Kripke completeness revisited

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Abstract

The evolution of completeness proofs for modal logic with respect to the possible world semantics is studied starting from an analysis of Kripke's original proofs from 1959 and 1963. The critical reviews by Bayart and Kaplan and the emergence of Henkin-style completeness proofs are detailed. It is shown how the use of a labelled sequent system permits a direct and uniform completeness proof for a wide variety of modal logics that is close to Kripke's original arguments but without the drawbacks of Kripke's or Henkin-style completeness proofs.

Introduction

The question about the ultimate attribution for what is commonly called Kripke semantics has been exhaustively discussed in the literature, recently in two surveys (Copeland 2002 and Goldblatt 2005) where the rôle of the precursors of Kripke semantics is documented in detail.

All the anticipations of Kripke's semantics have been given ample credit, to the extent that very often the neutral terminology of "relational semantics" is preferred. The following quote nicely summarizes one representative standpoint in the debate:

> As mathematics progresses, notions that were obscure and perplexing become clear and straightforward, sometimes even achieving the status of "obvious." Then hindsight can make us all wise after the event. But we are separated from the past by our knowledge of the present, which may draw us into "seeing" more than was really there at the time. (Goldblatt 2005, section 4.2)

We are not going to treat this issue here, nor discuss the parallel development of the related algebraic semantics for modal logic (Jonsson and Tarski 1951), but instead concentrate on one particular and crucial aspect in the history of possible worlds semantics, namely the evolution of completeness proofs for modal logic with respect to Kripke semantics.

Kripke published in 1959 a proof of completeness for first-order **S5** and in 1963[1] an extension of the method to cover the propositional modal systems **T**, **S4**, **S5**, and **B**. His method employed a generalization of Beth tableaux and completeness was established in a direct and explicit way by showing how a failed search for a countermodel gives a proof.

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[1] The results of Kripke's 1963 paper had already been announced in an abstract published in 1959.

Kripke's proof was criticized in a review by Kaplan as lacking in rigor and as making excessive use of "intuitive" arguments on the geometry of tableau proofs. Kaplan suggested a different, more "mathematical" and more elegant approach based on an adaptation of Henkin's completeness proof for classical logic. Indeed, a Henkin-style completeness proof for **S5** had already been published in 1959 by Bayart and other proofs were published at the time of the review or shortly after (Makinson 1966, Cresswell 1967).

Henkin-style completeness proofs were since then preferred in the literature on modal logic, even for labelled systems. The explicit character of Kripke's original proof, that constructs countermodels for unprovable formulas, is lost with the Henkin approach.

The purported mathematical elegance of a proof of completeness in the Henkin style resembles a well-designed trick. The proof gives no way to obtain derivability from validity, nor does it show how to construct a countermodel for underivable propositions. For modal logic other specific problems arise. For instance, in systems with an irreflexive accessibility relation, as those needed in temporal logic, the canonical accessibility relation need not be irreflexive; Some extra devices, such as the one called bulldozing have to be used to obtain an irreflexive frame from the canonical one (cf. Bull and Segerberg 1984, 2001).

The criticism of insufficient formalization in Kripke's original argument can be overcome by the use of a system that embodies Kripke semantics in an explicit way, through the use of a labelled syntax. In Kripke, the ramified structure of systems of sets of alternative tableaux contains the semantics in the form of geometric conditions on tree-structures in proofs. We show how this structure can be replaced by a simple labelled sequent system. Completeness is established with a Schütte-style construction of an exhaustive proof search in the labelled system: either a proof or a countermodel is found. The countermodel is extracted directly from the labels used in a non-conclusive branch in the search tree. The problems mentioned above with the treatment of negative properties of the accessibility relation, such as irreflexivity, simply do not arise.

The contents of the paper are as follows: Section 1 presents a background on modal logic and its Kripke semantics; It can be skipped by readers already familiar with modal logic. In Section 2 we present a re-reading of Kripke's original completeness proofs, as published in the papers from 1959 and 1963, respectively, and the reviews to these papers by Bayart and Kaplan. In Section 3 we give a sketch of a Henkin-style completeness proof for modal logic. In Section 4 we present our method for obtaining labelled sequent calculi with good structural properties for all modal logics characterized by a relational semantics. The completeness proof is presented is Section 5.

1. Background on modal logic and its Kripke semantics

Traditional Kripke completeness is concerned with systems of normal modal logics, that is, systems obtained as extensions of basic modal logic. In this Section we shall recall the basic definitions and the standard notions of what is nowadays regarded as "Kripke semantics."

1.1. The language and axioms of modal logic

In modal logic, we start from the language of propositional logic and add to it the two modal operators $\Box$ and $\Diamond$, to form from any given formula A the formulas $\Box A$ and $\Diamond A$. These are read as "necessarily A" and "possibly A," respectively.

A system of modal logic can be an extension of intuitionistic or classical propositional logic. In the latter, the notions of necessity and possibility are usually connected by the equivalence $\Box A \supset\subset \sim\Diamond \sim A$.

It is seen that necessity and possibility behave analogously to the quantifiers: In one interpretation, the necessity of A means that A holds in all circumstances, and the possibility of A means that A holds in some circumstances. The definability of possibility in terms of necessity is analogous to the definability of existence in terms of universality.

The system of **basic modal logic**, denoted by **K** in the literature, adds to the axioms of classical propositional logic the following:

Table 1. The system of basic modal logic

1. Axiom: $\Box(A \supset B) \supset (\Box A \supset \Box B)$,

2. Rule of necessitation: From A to infer $\Box A$.

One axiom and one rule is added to the axioms and rules of propositional logic. The rule of necessitation requires that the premiss be derivable in the axiomatic system, i.e., its contents are that if A is a theorem, also $\Box A$ is a theorem. If instead of axiomatic logic we start from a system of natural deduction, the following rules are added:

Table 2. Natural deduction for basic modal logic

$$\frac{\Box(A \supset B) \quad \Box A}{\Box B} \qquad \frac{A}{\Box A}$$

The second rule, called "necessitation" or "box introduction," requires a restriction: If from any formula A one could conclude $\Box A$, by first assuming A and then applying necessitation and implication introduction, one could conclude $A \supset \Box A$. Anything implies its own necessity, which clearly is wrong. In the axiomatic formulation, the premiss of necessitation was a theorem. In a natural deduction system, one requires that A be derivable with no open assumptions. If one thinks of the analogy between necessity and universal quantification, it appears that the restriction is analogous to the variable condition in the rule for introducing the universal quantifier. Inappropriate formulations of the rule of necessitation have caused considerable confusion in the literature and led many authors to the conclusion that the deduction theorem fails in modal logic (see Hakli and Negri 2008 for a thorough discussion of this issue).

The analogy between necessity and possibility and the quantifiers suggests other operators similar to those of modal logic. For example, whatever must be done is *obligatory*, whatever can be done is *permitted*. These two notions belong to **deontic** logic. Even more simply, we can read $\Box A$ as "always A" and $\Diamond A$ as "some time A," respectively, which gives rise to **tense logic**.

The early study of modal logic, to the late 1950s, consisted mainly of suggested axiomatic systems based on an intuitive understanding of the basic notions. Certain axiomatizations became standard and are collected here in the form of a table. All of them start with the axioms of classical propositional logic and the axioms of basic modal logic of table 1.

Table 3. Extensions of basic modal logic

	Axiom
T	$\Box A \supset A$
4	$\Box A \supset \Box\Box A$
E	$\Diamond A \supset \Box\Diamond A$
B	$A \supset \Box\Diamond A$
3	$\Box(\Box A \supset B) \vee \Box(\Box B \supset A)$
D	$\Box A \supset \Diamond A$
2	$\Diamond\Box A \supset \Box\Diamond A$
W	$\Box(\Box A \supset A) \supset \Box A$

Well-known extensions of basic modal logic are obtained through the addition of one or more of the above axioms to system **K**, for instance **K4** is obtained by adding 4, **S4** by adding T and 4, **S5** by adding T, 4, and E (or T, 4, and B), deontic **S4** and **S5** are obtained with the addition of axiom D to **S4** and **S5**, respectively. The addition of W gives what is known as the Gödel-Löb system. Axiom 2, also known as axiom M, gives the extension of **K4** and **S4** known as **K4.1** and **S4.1**, respectively. Axiom 3 is used for instance in the extension **S4.3** of system **S4**.

The study of modal logic was completely changed in the late 1950s through the invention of a **relational semantics** of modal logic to which we now turn.

1.2. Kripke semantics

What is known as Kripke semantics, also known under the neutral term relational semantics, was presented by Saul Kripke in 1959 for the modal logic **S5**. It was modified later to accommodate also other modal logics and intuitionistic logic (Kripke 1963, 1965). The idea had several significant anticipations in the work of Arnould Bayart, Rudolf Carnap, Jaakko Hintikka, Stig Kanger, Richard Montague, Arthur Prior, and others. Questions about the originality and ultimate attribution for the invention of Kripke semantics have raised a considerable debate. We shall not take any position on these issues here, but refer to Goldblatt (2005) for an in-depth discussion.

The basic idea of the semantics is that a proposition is necessary if and only if it is true in all "possible worlds." The idea is made precise as follows:

A **Kripke frame** is a set W, the elements of which are called **possible worlds**, together with an **accessibility relation** R, that is, a binary relation between elements of W. A Kripke frame becomes a **Kripke model** when a **valuation** is given. A valuation val takes a world w and an atomic formula P and gives as value 0 or 1, to determine which atomic formulas are true at what particular worlds. The notation is

$w \Vdash P$ whenever $val(w, P) = 1$.

It is read as: *formula P is true at world w*, alternatively as *w forces P*. If $val(w, P) = 0$, we write $w \nVdash P$. A valuation is just like a line in a truth table, except that it is indexed by a world. If there is just one world, we have essentially the truth-table semantics of classical propositional logic. Valuations are supposed to be actually given, not just to exist in some abstract sense, so we have $w \Vdash P$ or $w \nVdash P$ for each atom P.

Valuations are extended in a unique way to arbitrary formulas by means of inductive clauses. For the propositional connectives, the inductive extension is straightforward:

Table 4. Valuations for the connectives

$w \Vdash A \& B$ whenever $w \Vdash A$ and $w \Vdash B$,
$w \Vdash A \lor B$ whenever $w \Vdash A$ or $w \Vdash B$
$w \Vdash A \supset B$ whenever from $w \Vdash A$ follows $w \Vdash B$
$w \Vdash \bot$ for no w.

It was assumed above that it is decidable if an atomic formula is forced at a given world. The same property holds then for arbitrary formulas, by the inductive clauses of table 4. Further, if $w \nVdash A$, then $w \Vdash \sim A$. To prove this, assume $w \Vdash A$. We have a contradiction (in fact, 0=1), so that $w \Vdash \bot$. Therefore, by the inductive clause for implication, $w \Vdash \sim A$.

Definition 1.1. *Given a Kripke frame W, formula A is* **valid in** *W if, for every valuation, $w \Vdash A$ for every world w in W.*

The central idea in Kripke's semantics for modal logic is that a formula of the form $\Box A$ is true at w if A is true at all worlds accessible from w through the relation R:

$w \Vdash \Box A$ if and only if for all o, $o \Vdash A$ follows from wRo.

The second key insight of Kripke semantics is that the axioms of different systems of modal logic correspond to special properties of the accessibility relation. Let us take what is probably the simplest example, namely a **reflexive** frame: We assume the accessibility relation to be reflexive. The condition corresponds to axiom T of table 3:

$w \Vdash \Box A \supset A$ for every world w.

To see this, assume $w \Vdash \Box A$. Then $o \Vdash A$ for every o accessible from w, in particular, by reflexivity, for w itself, so $w \Vdash A$. Therefore $w \Vdash \Box A \supset A$. On the other hand, it is easily seen that a frame that validates $\Box A \supset A$ has to be reflexive, so that reflexivity of the accessibility relation is equivalent to having a modal system with axiom T.

Similarly, it is seen that $\Box A \supset \Box \Box A$ is valid in every transitive frame and that every frame validating it has to be transitive. We say that there is a **correspondence** between a modal axiom and a property of the accessibility relation.

Observe that the defining axiom of the system of basic modal logic **K**, $\Box(A \supset B) \supset (\Box A \supset \Box B)$, is valid in every frame.

Table 6 of Section 4 gives a list of common modal axioms together with their corresponding frame conditions.

2. Kripke's original completeness proofs

In this Section we shall analyze the content of Kripke's original completeness proofs for systems of modal logic, as published in 1959 and 1963, and the criticisms that were raised by Bayart and Kaplan in their reviews, both published in 1966.

2.1. A completeness theorem in modal logic

In his paper of 1959, the young Saul Kripke presented a completeness theorem for 1st-order **S5** with equality.

The starting point is a Hilbert-style axiomatization obtained from Rosser's 1953 first-order predicate calculus with equality, with the addition of the following axiom schemes and rules of inference:

A1: $\Box A \supset A$

A2: $\sim \Box A \supset \Box \sim \Box A$

A3: $\Box(A \supset B) \supset (\Box A \supset \Box B)$

R1: If $\vdash A$ and $\vdash A \supset B$ then $\vdash B$

R2: If $\vdash A$ then $\vdash \Box A$

Given a non-empty domain D and a formula A, a **complete assignment** for A in D is a function which to every free individual variable assigns an element of D, to every propositional variable either T or F, and to n-ary predicates $P(x_1, \ldots, x_n)$ n-tuples of D.

A **model** of A in D is a pair (G, K) where G is a complete assignment in a set K of assignments, such that every member of K agrees with G on the assignment of free variables of A. The evaluation of an element H of K on an arbitrary subformula of A is obtained inductively in the usual way from the assignment of individual and propositional variables and of predicates. For example, $P(x_1, \ldots, x_n)$ is true in the model for the assignment H if the values $\alpha_1, \ldots, \alpha_n$ assigned to the variables belong to the subset of n-tuples assigned to the n-ary predicate P; $\forall x B$ is true if $B(x)$ is true for every assignment of x in D; $\Box B$ is true if B true under every assignment in K.

A formula A is **valid** in (G, K) if it is assigned T by G, valid in D if it is valid in every model in D, **satisfiable** if it is valid for some model based on D, and **universally valid** if valid on every non-empty domain.

The intuitive idea here is that a proposition is necessary if and only if it is true in all possible worlds; All possible worlds are just all possible evaluations, the real world being represented by G and the other members of K representing possible worlds:

> The basis of the informal analysis which motivated these definitions is that a proposition is necessary if and only if it is true in all "possible worlds". (It is not necessary for our present purposes to analyze the concept of a "possible world" any further.) ... In modal logic, however, we wish to know not only about the

real world but about other conceivable worlds; P may be true in the real world but false in some imaginable one, and similarly for $P(x_1, ..., x_n)$. Thus we are led not to a single assignment but to a set K of assignments, all but one of which represent worlds which are conceivable but not actual; the assignment representing the actual world is singled out as G, and the pair (G, K) is said to form a model of A. (Kripke 1959, p. 2)

Kripke shows, using an adaptation to modal logic of Beth's method of semantical tableaux, that a formula is derivable in **S5** if and only if it is universally valid. The tableau method is presented as a test of semantical entailment from $A_1 \& \ldots \& A_n$ to B through a systematic search for a countermodel in which $A_1, \ldots, A_n$ are valid but B is not. The construction produces a system of alternative sets of tableaux, each set containing a main tableau and subsidiary tableaux.

The rules for tableaux are the familiar ones, with the splitting into alternative tableaux in the case of conjunction in the right (and disjunction in the left if a full language is used). The rules for necessity are:

Yl. If $\Box A$ appears in the left column of a tableau, then we put A in the left column of every tableau of the set

Yr. If $\Box A$ appears in the right column of a tableau, then we introduce a new auxiliary tableau which is started out by putting A in its right column.

A tableau is **closed** if and only if either a formula occurs in both of its columns, or $a = a$, for some variable a, occurs in its right column. A set of tableaux is closed if and only if at least one of its members is closed. A system is closed if and only if all its alternative sets are closed.

Theorem 1: B is semantically entailed by $A_1 \& \ldots \& A_n$ if and only if the construction beginning with $A_1 \& \ldots \& A_n$ in a left column and B in a right column is closed.

The proof is divided in two parts, the first part (Lemma 1, validity) shows that if B is not semantically entailed by $A_1 \& \ldots \& A_n$, then the tableau construction cannot be closed.

If B is not semantically entailed by $A_1 \& \ldots \& A_n$, there is a model (G, K) on D such that $A_1, \ldots, A_n$ are true and B false in it. The inductive clauses for valuations match the tableaux rules in such a way that they preserve countermodels, so in the end, since the construction is closed, every alternative set contains a tableau which either has a formula in both columns or has $a = a$ in the right column. So there would be some formula which is at the same time true and false in the model, or $a = a$ would be false, a contradiction.

The information on the existence of a countermodel is directly transferred to a constraint on the tableau.

The second part (Lemma 2, completeness) shows that if the tableau construction is not closed, then a countermodel is found.

Lemma 2: If the construction starting with $A_1, \ldots, A_n$ on the left and B on the right is not closed, then B is not semantically entailed by $A_1, \ldots, A_n$.

The proof of Lemma 2 is obtained by choosing one of the alternative sets which is not closed and by defining a suitable countermodel on the basis of that. As domain D the set of free variables in the latter set is taken. The assignment is defined as follows: Every free variable except those eliminated by the rule of substitution for identity (Il) is assigned to itself. Free variables eliminated by rule Il are assigned the variable that replaces them. Propositional variables occurring on the left of the tableaux of the chosen set are assigned T, those occurring on the right are assigned F. Predicates P^n of n variables are assigned the set of n-tuples $(x_1, \ldots, x_n)$ of variables such that $P^n(x_1, \ldots, x_n)$ appears on the left in the tableaux of the set. It is then shown by induction on formulas that every formula occurring on the left (resp. right) is assigned T (resp. right).

After proving a Löwenheim-Skolem result, by which a formula is satisfiable in a finite or denumerable domain if it is satisfiable in a nonempty domain, Kripke proceeds with the proof of completeness with respect to the original Hilbert-type system. For importing the completeness result proved for the tableau system, a form of a deduction theorem is proved. If we start a construction with $A_1, \ldots, A_n$ in the left column and B on the right of a tableau (initial stage), after m applications of the rules there are finitely many tableaux, and this is called the $(m+1)$th stage of the construction. Each stage is put in correspondence with an equivalent *characteristic formula*: The characteristic formula of a given tableau with $A_1, \ldots, A_m$ in the left and $B_1, \ldots, B_n$ in the right is $A_1 \& \ldots \& A_n \& \sim B_1 \& \sim B_m$; the characteristic formula of any of the alternative sets at a given stage is $\exists x_1 \ldots \exists x_p (A \& \diamond B_1 \& \ldots \& \diamond B_q)$ where A is the characteristic formula of the main tableau of the set and $B_1, \ldots, B_q$ are the characteristic formulas of the auxiliary tableaux of the set and $x_1, \ldots, x_p$ are the free variables of $A \& \diamond B_1 \& \ldots \& \diamond B_q$. Finally, the characteristic formula of a stage is $D_1 \vee \cdots \vee D_r$, where $D_1, \ldots, D_r$ are the characteristic formulas of the alternative sets of the stage. Then it is shown (Lemma 4) that if A is the characteristic formula of the initial stage and B is the characteristic formula of any stage, then $A \supset B$ is provable in the given Hilbert system $\mathbf{S5}^{*=}$ ($\mathbf{S5}$ with equality and quantifiers).

The proof of completeness for $\mathbf{S5}^{*=}$ (Theorem 5) can be summarized as follows: If A is universally valid, then the tableau construction beginning with A in the right column is closed. If B is the characteristic formula of the earliest stage at which the closure holds, by Lemma 4 the implication between the closure formula of the initial stage and B is derivable in $\mathbf{S5}^{*=}$, that is, $\vdash \exists a_1 \ldots a_p \sim A \supset B$. It is detailed how the fact that B is the characteristic formula of a stage of closure gives $\vdash \sim B$, from which $\vdash \sim \exists a_1 \ldots a_p \sim A$, and therefore $\vdash A$ follows.

Theorem 6 establishes validity: If $\vdash A$ in $\mathbf{S5}^{*=}$, A is universally valid. The proof consists in observing that the axioms of $\mathbf{S5}^{*=}$ are universally valid and that modus ponens preserves universal validity. As for the rule of necessitation, if A is universally valid, the tableau construction starting with A on the right closes, hence also the tableau construction starting with $\Box A$ closes, so universal validity of $\Box A$ follows by Theorem 1.

Kripke proceeds with defining truth tables for $\mathbf{S5}$ and establishing that a formula is a

tautology if and only if it is universally valid. In the final part, second order quantification is treated.

2.2. Semantical analysis of modal logic I. Normal modal propositional calculi

In Kripke (1963) the results of Kripke (1959) were extended to various systems of modal logic: Gödel-Feys-von Wright's **M** (**T**), the *Brouwersche* system **B**, and Lewis' **S4** and **S5**.

The core of normal modal systems is taken to consist of axioms A1 and A3 and rules R1 and R2 (see previous Section), which give the system **M**, alternatively called system **T**. System **S4** is obtained by the addition of

A4: $\Box A \supset \Box\Box A$,

the Brouwersche system by the addition of

$A \supset \Box\Diamond A$,

and **S5** by the addition of

A2: $\sim\Box A \supset \Box \sim\Box A$.

The novelty here is the explicit appearance of the accessibility relation:

> A *normal model structure* (n.m.s) is an ordered triple $(G, K, \mathbf{R})$ where K is a non-empty set, $G \in K$, and $\mathbf{R}$ is a reflexive relation defined on K. If $\mathbf{R}$ is transitive, we call the n.m.s. an **S4** *model structure*; if $\mathbf{R}$ is symmetric, we call it a BROUWER*sche model structure*; if $\mathbf{R}$ is an equivalence relation, we call it an **S5** *model structure*. (Kripke 1963, p. 68)

An **M** (**S4**, **S5**, Brouwersche) model for a formula A is given by a binary function Φ that has as arguments the propositional variables P of A and the elements H of K, and as range the truth values T, F. The function is extended in a unique way to all subformulas of A by the following inductive clauses:

> $\Phi(B\&C, H) = T$ if and only if $\Phi(B, H) = T$ and $\Phi(C, H) = T$
> $\Phi(\sim B, H) = T$ if and only if $\Phi(B, H) = F$
> $\Phi(\Box B, H) = T$ if and only if $\Phi(B, H') = T$ for all H' such that HRH'

Truth (falsity) of A in a model is defined as truth (resp. falsity) in the *real* world G, $\Phi(A, G) = T$ (resp. $\Phi(A, G) = F$); validity as truth in all models, and satisfiability as truth in at least one of them.

The relation of the new definitions to those in Kripke (1959) is given in section 2.1 (Informal explanation). Two main differences arise: Whereas in the previous work worlds were identified with complete assignments, here the two notions are distinct, to the effect that there can be worlds in which the same truth values are assigned to the atomic formulas. The second novelty is the relation $\mathbf{R}$, for which the following informal explanation is given:

Intuitively we interpret the relation $\mathbf{R}$ as follows: Given any two worlds $H_1, H_2 \in K$, we read "$H_1 \mathbf{R} H_2$" as "H_2 is possible relative to H_1", "possible in H_1", or "related to H_1"; that is to say, every proposition true in H_2 is to be possible in H_1.[2] (Kripke 1963, p. 70)

On the basis of this reading, modal axioms are related to properties of the accessibility relation: Transitivity is shown to correspond to $\Diamond\Diamond A \supset \Diamond A$, and symmetry to $A \supset \Box\Diamond A$

In section 2.2 a *connected* model structure is defined as one in which all the possible worlds are related through the transitive closure $\mathbf{R}^*$ (the *ancestral*, in Kripke's terminology) of $\mathbf{R}$. Here Kripke shows that every satisfiable formula has a connected model, or equivalently, that every non-valid formula has a connected countermodel. Given a model with valuation Φ on $(G, K, \mathbf{R})$, a connected countermodel is defined by the restriction of Φ and $\mathbf{R}$ to the set of worlds accessible form the "real world" G through the transitive closure of $\mathbf{R}$. By the restriction to connected models an equivalence relation gives the same models as a total relation, so the treatment of Kripke (1959) for $\mathbf{S5}$, without the accessibility relation, can be seen as a special case of the new one.

A further reduction is prepared for with the definition of a *tree* as a triple (G, K, S) where S is a binary relation on K, G has no predecessor with respect to S, and every other element of K has a unique predecessor.

In section 3, semantic tableaux are presented as a generalization of the tableaux of Kripke (1959). A tableau construction gives at each stage a system of *alternative* sets of tableaux, each containing a *main* tableau and *auxiliary* tableaux.

The completeness proof is given by a systematic search of a countermodel; If no countermodel is found, the formula is valid. The procedure for a formula of the form $A_1 \& \ldots \& A_m \supset B_1 \vee \cdots \vee B_n$ starts by imposing $A_1, \ldots, A_m$ to be true in the model and $B_1, \ldots, B_n$ false, that is in putting $A_1, \ldots, A_m$ to the left and $B_1, \ldots, B_n$ to the right of the main tableau. The rules for the tableau construction transform the requirement into equivalent conditions on more elementary formulas. The rules can either produce a continuation of the same tableau (rules for negation, conjunction, and necessity on the left) or, in the case of right conjunction, the splitting of the tableau into alternative sets of tableaux $t_1, t_2 \ldots$. The splitting corresponds to the fact that in order to falsify a conjunction it is enough to falsify one of the conjuncts. For necessity on the right, the tableau construction proceeds by creating from the given tableau t that contains a formula $\Box A$ on the right, a new auxiliary tableau t' such that tRt'. For specific modal systems, additional properties are assumed on R. These properties are not made a formal part of the syntax in the tableau construction.

Kripke did not devise a formal notation to fully describe his tableau construction and wrote in fact:

I hope that this explanation makes the process clear intuitively; the formal statement is rather messy... (Kripke 1963, p. 73)

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[2]Observe that this informal reading imposes the definition of the canonical accessibility relation if possible worlds are Henkin sets (see Section 3 below).

A tableau is closed if the same formula appears on both sides of the tableau, a set of tableaux is closed if some tableau in it is closed, and a system is closed if each of its alternative sets is closed. A construction is closed if at some stage a closed system of alternative sets appears.

Some additional restrictions are posed in order to facilitate the tableau construction, for instance, a rule need not be applied if it produces a formula that is already in the tableau.[3] Another optimization in the tableau construction is the irrelevance of the order of application of the rules, so that no special strategy is needed (but see Bayart's objection in Section 2.3 below for the predicate case).

The procedure is clarified by an example that shows how the tableau procedure works for the formula $\Box(A \mathbin{\&} B) \supset \Box(\Box A \mathbin{\&} \Box B)$: A closed S4 tableau construction is obtained, that is, the search for a countermodel fails and therefore the formula is valid.

In section 3.2, Kripke proves the completeness of the tableau procedure with respect to the semantics.

First (Lemma 1) he shows validity: If the construction for A is closed, then A is valid. This is proved by contradiction: If A is not valid, there exists a valuation Φ on a model structure $(G, K, \mathbf{R})$ with $\Phi(A, G) = F$. It is then shown by induction on the stages of the tableau construction, started with A on the right, that each stage can be put in correspondence with worlds in the model and the relation that links auxiliary tableaux with the relation in the model, and that formulas on the right are false and those on the left true under the assignment Φ. The construction is closed, so a contradiction follows because there would be a formula both true and false under Φ.

Lemma 2 proves completeness, again by contradiction: If the construction for A is not closed, then A is not valid.

The proof can be summarized as follows: The tableau construction has a tree structure that can either be finite or infinite. If the construction is finite it has, because it is not closed, at least one alternative set $\mathcal{S}_0$ that is not closed. Then a countermodel $(G, K, \mathbf{R})$ is defined by taking for K the set $\mathcal{S}_0$, for $\mathbf{R}$ the relation R between elements of $\mathcal{S}_0$, and for G the main tableau of $\mathcal{S}_0$. The valuation Φ is defined by putting $\Phi(P, H) = T$ if P appears on the left side of H, and $\Phi(P, H) = F$ otherwise. It is then shown by induction on formulas that for arbitrary formulas A we have $\Phi(A, H) = T$ if A appears on the left of H, and $\Phi(A, H) = F$ otherwise. Therefore A is false under Φ.

In case the tableau construction is infinite, the proof is a bit more complex and König's lemma is used to extract an infinite path that is used to define the countermodel.

By the completeness proof, the previously established reduction to connected structures is strengthened to a reduction to trees. The reduction is used for obtaining a reformulation of the tableau rules in which relation R is replaced by a unary successor relation S. The tableau rules are thus modified so that the properties of the relation R become part of the rules. For example, the left rule for $\Box$ that subsumes transitivity is as follows:

Yl: If $\Box A$ appears on the left of a tableau t_1, we put A on the left of t_1 and put $\Box A$ on the left of any tableau t_2 such that $t_1 R t_2$. (Kripke 1963, p. 81)

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[3]Observe the analogy to the search for minimal derivations in Gentzen systems with height-preserving contraction, where a rule need not be applied if it leads to a duplication of formulas in a sequent.

A similar modification is given for the Brouwerische system, which incorporates symmetry of R.

Kripke defines two tableaux t_1 and t_2 to be *contiguous* if $t_1 S t_2$ or $t_2 S t_1$ and observes that, because of the new formulation, application of a rule to a tableau affects only the tableaux contiguous with it.[4]

In section 4, completeness of the Hilbert systems is proved. The proof of validity (consistency) reduces to the immediate task of verifying that the axioms are valid and that the rules preserve validity.

For completeness, first the definition of a characteristic formula of a tableau is given as in Kripke (1959), and the lemma already proved in Kripke (1959) for **S5** (Lemma 4) is extended to the systems here considered. If the tableau procedure started with A is closed, we find for each of the alternative sets $\mathcal{S}_j$ a stage of closure, that is, with the same formula both on the left and on the right of the tableau, so the characteristic formula D_j contains the conjuction of a formula and its negation. By the above mentioned lemma, we have $\vdash \sim A \supset D_1 \vee \cdots \vee D_m$ and since for all j, $\vdash \sim D_j$, we have $\vdash \sim (D_1 \vee \cdots \vee D_m)$, and therefore, because all the systems are extensions of classical logic, we get $\vdash A$.

The rest of the paper contains proofs of decidability for all the systems considered, obtained by means of a bound in the tableau proof search procedure, a section on matrices to establish independence results, and a proof, by the method later called "glueing of Kripke models," of the modal disjunction property. The last one had already been proved by McKinsey-Tarski (1948) and Lemmon (1960) using algebraic semantics.

2.3. Reviews

Both of Kripke's papers were carefully reviewed. In 1966, a review of Kripke (1959) by Arnould Bayart appeared. After a detailed summary of the basic definitions and results of the paper, the reviewer observed a lack of determinism in the tableau rules for the quantifiers and suggested the introduction of a control mechanism to avoid dead-ends:

> An objection against both the proof and the statement of theorem 1 is that, at each step of the construction of a system of tableaux, several possibilities generally occur so that different end results can be reached. If one starts with bx and $\sim bx$ in the left column and with $(x)bx$ in the right column, one obtains a closed tableau by working on $\sim bx$, and one obtains an infinite not closed tableau by working on $(x)bx$. The rules for constructing tableaux should be supplemented by a rule imposing some definite choice at each step and guaranteeing that each formula appearing in a not closed branch of the construction will at some moment become the object of an application of a construction rule and that the rules for universal quantification in a left column will be applied for all individual variables appearing in the branch.

The same year, Kripke (1963) was reviewed by David Kaplan who, even if he praised Kripke's result, found that the development lacked in rigor:

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[4]This system of rules thus enjoys the remarkable property nowadays called *locality*.

Although the author extracts a great deal of information from his tableau constructions, a completely rigorous development along these lines would be extremely tedious. As a consequence a number of small gaps must be filled by the reader's geometrical intuition ... The dangers inherent in relying on intuition are illustrated by the author's need to correct a fallacious proof in A completeness theorem in modal logic ... the author criticizes another writer's faulty version of the rule; but his own formulation also requires amendment ... The proofs of the decision procedures seemed to the reviewer excessively intuitive even within the allowable space, and the proof for S4 contains an error (which can be corrected...).

Kaplan went on suggesting an alternative approach to the completeness proof for modal logic:

The reviewer believes that future research will bring considerably simpler more rigorous proofs which avoid the tableau technique. In fact the interesting half of the main theorem can be established by using the technique of Henkin ...

After a sketch of the idea of the Henkin-style proof of completeness for modal logic, Kaplan observed that the proof was suggested to him by Dana Scott and that the argument was already foreshadowed in Kanger (1957). The review also witnesses the already existent debate on the ultimate attribution of the possible world semantics. The contributions of Carnap, Kanger, Hintikka, and Montague are mentioned as important anticipations. The review ends with words of praise for the paper as one "among the most important contributions to the study of modal logic," followed by a list of corrections to about 20 misprints.

3. Henkin-style completeness proof

Kaplan's review of 1966 gave the guiding ideas of the adaptation of the proof of Henkin (1949) to modal logic as suggested to him by Lemmon and Scott.

Henkin-style completeness proofs for various systems of modal logic with respect to the relational semantics were published at the same time (Makinson 1966), or shortly after.[5] Indeed, Cresswell refers to the papers by Arnould Bayart (1958, 1959), who was apparently the first to have given a proof of completeness in this style for modal logic. Bayart considered second-order **S5**. In the first of the two papers, he proved validity (in French, *correction*), in the second quasi-completeness (in French, *quasi-adequation*), the quasi being referred to the limitations for the second order case. The papers by Bayart are difficult to read; They use Polish notation and an archaic style of exposition, which may explain why they were little known. Instead of tableaux, they use a system of sequent calculus with invertible rules. The use of the possible worlds semantics is independent of Kripke's, and declared by the author to have been inspired by Leibniz. Bayart himself did not mention his alternative approach to completeness for **S5** in his review of Kripke (1959).

Henkin-style completeness proofs seem to have been unanimously considered superior to the proofs originally devised by Kripke. Kaplan's criticism of Kripke (1963) was confirmed in Makinson's review of Kripke (1965).

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[5]In 1967 a paper by Cresswell with a completeness proof for T and S with the Barcan formula appeared.

It should be remarked that alternative and simpler proofs have since been constructed for these completeness theorems, in which maximal consistent set constructions do the work of Kripke tableaux. (Makinson, 1970)

In addition, a reviewer of Makinson's paper wrote:

> The proof, which makes no use of the axiom of choice, and which is nicely worked out in detail, is Henkin-style and thus avoids the Beth-Hintikka technique of semantic tableaux employed by Kripke... (Åqvist 1970, p. 136)

Before discussing the relative advantages of the two approaches, we sketch the structure of Henkin-style completeness proofs for modal logic.

We recall that a frame is $\mathcal{F}$ is a non-empty set S endowed with a binary relation R. A model $\mathcal{M}$ is given by a frame together with an assignment V of atomic formulas to subsets of S. Often the forcing relation notation $\mathcal{M} \Vdash_v P$ is used for $v \in V(P)$. The assignment V is extended to arbitrary formulas by the standard inductive clauses, for example:

$\mathcal{M} \Vdash_v A \supset B$ if from $\mathcal{M} \Vdash_v A$, $\mathcal{M} \Vdash_v B$ follows
$\mathcal{M} \Vdash_v \Box A$ if $\mathcal{M} \Vdash_w A$ for all w such that vRw

Validity in a model is defined by truth in every world:

$\mathcal{M} \Vdash A$ if $\mathcal{M} \Vdash_v A$ for all $v \in S$

Validity in a frame is defined by validity in every model based upon the frame:

$\mathcal{F} \Vdash A$ if $\mathcal{M} \Vdash A$ for all V

If $\mathcal{C}$ is a class of frames, A is valid in the class of frames, $\mathcal{C} \Vdash A$, if it is valid in every frame of the class, that is, $\mathcal{F} \Vdash A$ for all $\mathcal{F}$ in $\mathcal{C}$. For example, $\Box(A \supset B) \supset (\Box A \supset \Box B)$ is valid in all frames; $\Box A \supset \Box\Box A$ is valid in all transitive frames.

Let $\mathcal{L}$ be a normal modal logic. $\mathcal{L}$ is usually defined by the set of propositional tautologies plus the axiom $\Box(A \supset B) \supset (\Box A \supset \Box B)$ and closure under the rules of modus ponens and necessitation.

The deducibility relation is defined implicitly by: $\vdash_{\mathcal{L}} A$ if $A \in \mathcal{L}$.

$\mathcal{L}$ is *sound* with respect to $\mathcal{C}$ if from $\vdash_{\mathcal{L}} A$, $\mathcal{C} \Vdash A$ follows.
$\mathcal{L}$ is *complete* with respect to $\mathcal{C}$ if from $\mathcal{C} \Vdash A$, $\vdash_{\mathcal{L}} A$ follows.

Soundness is proved by a straightforward induction on the derivation of A in $\mathcal{L}$ and we need not go into the details here. If $\mathcal{L}$ has additional axioms, then it is proved that they are valid in the class of frames considered.

Completeness is proved by the canonical model construction. From $\mathcal{L}$ a special model is built in which validity and derivability coincide. We start with the proof for classical logic.

A set of formulas Δ is a *maximal set* if it is consistent and has no consistent extension. We recall that a set of formulas is consistent if for no finite subset Δ_0 of Δ, $\Delta_0 \vdash_{\mathcal{L}} \bot$. An equivalent characterization for a maximal set Δ requires that Δ is consistent and for every A, either A or $\sim A$ is in Δ.

By Lindenbaum's Lemma, every consistent set of formulas Γ can be extended to a maximal consistent set: One starts from an enumeration of the formulas in the language, $A_0, \ldots, A_n, \ldots$, and defines inductively a chain of sets of formulas as follows:

$\Delta_0 \equiv \Gamma$
$\Delta_{n+1} \equiv \Delta_n \cup \{A_n\}$ if $\Delta_n \vdash_{\mathcal{L}} A_n$
$\Delta_{n+1} \equiv \Delta_n \cup \{\sim A_n\}$ otherwise

It is not difficult to verify that $\Delta \equiv \bigcup_{n \geq 0} \Delta_n$ is a maximal consistent set that contains Γ.

By the construction of a maximal set containing a set of formulas Γ, the following hold:

1. Maximal sets are deductively closed.
2. If $\Gamma \nvdash A$, then there exists a maximal set that contains Γ but not A.

The valuation in the *canonical model* is defined by putting $\Delta \Vdash P$ if $P \in \Delta$. It is then shown by induction on formulas that also for arbitrary formulas A we have $\Delta \Vdash A$ if $A \in \Delta$. By taking the contrapositive of 2. above, we have:

Completeness. If $\Gamma \models A$, then $\Gamma \vdash A$.

The argument is augmented as follows to cover modal logic:

The canonical model is a Kripke model in which the nodes are maximal consistent sets of formulas, the accessibility relation is such that two nodes Γ, Δ are related if all the necessary truths in the former are in the latter, and a formula is forced at a node if it belongs to that node. The notation is:

$$\mathcal{M}^{\mathcal{L}} \equiv (S^{\mathcal{L}}, R^{\mathcal{L}}, V^{\mathcal{L}})$$

Here

$S^{\mathcal{L}} \equiv \{\Gamma : \Gamma \text{ is } \mathcal{L}\text{-maximal consistent}\}$
$\Gamma R^{\mathcal{L}} \Delta$ if for all A, $\Box A$ in Γ implies A in Δ
$V^{\mathcal{L}}(P) \equiv \{\Gamma : P \in \Gamma\}$

We have:

Truth Lemma. $\mathcal{M}^{\mathcal{L}} \Vdash_{\Gamma} A$ if and only if $A \in \Gamma$.

The proof is by induction on A, the only non-trivial case being the one of a modalized formula. The case follows from the definition of validity in the model, from the definition of $R^{\mathcal{L}}$, and from the fact that maximal consistent sets are deductively closed.

To prove that validity and derivability coincide in the canonical model, that is,

$$\mathcal{M}^{\mathcal{L}} \Vdash A \text{ if and only if } \vdash_{\mathcal{L}} A$$

it is enough to prove the more general:

Lemma. $\Gamma \models_{\mathcal{M}^{\mathcal{L}}} A$ if and only if $\Gamma \vdash_{\mathcal{L}} A$.

The left-hand side of the latter amounts to the fact that every maximal set that contains Γ also contains A, so the equivalence immediately follows from the properties 1. and 2. above.

Observe that the proof gives no way to obtain derivability from validity, nor does it show how to construct a countermodel for underivable propositions. The explicit character of

Kripke's original proof, that constructs countermodels for unprovable formulas, is lost with the Henkin approach.

In the next two Sections we review the definition and properties of our labelled calculi, and show how they can be used to make rigorous and generalize Kripke's original completeness proof.

4. A sequent system with internalized Kripke semantics

The development of structural proof theory has led to the remarkable class of sequent calculi, called **G3**-calculi, in which all of the structural rules–weakening, contraction, and cut–are admissible. We shall present a method for obtaining similarly behaving labelled sequent calculi for modal logics. In these, all the structural rules are admissible; They support, whenever possible, proof search, and have a simple and uniform syntax that allows easy proofs of metatheoretic results, such as those reviewed in the previous Sections.

We shall present in this Section a sequent system for the basic modal logic **K** with rules for the modalities $\Box$ and $\Diamond$. These rules are obtained through a meaning explanation in terms of the possible worlds semantics and an inversion principle. The modal logic **K** is characterized by arbitrary frames and restrictions on the class of frames that characterize a given modal logic amount to the addition of certain frame properties to our sequent calculus. These properties are added in the form of mathematical rules, following the method of extension of sequent calculus presented in chapter 6 of Negri and von Plato (2001). All the extensions are thus obtained in a modular way. As a consequence, the structural properties of the resulting calculi can be established in one theorem for all systems. A basic knowledge of sequent calculus, for example Negri and von Plato (2001, chapter 3), is sufficient for what follows.

4.1. Basic modal logic

Basic modal logic is formulated as a labelled sequent calculus through an internalization of the possible worlds semantics within the syntax. First we enrich the language so that sequents are expressions of the form $\Gamma \to \Delta$ where the multisets Γ and Δ consist of **relational atoms** wRo and **labelled formulas** $w : A$, the latter corresponding to the forcing $w \Vdash A$ in Kripke models. Here w, o range over a set W of labels/possible worlds and A is any formula in the language of propositional logic extended by the modal operators of necessity and possibility, $\Box$ and $\Diamond$.

The rules for each connective/modality are obtained from its meaning explanation in terms of the relational semantics: The inductive definition of forcing for a modal formula is:

$w \Vdash \Box A$ *whenever for all o, from wRo follows $o \Vdash A$.*

The definition gives:

If $o : A$ can be derived for an arbitrary o accessible from w, then $w : \Box A$ can be derived.

Formally, we have the rule

$$\frac{wRo, \Gamma \to \Delta, o : A}{\Gamma \to \Delta, w : \Box A} \, R\Box$$

In the rule, the arbitrariness of o becomes the variable condition that o must not occur in Γ, Δ.

Reading the semantical explanation in the other direction, we have that $w \Vdash \Box A$ and wRo give $o \Vdash A$. A corresponding rule for the antecedent side is:

$$\frac{o : A, w : \Box A, wRo, \Gamma \to \Delta}{w : \Box A, wRo, \Gamma \to \Delta} \, L\Box$$

The rules for $\Diamond$ are obtained similarly from the semantic explanation

$$w : \Diamond A \quad whenever \quad for \; some \quad o, \; wRo \; and \; o : A.$$

The rules of sequent calculus for the propositional connectives are obtained by a labelling of the active formulas with the same label in the premises and conclusion of each rule of the calculus **G3cp** (cf. Negri and von Plato 2001, p. 49). The following sequent calculus **G3K** for basic modal logic is thus obtained:

Table 5. The sequent calculus G3K

Initial sequents:

$w : P, \Gamma \to \Delta, w : P$ $wRo, \Gamma \to \Delta, wRo$

Propositional rules:

$$\frac{w : A, w : B, \Gamma \to \Delta}{w : A\&B, \Gamma \to \Delta} \, L\& \qquad\qquad \frac{\Gamma \to \Delta, w : A \quad \Gamma \to \Delta, w : B}{\Gamma \to \Delta, w : A\&B} \, R\&$$

$$\frac{w : A, \Gamma \to \Delta \quad w : B, \Gamma \to \Delta}{w : A \vee B, \Gamma \to \Delta} \, L\vee \qquad \frac{\Gamma \to \Delta, w : A, w : B}{\Gamma \to \Delta, w : A \vee B} \, R\vee$$

$$\frac{\Gamma \to \Delta, w : A \quad w : B, \Gamma \to \Delta}{w : A \supset B, \Gamma \to \Delta} \, L\supset \qquad \frac{w : A, \Gamma \to \Delta, w : B}{\Gamma \to \Delta, w : A \supset B} \, R\supset$$

$$\frac{}{w :\perp, \Gamma \to \Delta} \, L\perp$$

Modal rules:

$$\frac{o : A, w : \Box A, wRo, \Gamma \to \Delta}{w : \Box A, wRo, \Gamma \to \Delta} \, L\Box \qquad\qquad \frac{wRo, \Gamma \to \Delta, o : A}{\Gamma \to \Delta, w : \Box A} \, R\Box$$

$$\frac{wRo, o : A, \Gamma \to \Delta}{w : \Diamond A, \Gamma \to \Delta} \, L\Diamond \qquad\qquad \frac{wRo, \Gamma \to \Delta, w : \Diamond A, o : A}{wRo, \Gamma \to \Delta, w : \Diamond A} \, R\Diamond$$

In the first initial sequent, P is an arbitrary atomic formula. In $R\Box$ and in $L\Diamond$, o is a fresh label. Observe that atoms of the form wRo in the right-hand side of sequents are never active in the logical rules nor in the rules that extend the logical calculus. Moreover, the modal

axioms that correspond to the properties of the accessibility relation are derived from their rule presentations alone. As a consequence, initial sequents of the form $wRo, \Gamma \to \Delta, wRo$ are needed only for deriving properties of the accessibility relation, namely, the axioms that correspond to the rules for R given below. Thus such initial sequents can as well be left out from the calculus without impairing the completeness of the system.

4.2. Extensions

Our aim is to extend the above basic calculus so that the structural properties of the extensions are automatically guaranteed. This will follow from the form of the axioms that characterize the extensions. The following table continues table 3 with the frame properties of modal axioms:

Table 6. Modal axioms with corresponding frame properties

	Axiom	Frame property
T	$\Box A \supset A$	$\forall w \; wRw$ reflexivity
4	$\Box A \supset \Box\Box A$	$\forall wor(wRo \,\&\, oRr \supset wRr)$ transitivity
E	$\Diamond A \supset \Box\Diamond A$	$\forall wor(wRo \,\&\, wRr \supset oRr)$ euclideanness
B	$A \supset \Box\Diamond A$	$\forall wo(wRo \supset oRw)$ symmetry
3	$\Box(\Box A \supset B) \vee \Box(\Box B \supset A)$	$\forall wor(wRo \,\&\, wRr \supset oRr \vee rRo)$ connectedness
D	$\Box A \supset \Diamond A$	$\forall w \exists o \; wRo$ seriality
2	$\Diamond\Box A \supset \Box\Diamond A$	$\forall wor(wRo \,\&\, wRr \supset \exists l(oRl \,\&\, rRl))$ directedness
W	$\Box(\Box A \supset A) \supset \Box A$	no infinite R-chains + transitivity

The frame properties in the first group (T, 4, E, B, 3) are universal axioms, those in the second group are what are known as geometric implications (cf. Negri 2003), whereas the last one is not expressible as a first-order property.

The systems **T**, **K4**, **KB**, **S4**, **B**, **S5**, ... are obtained by adding one or more axioms to the system **K**. Sequent calculi are obtained by adding to the system **G3K** the rules that correspond to the properties of the accessibility relation that characterize their frames. For instance, a sequent calculus for **S4** is obtained by adding to **G3K** the rules that correspond to the axioms of reflexivity and transitivity of the accessibility relation:

$$\frac{wRw, \Gamma \to \Delta}{\Gamma \to \Delta} \; Ref \qquad \frac{wRr, wRo, oRr, \Gamma \to \Delta}{wRo, oRr, \Gamma \to \Delta} \; Trans$$

A system for **S5** is obtained by adding also the rule that corresponds to symmetry:

$$\frac{oRw, wRo, \Gamma \to \Delta}{wRo, \Gamma \to \Delta} \; Sym$$

The rule for euclideanness is

$$\frac{oRr, wRo, wRr, \Gamma \to \Delta}{wRo, wRr, \Gamma \to \Delta} \; Eucl$$

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If o is substituted for r in *Eucl*, a duplication wRo, wRo is produced in the premiss and in the conclusion. The same happens in *Trans* if $w \equiv o \equiv r$. Contracted instances of these rules must be added to the system:

$$\frac{wRw, wRw, \Gamma \to \Delta}{wRw, \Gamma \to \Delta} \; Trans^* \qquad \frac{oRo, wRo, \Gamma \to \Delta}{wRo, \Gamma \to \Delta} \; Eucl^*$$

Both of the contracted rules are instances of rule *Ref*, therefore, in order to have the rule of contraction admissible, they have to be added into systems that do not contain rule *Ref*. Similar additions must be made for all extensions by rules that have instances with two occurrences of the same relational atom in the conclusion. The condition we require to be satisfied by this addition is called *closure condition* (cf. section 6.1 of Negri and von Plato 2001). The closure condition is unproblematic because it requires only a bounded, very small number (usually one or two) of rules to be added. This is general and uniform even if, as seen above, there are contracted rules that may turn out to be superfluous in some systems.

Extensions are obtained in a modular way for all possible combinations of properties.

G3T = **G3K** + *Ref*
G3K4 = **G3K** + *Trans*
G3KB = **G3K** + *Sym*
G3S4 = **G3K** + *Ref* + *Trans*
G3TB = **G3K** + *Ref* + *Sym*
G3S5 = **G3K** + *Ref* + *Trans* + *Sym*

A system for **deontic logic** is obtained by the addition of the geometric rule *Ser*:

$$\frac{wRo, \Gamma \to \Delta}{\Gamma \to \Delta} \; Ser$$

Here the variable condition is $o \notin \Gamma, \Delta$.

Directedness is another property that follows the pattern of a geometric implication, and it is converted into the rule

$$\frac{oRl, rRl, wRo, wRr, \Gamma \to \Delta}{wRo, wRr, \Gamma \to \Delta} \; Dir$$

The variable condition is $l \notin wRo, wRr, \Gamma, \Delta$.

The property that corresponds to axiom W, needed for provability logic, can be incorporated in the system through a modification of the rules for $\Box$ (cf. section 5 of Negri 2005).

4.3. Structural properties

Let **G3K*** be any extension of **G3K** by rules for the accessibility relation that follow the *regular rule scheme* for extensions of sequent calculus (as in chapter 6 of Negri and von Plato 2001) or the more general *geometric rule scheme* (as in Negri 2003). The following properties can be established uniformly for all systems that belong to the class **G3K***. We refer to Negri (2005) for the proofs.

Lemma 4.1. *Sequents of the form*

$$w : A, \Gamma \to \Delta, w : A$$

with A an arbitrary modal formula are derivable in **G3K***.

To prove the correspondence between our systems and their Hilbert-style presentations, it is necessary to show that the characteristic axioms are derivable and the systems closed under the rules of necessitation and *modus ponens*. The latter will be a consequence of admissibility of cut.

Lemma 4.2. *For arbitrary A and B, the sequent*

$$\to w : \Box(A \supset B) \supset (\Box A \supset \Box B)$$

is derivable in **G3K***.

The rule of necessitation,

$$\frac{\to w : A}{\to w : \Box A}$$

is a context-dependent rule, as it requires both the antecedent and succedent contexts to be empty. As an explicit rule, it would impair the flexibility of the systems in the permutations that are needed for proving cut elimination; However, we do not need to add any such rule because we can show that it is admissible. To prove this, we exploit the first-order features of the system to show a lemma about substitution.

Substitution of labels is defined in the obvious way for relational atoms and labelled formulas and is extended to multisets componentwise. We have

Lemma 4.3. *If* $\Gamma \to \Delta$ *is derivable in* **G3K***, *then* $\Gamma(o/w) \to \Delta(o/w)$ *is also derivable, with the same derivation height.*

Theorem 4.4. *The rules of weakening*

$$\frac{\Gamma \to \Delta}{w : A, \Gamma \to \Delta} LW \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, w : A} RW$$

$$\frac{\Gamma \to \Delta}{wRo, \Gamma \to \Delta} LW \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, wRo} RW$$

are height-preserving admissible in **G3K***.

Corollary 4.5. *The necessitation rule is admissible in* **G3K***.

We also obtain a very useful property of a sequent calculus, namely:

Lemma 4.6. *All the rules of* **G3K*** *are height-preserving invertible.*

The most important structural property of our calculi, besides cut-admissibility, is height-preserving admissibility of contraction. First observe that there are, *a priori*, four contraction rules, namely left and right contraction for expressions of the forms $w : A$ and wRo. Explicitly stated, the rules of left and right contraction are:

$$\frac{w:A, w:A, \Gamma \to \Delta}{w:A, \Gamma \to \Delta} \; LC \qquad \frac{wRo, wRo, \Gamma \to \Delta}{wRo, \Gamma \to \Delta} \; LC_R$$

$$\frac{\Gamma \to \Delta, w:A, w:A}{\Gamma \to \Delta, w:A} \; RC \qquad \frac{\Gamma \to \Delta, wRo, wRo}{\Gamma \to \Delta, wRo} \; RC_R$$

Observe that rule RC_R is not needed in case we use the calculus without the initial sequent $wRo, \Gamma \to \Delta, wRo$.

Theorem 4.7. *The rules of contraction are height-preserving admissible in* **G3K***.

Also cut can take two forms, namely

$$\frac{\Gamma \to \Delta, w:A \quad w:A, \Gamma' \to \Delta'}{\Gamma, \Gamma' \to \Delta, \Delta'} \; Cut$$

and

$$\frac{\Gamma \to \Delta, wRo \quad wRo, \Gamma' \to \Delta'}{\Gamma, \Gamma' \to \Delta, \Delta'} \; Cut_R$$

However,Cut_R is not needed if the variant of **G3K** without the initial sequent $wRo, \Gamma \to \Delta, wRo$ is used.

We have:

Theorem 4.8. *The cut rule is admissible in* **G3K***.

5. Kripke completeness revisited

Kripke's original proof of completeness for modal logic used a direct construction of a Beth tree from a failed proof search. In later proofs, Kripke countermodels had nodes built from Henkin sets of formulas and extra devices that impose additional properties on the accessibility relation that are not automatically captured by the Henkin construction.[6] Kripke used tableaux in which the semantical element was hidden in their tree structure and therefore had to use some not fully formalized arguments in his completeness proofs. We show that, for the labelled calculus introduced in the previous Section, we can give a completeness proof close to Kripke's original argument but without any appeal to geometric intuition. For every sequent, the proof search either ends in a proof or fails, and the failed proof tree gives a Kripke countermodel.

5.1. Soundness

We reformulate first the semantical notions of Section 2 so that they apply to our labelled calculi:

Definition 5.1. *Let K be a frame with an accessibility relation $\mathcal{R}$ that satisfies the properties* $\ast$. *Let W be the set of variables (labels) used in derivations in* **G3K***. *An* **interpretation**

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[6]Such devices include for example "bulldozing" methods for imposing irreflexivity.

*of the labels W in frame K is a function $[\![\cdot]\!] : W \to K$. A **valuation** of atomic formulas in frame K is a map $\mathcal{V} : AtFrm \to \mathcal{P}(K)$ that assigns to each atom P the set of nodes of K in which P holds; the standard notation for $k \in \mathcal{V}(P)$ is $k \Vdash P$.*

Valuations are extended to arbitrary formulas by the following inductive clauses:

$k \Vdash \bot$ for no k,
$k \Vdash A\&B$ if $k \Vdash A$ and $k \Vdash B$,
$k \Vdash A \vee B$ if $k \Vdash A$ or $k \Vdash B$,
$k \Vdash A \supset B$ if $k \Vdash A$ implies $k \Vdash B$,
$k \Vdash \Box A$ if for all k', from $k\mathcal{R}k'$ follows $k' \Vdash A$,
$k \Vdash \Diamond A$ if there exists k' such that $k\mathcal{R}k'$ and $k' \Vdash A$.

Definition 5.2. *A sequent $\Gamma \to \Delta$ is **valid for an interpretation and a valuation** in K if for all labelled formulas $w : A$ and relational atoms oRr in Γ, whenever $[\![w]\!] \Vdash A$ and $[\![o]\!]\mathcal{R}[\![r]\!]$ in K, then for some $l : B$ in Δ, $[\![l]\!] \Vdash B$. A sequent is **valid** if it is valid for every interpretation and every valuation in a frame.*

Theorem 5.3. *If the sequent $\Gamma \to \Delta$ is derivable in **G3K***, then it is valid in every frame with the properties $*$.*

Proof. By induction on the derivation of $\Gamma \to \Delta$ in **G3K***. If it is an initial sequent, then there is a labelled atom $w : P$ both in Γ and in Δ so the claim is obvious, and similarly if the sequent is conclusion of $L\bot$ since for no valuation can $\bot$ be forced at any node.

If $\Gamma \to \Delta$ is a conclusion of a propositional rule, assume the rule is $L\&$ with the premiss $w : A, w : B, \Gamma' \to \Delta$. Assume that for an arbitrary assignment and interpretation, all the formulas in Γ are valid. Since $[\![w]\!] \Vdash A\&B$ is equivalent to $[\![w]\!] \Vdash A$ and $[\![w]\!] \Vdash B$, the inductive hypothesis, i.e., validity of $w : A, w : B, \Gamma' \to \Delta$ for every interpretation, gives the desired conclusion.

If $\Gamma \to \Delta$ is a conclusion of a modal rule, say $R\Box$, with the premiss $wRo, \Gamma' \to \Delta', o : A$, assume by the induction hypothesis that the premiss is valid. Let $[\![\cdot]\!]$ be an arbitrary interpretation that validates all the formulas in Γ'. We claim that one of the formulas in Δ' or $w : \Box A$ is valid under this intepretation. Let k be an arbitrary element of K such that $[\![w]\!]Rk$; let $[\![\cdot]\!]'$ be the interpretation identical to $[\![\cdot]\!]$ except possibly on o, where we set $[\![o]\!]' \equiv k$. Clearly $[\![\cdot]\!]'$ validates all the formulas in the antecedent of the premiss, so it validates a formula in Δ' or $o : A$ (the alternative being independent of the choice of $[\![o]\!]'$). In the former case we have that also $[\![\cdot]\!]$ validates a formula in Δ', in the latter that $[\![\cdot]\!]$ validates $w : \Box A$.

If the sequent is a conclusion of a mathematical rule without eigenvariables, let the rule be for instance *Trans*:

$$\frac{wRr, wRo, oRr, \Gamma \to \Delta}{wRo, oRr, \Gamma \to \Delta}$$

Let $[\![w]\!]\mathcal{R}[\![o]\!]$ and $[\![o]\!]\mathcal{R}[\![r]\!]$. Since $\mathcal{R}$ satisfies transitivity by assumption, we have $[\![w]\!]\mathcal{R}[\![r]\!]$, so validity of the premiss gives validity of the conclusion.

If the sequent is a conclusion of a mathematical rule with eigenvariables, let the rule be

for instance *Directedness*

$$\frac{oRl, rRl, wRo, wRr, \Gamma \to \Delta}{wRo, wRr, \Gamma \to \Delta}$$

Here l is an eigenvariable. Since by hypothesis the frame is directed, if $[\![w]\!]\mathcal{R}[\![o]\!]$ and $[\![w]\!]\mathcal{R}[\![r]\!]$, there exists d such that $[\![o]\!]\mathcal{R}d$ and $[\![r]\!]\mathcal{R}d$. The premiss is valid for all interpretations, in particular for one that coincides with $[\![\cdot]\!]$ on all labels, except possibly on l where it is assigned value d (this choice is possible because l is an eigenvariable). It follows that one of the formulas in Δ holds under this interpretation. QED.

5.1. Completeness

The proof of completeness follows the pattern of the proof of completeness for predicate logic, as in Negri and von Plato (2001, section 4.4).

The idea we pursue with the labelled system is the same as in Kripke's proof, but instead of looking for a failed search of a countermodel, we look directly for a proof: To see whether a formula is derivable, we check if it is universally valid, that is, valid at an arbitrary world for an arbitrary valuation, $w \Vdash A$. This is translated to a sequent $\to w : A$ in our calculus. The rules of the calculus applied backwards give equivalent conditions until the atomic components of A are reached. It can happen that we find a proof, or that we find that a proof does not exist either because we reach a stage where no rule is applicable, or because we go on with the search forever. In the two latter cases the attempted proof itself gives a countermodel.

Theorem 5.4. *Let $\Gamma \to \Delta$ be a sequent in the language of **G3K***. Then either the sequent is derivable in **G3K*** or it has a Kripke countermodel with properties *.*

Proof. We define for an arbitrary sequent $\Gamma \to \Delta$ in the language of **G3K*** a reduction tree by applying the rules of **G3K*** root first in all possible ways. If the construction terminates we obtain a proof, else the tree becomes infinite. By König's lemma an infinite tree has an infinite branch that is used to define a countermodel to the endsequent.

1. Construction of the reduction tree: The reduction tree is defined inductively in stages as follows:

Stage 0 has $\Gamma \to \Delta$ at the root of the tree. Stage $n > 0$ has two cases:

Case I: If every topmost sequent is an initial sequent or a conclusion of $L\bot$ or of a zero-premiss mathematical rule, the construction of the tree ends.

Case II: If not every topmost sequent is an initial sequent or a conclusion of $L\bot$ or of a zero-premiss mathematical rule, we continue the construction of the tree by writing above those topsequents that are not initial, nor conclusions of $L\bot$ or of a zero-premiss mathematical rule, other sequents that are obtained by applying root-first the rules of **G3K*** whenever possible, in a given order.

There are $10 + r$ different stages, 10 for the rules of the basic modal systems, r for the mathematical rules. At stage $n = 10 + r + 1$ we repeat stage 1, at stage $n = 10 + r + 1$ we repeat stage 2, and so on for every n.

We start, for $n = 1$, with $L\&$: For each topmost sequent of the form

$$w_1 : B_1 \& C_1, \ldots, w_m : B_m \& C_m, \Gamma' \to \Delta$$

where $B_1 \& C_1, \ldots, B_m \& C_m$ are all the formulas in Γ with a conjunction as the outermost logical connective, we write

$$w_1 : B_1, w_1 : C_1, \ldots, w_m : B_m, w_m : C_m, \Gamma' \to \Delta$$

on top of it. This step corresponds to applying root first m times rule $L\&$.

For $n = 2$, we consider all the sequents of the form

$$\Gamma \to w_1 : B_1 \& C_1, \ldots, w_m : B_m \& C_m, \Delta'$$

where $w_1 : B_1 \& C_1, \ldots, w_m : B_m \& C_m$ are all the labelled formulas in the succedent with a conjunction as the outermost logical connective. We write on top of them the 2^m sequents

$$\Gamma \to w_1 : D_1, \ldots, w_m : D_m, \Delta'$$

where D_i is either B_i or C_i and all possible choices are taken. This is equivalent to applying $R\&$ root first successively with principal labelled formulas $w_1 : B_1 \& C_1, \ldots, w_m : B_m \& C_m$.

For $n = 3$ and 4 we consider $L\vee$ and $R\vee$ and define the reductions symmetrically to the cases $n = 2$ and $n = 1$, respectively.

For $n = 5$, for each topmost sequent that has the labelled formulas $w_1 : B_1 \supset C_1, \ldots, w_m : B_m \supset C_m$ with implication as the outermost logical connective in the antecedent, Γ' the other formulas, and succedent Δ, we write on top of it the 2^m sequents

$$w_{i_1} : C_{i_1}, \ldots, w_{i_k} : C_{i_k}, \Gamma' \to w_{j_{k+1}} : B_{j_{k+1}}, \ldots, w_{j_m} : B_{j_m}, \Delta$$

Here $i_1, \ldots, i_k \in \{1, \ldots, m\}$ and $j_{k+1}, \ldots, j_m \in \{1, \ldots, m\} - \{i_1, \ldots, i_k\}$. This step, perhaps less transparent because of the double indexing, corresponds to the root-first application of rule $L\supset$ with principal formulas $w_1 : B_1 \supset C_1, \ldots, w_m : B_m \supset C_m$.

For $n = 6$, we consider all the labelled sequents that have implications in the succedent, say $w_1 : B_1 \supset C_1, \ldots, w_m : B_m \supset C_m$, and Δ' the other formulas, and write on top of them

$$w_1 : B_1, \ldots, w_m : B_m, \Gamma \to w_1 : C_1, \ldots, w_m : C_m, \Delta'$$

that is, apply root first m times rule $R\supset$.

For $n = 7$, we consider all topsequents with modal formulas $w_1 : \Box B_1, \ldots, w_m : \Box B_m$ and relational atoms $w_1 R o_1, \ldots, w_m R o_m$ in the antecedent, and write on top of these sequents the sequents

$$o_1 : B_1, \ldots, o_m : B_m, w_1 : \Box B_1, \ldots, w_m : \Box B_m, w_1 R o_1, \ldots, w_m R o_m, \Gamma' \to \Delta$$

that is, apply m times rule $L\Box$.

For $n = 8$, let $w_1 : \Box B_1, \ldots, w_m : \Box B_m$ be all the formulas with $\Box$ as the outermost connective in the succedent of topsequents of the tree, and let Δ' be the other formulas. Let

$r_1, \ldots, r_m$ be fresh variables, not yet used in the reduction tree, and write on top of each sequent the sequent

$$w_1 R r_1, \ldots, w_m R r_m, \Gamma \to \Delta, r_1 : B_1, \ldots, r_m : B_m$$

that is, apply m times rule $R\square$.

For $n = 9$, let $w_1 : \Diamond B_1, \ldots, w_m : \Diamond B_m$ be all the formulas with $\Diamond$ as the outermost connective in the antecedent of topsequents of the tree, and let Γ' be the other formulas. Let $l_1, \ldots, l_m$ be fresh variables, and write on top of each sequent the sequent

$$w_1 R l_1, \ldots, w_m R l_m, l_1 : B_1, \ldots, l_m : B_m, \Gamma' \to \Delta$$

that is, apply m times rule $L\Diamond$.

For $n = 10$, consider all topsequents with modal formulas $w_1 : \Diamond B_1, \ldots, w_m : \Diamond B_m$ in the succedent and relational atoms $w_1 R o_1, \ldots, w_m R o_m$ in the antecedent, and write on top of these sequents the sequents

$$w_1 R o_1, \ldots, w_m R o_m, \Gamma \to \Delta', w_1 : \Diamond B_1, \ldots, w_m : \Diamond B_m, o_1 : B_1, \ldots, o_m : B_m$$

that is, apply m times rule $R\Diamond$.

Finally, for $n = 10 + j$, we consider the generic case of a mathematical rule, that is, a rule for the relation R. For systems with the *subterm property*,[7] the mathematical rules need to be instantiated only on terms in the conclusion or on eigenvariables. Thus, if the system contains rule *Ref*, instances of that rule consist in adding to the antecedent all the relational atoms wRw for w in $\Gamma \to \Delta$; With a rule with eigenvariables, such as seriality, the step for that rule adds all the atoms of the form wRo for w in $\Gamma \to \Delta$ and o a fresh variable. Observe that because of height-preserving substitution and height-preserving admissibility of contraction, once a rule with eigenvariables has been considered, it need not be instantiated again on the same principal formulas. If it is a rule such as *Trans*, consider all the sequents with a pair of atoms of the form wRo, oRr in the antecedent and write on top of them the sequents with the atoms wRr added.

For any n, for each sequent that is neither initial, nor conclusion of $L\bot$, nor of a zero-premiss mathematical rule, nor treatable by any one of the above reductions, we write the sequent itself above it.

If the reduction tree is finite, all its leaves are initial or conclusions of $L\bot$, or of zero-premiss mathematical rules, and the tree, read from the leaves to the root, yields a derivation.

2. Construction of the countermodel: If the reduction tree is infinite, it has an infinite branch. Let $\Gamma_0 \to \Delta_0 \equiv \Gamma \to \Delta, \Gamma_1 \to \Delta_1 \ldots, \Gamma_i \to \Delta_i, \ldots$ be one such branch. Consider the sets of labelled formulas and relational atoms

$$\mathbf{\Gamma} \equiv \bigcup_{i \geq 0} \Gamma_i \qquad \mathbf{\Delta} \equiv \bigcup_{i \geq 0} \Delta_i$$

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[7] Cf. section 6 of Negri 2005.

We define a Kripke model that forces all the formulas in $\mathbf{\Gamma}$ and no formula in $\mathbf{\Delta}$ and is therefore a countermodel to the sequent $\Gamma \to \Delta$.

Consider the frame K the nodes of which are all the labels that appear in the relational atoms in $\mathbf{\Gamma}$, with their mutual relationships expressed by the wRo's in $\mathbf{\Gamma}$. Clearly, the construction of the reduction tree imposes the frame properties of the countermodel, for instance, in the system $\mathbf{G3S4}$, the constructed frame is reflexive and transitive. The model is defined as follows: For all atomic formulas $w : P$ in $\mathbf{\Gamma}$, we stipulate that $w \Vdash P$ in the frame, and for all atomic formulas $o : Q$ in $\mathbf{\Delta}$ we stipulate that $o \nVdash Q$. Since no sequent in the infinite branch is initial, this choice can be coherently made, for if there were the same labelled atom in $\mathbf{\Gamma}$ and in $\mathbf{\Delta}$, then, since the sequents in the reduction tree are defined in a cumulative way, for some i there would be a labelled atom $w : P$ both in the antecedent and in the succedent of $\Gamma_i \to \Delta_i$.

We then show inductively on the weight of formulas that A is forced in the model at node w if $w : A$ is in $\mathbf{\Gamma}$ and A is not forced at node w if $w : A$ is in $\mathbf{\Delta}$. Therefore we have a countermodel to the endsequent $\Gamma \to \Delta$.

If A is $\perp$, it cannot be in $\mathbf{\Gamma}$ because no sequent in the branch contains $w : \perp$ in the antecedent, so it is not forced at any node of the model.

If A is atomic, the claim holds by the definition of the model.

If $w : A \equiv w : B \& C$ is in $\mathbf{\Gamma}$, there exists i such that $w : A$ appears first in Γ_i, and therefore, for some $l \geq 0$, $w : B$ and $w : C$ are in Γ_{i+l}. By the induction hypothesis, $w \Vdash B$ and $w \Vdash C$, and therefore $w \Vdash B \& C$.

If $w : A \equiv w : B \& C$ is in $\mathbf{\Delta}$, consider the step i in which the reduction for A applies. This gives a branching, and one of the two branches belongs to the infinite branch, so either $w : B$ or $w : C$ is in $\mathbf{\Delta}$, and therefore by the inductive hypothesis, $w \nVdash B$ or $w \nVdash C$, and therefore $w \nVdash B \& C$.

If $w : A \equiv w : B \vee C$ is in $\mathbf{\Gamma}$, we reason similarly to the case of $w : A \equiv w : B \& C$ in $\mathbf{\Delta}$.

If $w : A \equiv w : B \vee C$ is in $\mathbf{\Delta}$, we argue as with $w : A \equiv w : B \vee C$ in $\mathbf{\Gamma}$.

If $w : A \equiv w : B \supset C$ is in $\mathbf{\Gamma}$, then either $w : B$ is in $\mathbf{\Delta}$ or $w : C$ is in $\mathbf{\Gamma}$. By the inductive hypothesis, in the former case $w \nVdash B$, and in the latter $w \Vdash C$, so in both cases $w \Vdash B \supset C$.

If $w : A \equiv w : B \supset C$ is in $\mathbf{\Delta}$, then for some i, $w : B \in \Gamma_i$ and $w : C \in \Delta_i$, so by the inductive hypothesis $w \Vdash B$ and $w \nVdash C$, so $w \nVdash B \supset C$.

If $w : A \equiv w : \Box B$ is in $\mathbf{\Gamma}$, we consider all the relational atoms wRo that occur in $\mathbf{\Gamma}$. If there is no such atom, then the condition that for all o accessible from w in the frame, $o \Vdash B$ is vacuously satisfied, and therefore $w \Vdash \Box B$ in the model. Else, for any occurrence of wRo in $\mathbf{\Gamma}$ we find, by the construction of the reduction tree, an occurrence of $o : B$ in $\mathbf{\Gamma}$. By the inductive hypothesis, $o \Vdash B$, and therefore $w \Vdash \Box B$ in the model.

If $w : A \equiv w : \Box B$ is in $\mathbf{\Delta}$, consider the step at which the reduction for $w : A$ applies. We then find $o : B$ in $\mathbf{\Delta}$ for some o with wRo in $\mathbf{\Gamma}$. By the induction hypothesis, $o \nVdash B$, and therefore $w \nVdash A$.

The cases of $w : A \equiv w : \Diamond B$ in $\mathbf{\Gamma}$ and of $w : A \equiv w : \Diamond B$ in $\mathbf{\Delta}$ are symmetric to those of $w : A \equiv w : \Box B$ in $\mathbf{\Delta}$ and of $w : A \equiv w : \Box B$ in $\mathbf{\Gamma}$, respectively. QED.

Corollary 5.5. *If a sequent $\Gamma \to \Delta$ is valid in every Kripke model with the frame properties*

$*$, *then it is derivable in the system* **G3K***.

In case the system has an irreflexive accessibility relation, we have a zero-premiss mathematical rule of the form

$$wRw, \Gamma \rightarrow \Delta$$

A sequent of this form cannot appear in the infinite branch, and therefore the countermodel will be irreflexive by construction. The problem with properties such as irreflexivity in the Henkin-style completeness proof thus disappears with our approach.

Conclusion and further work

We have reviewed here two main styles of completeness proofs for modal logic, Kripke's original proofs and Henkin-style proofs, and discussed their relative merits. Although Kripke's original proofs were more informative, Henkin-style proofs have been preferred in the literature on modal logic because of the difficulties in formalizing Kripke's original proof.

There are two main trends in the recent literature on the proof theory of modal logic: one that enriches the language of sequents by the use of labels (cf. Negri 2007 for references to the vast literature), another that avoids the use of labels. Recent variants of the latter approach include the systems of *nested sequents* (Kashima 1994), *tree-sequents* (Cerrato 1996), *deep sequents* (Brünnler 2006, Stouppa 2007), and *tree-hypersequents* (Poggiolesi 2008). These works can be regarded as formalizations of Kripke's original approach even if they do not explicitly refer to Kripke's own contributions. Also, the treatment of modal systems with geometric frame conditions has so far remained out of their scope.

Section 1.5 of Boretti (2008) contains a useful methodological discussion of labelled and unlabelled systems. In her words, "whereas the semantic notions are explicitly internalised into the labelled calculi in the form of the syntactical counterparts of forcing ($x : A$) and accessibility relation (xRy), tree-hypersequents and deep sequent systems hide their relational semantics under a more complex syntax."

We have presented here a labelled sequent system that simplifies Kripke's tableau method thanks to the fact that the accessibility relation is an explicit part of the syntax and not an implicit property of proof-trees. A wide class of modal systems is covered and a uniform, simple, and direct proof of completeness obtained that does not present the shortcomings of the original Kripke proofs, nor the limitations of Henkin-style proofs. Completeness proofs for first-order modal logic and for provability logic, along the lines of the method presented here, appear in Negri and von Plato (2008). A similar treatment for non-normal modal logics should not present any extra difficulty, and is left to future work.

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