Publications, 2009.
[29] S. Negri and J. von Plato. Cut elimination in the presence of axioms. *The Bulletin of Symbolic Logic*, **4**, 418–435, 1998.
[30] S. Negri and J. von Plato. *Structural Proof Theory*. Cambridge University Press, 2001.
[31] S. Negri and J. von Plato. *Proof Analysis*. Cambridge University Press, 2011.
[32] H. Perdry and P. Schuster. Noetherian orders. *Mathematical Structures in Computer Science*, **21**, 111–124, 2011.
[33] W. Rautenberg. Modal tableau calculi and interpolation. *Journal of Philosophical Logic*, **12**, 403–423, 1983.
[34] G. Sambin and S. Valentini. The modal logic of provability. The sequential approach, *Journal of Philosophical Logic*, **11**, 311–342, 1982.
[35] K. Schütte. Ein System des verknüpfenden Schliessens. *Archiv für mathematische Logik und Grundlagenforschung*, **2**, 55–67, 1956.
[36] K. Segerberg. An essay in classical modal logic. *Filosofiska Föreningen och Filosofiska Institutionen vid Uppsala Universiteit*, 1971.
[37] R. Solovay. Provability interpretations of modal logic. *Israel Journal of Mathematics*, **25**, 287–304, 1976.
[38] G. Takeuti. *Proof Theory*, 2nd edn. North-Holland, 1987.
[39] A. Troelstra and H. Schwichtenberg. *Basic Proof Theory*, 2nd edn. Cambridge, 2000.

Received 14 October 2012