On the finitization of Priorean linear time

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ABSTRACT. A labelled sequent calculus for Priorean linear time is defined through the method of internalization of the possible-worlds semantics into the syntax. The calculus has good structural properties, such as syntactic cut elimination, but requires an infinitary mathematical rule stating that between any two points there are only finitely many points. By replacing the infinitary rule with two weaker finitary rules a system for non-standard discrete frames is obtained. A conservativity result for an appropriate fragment of the original calculus is proved syntactically.

1 Introduction

The birth of temporal logic is closely connected with the name of Arthur Prior and his interest in classical philosophical problems, such as the conflict between fatalism and free will. The study of the answers given to this question by ancient philosophers including Aristotle and Diodorus Cronus, and medieval ones such as Ockham and Peter de Rivo gave him the idea to develop a logic of time on the model of the then nascent modal logic: Temporal operators for future and for past were to be formulated in analogy to the modalities $\Box$ and $\Diamond$ of necessity and possibility. Further operators were later introduced to denote the next and the previous moment (von Wright [23], Scott [21]). The introduction of the 'until' and 'since' operators into linear-time logic by Kamp [12] allowed the formulation of a more expressive temporal logic.

The importance of temporal logic increased greatly as a consequence of its application to computer science. Several versions of temporal logic have been considered, each reflecting the properties of the intended frames (linear, branching, circular, ...) or the presence or absence of past operators. In particular, Linear Time Logic (LTL) is a temporal logic without past operators that corresponds to discrete frames isomorphic to the natural numbers.

Propositional linear time logic is decidable, as shown for instance by Kesten *et al.* [13] with tableau methods, but the inherent presence of induction makes the development of a finitary proof system problematic. Decidability has not been, so far, established through terminating proof search for the whole logic, but only for fragments, as in the tableau system proposed by Schmitt and Goubalt-Larrecq [20]. Whereas tableau systems involve non-local rules, that is, global correctness conditions, systems of natural deduction or sequent calculus for full LTL typically either require a rule with an infinite number of premisses or are not normalizable/cut free ([5], [8]). Several attempts have been made in order to obtain a finitary cut-free calculus for LTL . A significant indirect contribution is found in Jäger *et al.* [11], where the finite model property is used to give an upper bound on the number of premisses of an infinitary rule, formally similar to the one used in temporal logic, for the logic of common knowledge. The semantic method allows to prove completeness for the calculus but not cut elimination.

We have a different goal in this work: Instead of trying to finitize the calculus for linear time, we identify a finitary fragment of the system. We use the method of internalization of the possible world semantics within the syntax of sequent calculi, as developed by Negri in [15, 16]. A labelled system G3LT for Priorean linear time is introduced, in Section 2.1, by adding to the basic calculus for temporal logic the mathematical rules that correspond to the properties of the intended class of frames. In particular, discreteness is given by an infinitary rule that states: If x is less than y, then x is the predecessor of y, or it is the predecessor of the predecessor of y, or ... and so on. Structural properties, such as the admissibility of weakening, contraction, and cut, are proved syntactically in Section 2.2, along the guidelines of the general method by Negri and von Plato [17].

A weaker system G3LT$_f$ is formulated in Section 3 by replacing the infinitary rule with two finitary counterparts that permit the splitting of an interval $[x, y]$ with an immediate successor of x and an immediate predecessor of y, respectively. Every sequent derivable in the finitary system is derivable in the infinitary one. The converse fails, but we identify a fragment of G3LT for which conservativity with respect to G3LT$_f$ is proved.

We conclude with a discussion of related literature.

2 A sequent calculus for Priorean linear time

Among the various versions of linear time logic, we consider here the calculus proposed by Prior in [19] (system 7.2, p. 178), which is characterized by the presence of both future and past operators: In addition to the traditional **G**, 'it is and always will be', and **H**, 'it is and always has been', also the next and last instant **T**, 'tomorrow', and **Y**, 'yesterday', are considered. If past operators are dropped, we obtain a system corresponding to the one commonly called unary LTL.

The view, developed after Prior's work, of temporal logic as a special modal logic, makes the use of Kripke semantics very natural. Kripke frames are interpreted as ordered sets of instants in the flow of time, with the accessibility relation being the order of temporal precedence. The syntax of temporal logic can thus be developed within the *method of internalization of Kripke semantics* for modal and non-classical logics: Semantic elements, such as possible worlds and accessibility relations, appear on a par with logical constants in systems of inference and the rules are directly generated from the semantic explanation of the logical constants. The systems of inference that result from this internalization are called *labelled systems*. From the extensive literature on labelled and hybrid systems (cf. [7] and the references discussed in [16]), we shall follow the method introduced by the second author in [15]. The treatment of temporal logic requires nontrivial extensions of the basic method and we shall therefore proceed with a self-contained presentation rather than relying on a general background (that can however be found in section 1 of [16]).

2.1 Logical and mathematical rules

Our sequent calculus for linear time is obtained as follows: The starting point is the cut- and contraction-free sequent calculus G3 that was introduced by Ketonen in the 1940's and recently systematically presented in [22]. In [17, 18] and in [14] a general method was presented for extending the basic logical sequent calculus without losing the structural properties such as admissibility of cut: Axioms for specific theories are suitably converted into inference rules to be added to the logical sequent calculus while preserving all the structural properties of the basic sequent system. For systems with internalized Kripke semantics the syntax of the calculus has to be enriched with labels and relations: Every formula in a sequent $\Gamma \Rightarrow \Delta$ is either a relational atomic formula $x \leq y$, $x \prec y$, $x = y$, or a labelled formula $x : A$. Intuitively, relational atoms and labelled formulas are the counterpart of the accessibility or equality relations and of the forcing relation $x \Vdash A$ of Kripke models, respectively.

The rules for the propositional connectives are analogous to the standard rules, with the active and principal formulas all marked by the same label x. For temporal operators, the rules are obtained from the meaning explanations in terms of their relational semantics:

$x \Vdash \mathbf{G}A$ (*resp.* $x \Vdash \mathbf{H}A$) *iff for all* y, $x \leq y$ (*resp.* $y \leq x$) *implies* $y \Vdash A$
$x \Vdash \mathbf{F}A$ (*resp.* $x \Vdash \mathbf{P}A$) *iff for some* y, $x \leq y$ (*resp.* $y \leq x$) *and* $y \Vdash A$
$x \Vdash \mathbf{T}A$ (*resp.* $x \Vdash \mathbf{Y}A$) *iff for all* y, $x \prec y$ (*resp.* $y \prec x$) *implies* $y \Vdash A$

The left-to-right direction in the explanation above justifies the left rules, the right-to-left direction the right rules. The rôle of the quantifiers is reflected in the variable conditions for rules $R\mathbf{G}$, $L\mathbf{F}$, $R\mathbf{T}$, $R\mathbf{H}$, $L\mathbf{P}$ and $R\mathbf{Y}$ below.

The logical rules for the calculus are given in Table 1. Observe that initial sequents are restricted to labelled atomic formulas $x : P$ or relational atoms At. This feature, common to all G3 systems of sequent calculus, is needed to ensure invertibility of the rules (Lemma 5) and other structural properties.

Initial sequents:

$$x : P, \Gamma \Rightarrow \Delta, x : P \qquad\qquad At, \Gamma \Rightarrow \Delta, At$$

Propositional rules:

$$\dfrac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A\&B, \Gamma \Rightarrow \Delta}\, L\& \qquad\qquad \dfrac{\Gamma \Rightarrow \Delta, x : A \quad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A\&B}\, R\&$$

$$\dfrac{x : A, \Gamma \Rightarrow \Delta \quad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta}\, L\vee \qquad\qquad \dfrac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B}\, R\vee$$

$$\dfrac{\Gamma \Rightarrow \Delta, x : A \quad x : B, \Gamma \Rightarrow \Delta}{x : A \supset B, \Gamma \Rightarrow \Delta}\, L\supset \qquad\qquad \dfrac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \supset B}\, R\supset$$

$$\dfrac{}{x : \perp, \Gamma \Rightarrow \Delta}\, L\perp$$

Temporal rules

$$\dfrac{y : A, x : \mathbf{G}A, x \leq y, \Gamma \Rightarrow \Delta}{x : \mathbf{G}A, x \leq y, \Gamma \Rightarrow \Delta}\, L\mathbf{G} \qquad\qquad \dfrac{x \leq y, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathbf{G}A}\, R\mathbf{G}$$

$$\dfrac{x \leq y, y : A, \Gamma \Rightarrow \Delta}{x : \mathbf{F}A, \Gamma \Rightarrow \Delta}\, L\mathbf{F} \qquad\qquad \dfrac{x \leq y, \Gamma \Rightarrow \Delta, x : \mathbf{F}A, y : A}{x \leq y, \Gamma \Rightarrow \Delta, x : \mathbf{F}A}\, R\mathbf{F}$$

$$\dfrac{y : A, x : \mathbf{T}A, x \prec y, \Gamma \Rightarrow \Delta}{x : \mathbf{T}A, x \prec y, \Gamma \Rightarrow \Delta}\, L\mathbf{T} \qquad\qquad \dfrac{x \prec y, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathbf{T}A}\, R\mathbf{T}$$

$$\dfrac{y : A, x : \mathbf{H}A, y \leq x, \Gamma \Rightarrow \Delta}{x : \mathbf{H}A, y \leq x, \Gamma \Rightarrow \Delta}\, L\mathbf{H} \qquad\qquad \dfrac{y \leq x, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathbf{H}A}\, R\mathbf{H}$$

$$\dfrac{y \leq x, y : A, \Gamma \Rightarrow \Delta}{x : \mathbf{P}A, \Gamma \Rightarrow \Delta}\, L\mathbf{P} \qquad\qquad \dfrac{y \leq x, \Gamma \Rightarrow \Delta, x : \mathbf{P}A, y : A}{y \leq x, \Gamma \Rightarrow \Delta, x : \mathbf{P}A}\, R\mathbf{P}$$

$$\dfrac{y : A, x : \mathbf{Y}A, y \prec x, \Gamma \Rightarrow \Delta}{x : \mathbf{Y}A, y \prec x, \Gamma \Rightarrow \Delta}\, L\mathbf{Y} \qquad\qquad \dfrac{y \prec x, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathbf{Y}A}\, R\mathbf{Y}$$

Rules $R\mathbf{G}$, $L\mathbf{F}$, $R\mathbf{T}$, $R\mathbf{H}$, $L\mathbf{P}$ and $R\mathbf{Y}$ have the condition that y is not in the conclusion.

Table 1. Logical rules for the system G3LT

In addition to the logical rules of Table 1, we have mathematical rules that correspond to the frame properties of accessibility relations.

Rules for Equality

$$\frac{x = x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \textit{EqRef}$$

$$\frac{y : P, x = y, x : P, \Gamma \Rightarrow \Delta}{x = y, x : P, \Gamma \Rightarrow \Delta}\ \textit{EqSubst} \qquad \frac{At(y), x = y, At(x), \Gamma \Rightarrow \Delta}{x = y, At(x), \Gamma \Rightarrow \Delta}\ \textit{EqSubst}_{\mathrm{At}}$$

Rules for the Order Relation

$$\frac{x \leq z, x \leq y, y \leq z, \Gamma \Rightarrow \Delta}{x \leq y, y \leq z, \Gamma \Rightarrow \Delta}\ \textit{Trans} \qquad \frac{x \leq x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \textit{Ref}$$

Rules for the Successor Relation

$$\frac{y = z, y \prec x, z \prec x, \Gamma \Rightarrow \Delta}{y \prec x, z \prec x, \Gamma \Rightarrow \Delta}\ \textit{UnPred} \qquad \frac{y = z, x \prec y, x \prec z, \Gamma \Rightarrow \Delta}{x \prec y, x \prec z, \Gamma \Rightarrow \Delta}\ \textit{UnSucc}$$

$$\frac{y \prec x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \textit{L-Ser} \qquad \frac{x \prec y, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \textit{R-Ser} \qquad \frac{x \leq y, x \prec y, \Gamma \Rightarrow \Delta}{x \prec y, \Gamma \Rightarrow \Delta}\ \textit{Inc}$$

Rules *L-Ser* and *R-Ser* have the condition that y is not in the conclusion.

The order relation $x \leq y$ is defined as the transitive and reflexive closure of the immediate successor relation $x \prec y$, that is,

$$x \leq y \ \equiv\ \exists n \in \mathbb{N}\, (x \prec^n y)$$

This means that if $x \leq y$, then y is reachable from x by iterating finitely many times the immediate successor relation.

The iterated successor relation is defined inductively by the following clauses, that result in the mathematical rules below:

$x \prec^0 y \ \equiv x = y$,
$x \prec^1 y \ \equiv x \prec y$,
$x \prec^{n+1} y \ \equiv\ \exists z(x \prec^n z \,\&\, z \prec y)$ for $n > 0$.

Rules for the Iterated Successor Relation

$$\frac{x \prec^n y, y \prec z, \Gamma \Rightarrow \Delta}{x \prec^{n+1} z, \Gamma \Rightarrow \Delta}\ \textit{LDef} \quad \frac{\Gamma \Rightarrow \Delta, x \prec^{n+1} z, x \prec^n y \quad \Gamma \Rightarrow \Delta, x \prec^{n+1} z, y \prec z}{\Gamma \Rightarrow \Delta, x \prec^{n+1} z}\ \textit{RDef}$$

Rule *LDef* has the condition that y is not in the conclusion.

Infinitary Rule

The left-to-right direction of the definition of $x \leq y$ as the transitive closure of $x \prec y$ gives the following infinitary rule

$$\frac{\{x \prec^n y, x \leq y, \Gamma \Rightarrow \Delta\}_{n \in \mathbb{N}}}{x \leq y, \Gamma \Rightarrow \Delta}\ T^{\omega}$$

The right-to-left direction gives, for every $n \in \mathbb{N}$, the following generalized form of rule *Inc*

$$\frac{x \leq y, x \prec^n y, \Gamma \Rightarrow \Delta}{x \prec^n y, \Gamma \Rightarrow \Delta}\, {\scriptstyle Inc_n}$$

This rule is admissible in our system by induction on n. The proof uses equality rules for $n = 0$, and *Trans* for the inductive case.

Finally, we observe that the *closure condition* required for admissibility of contraction (see e.g. [15] p. 510) does not bring to new rules in the system above since the contracted instances of *Trans* , *UnPred*, and *UnSucc* are special cases of *Ref* and *EqRef*.

2.2 Structural properties

Next we prove the structural properties of the system G3LT.

LEMMA 1. *Sequents of the form $x : A, \Gamma \Rightarrow \Delta, x : A$, with A an arbitrary modal formula, are derivable in* G3LT*.*

Proof. By induction on the length of the formula A. ■

In order to guarantee invertibility of all the rules, initial sequents with relational atoms as principal cannot be of the form $x \prec^n y, \Gamma \Rightarrow \Delta, x \prec^n y$ for $n > 1$. However, sequents of this form are easily derivable:

LEMMA 2. *Sequents of the form $x \prec^n y, \Gamma \Rightarrow \Delta, x \prec^n y$ are derivable in* G3LT *for all $n \in \mathbb{N}$.*

Proof. By induction on n. For $n = 0, 1$, observe that $x = y, \Gamma \Rightarrow \Delta, x = y$ and $x \prec y, \Gamma \Rightarrow \Delta, x \prec y$ are initial sequents. For the inductive case, assume a derivation of $x \prec^n z, z \prec y, \Gamma \Rightarrow \Delta, x \prec^{n+1} y, x \prec^n z$ with z different from x, y and not in Γ, Δ, and derive the claim for $n + 1$ by applying *RDef* with right premiss $x \prec^n z, z \prec y, \Gamma \Rightarrow \Delta, x \prec^{n+1} y, z \prec y$ and then *LDef*. ■

Substitution of labels is defined in the obvious way for relational atoms and labelled formulas, and extended to multisets componentwise. We have:

LEMMA 3. *If $\Gamma \Rightarrow \Delta$ is derivable in* G3LT*, then also $\Gamma(y/x) \Rightarrow \Delta(y/x)$ is derivable, with the same derivation height.*

Proof. By induction on the height h of the derivation. If $h = 0$, then $\Gamma \Rightarrow \Delta$ is either an initial sequent or a conclusion of $L\bot$. In both cases, the sequent $\Gamma(y/x) \Rightarrow \Delta(y/x)$ is also an initial sequent or a conclusion of $L\bot$.

Suppose that the claim holds for $h = n$, and consider the last rule applied in the derivation. If it is a propositional rule or a temporal or mathematical rule without a variable condition, apply the inductive hypothesis to the premiss(es) and then the rule. If the last rule is a rule with a variable condition, we need to avoid a clash with the eigenvariable: In that case, we apply twice the inductive hypothesis to the premiss(es) first to replace the eigenvariable with a fresh variable not appearing in the derivation, and then to perform the desired substitution. ■

In what follows, Greek lower case letters are used for denoting labelled and relational formulas.

THEOREM 4. *The rules of left and right weakening*

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\, LWk \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\, RWk$$

are height-preserving admissible in G3LT.

Proof. By induction on the height of the derivation. If $\Gamma \Rightarrow \Delta$ is an initial sequent or a conclusion of $L\bot$, also $\varphi, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, \varphi$ are. The cases of rules without variable condition are straightforward. If the last step is a rule with a variable condition, we first apply Lemma 3 to avoid a clash of variables and then the inductive hypothesis and the rule in question. ■

LEMMA 5. *All rules of* G3LT *are height-preserving invertible.*

Proof. The proof of height-preserving invertibility for propositional rules, for rule $LDef$, and for temporal rules with a variable condition is by induction on the height of derivation (clash of variables is avoided through the substitution lemma). The condition that $x \prec^n y, \Gamma \Rightarrow \Delta, x \prec^n y$ is not initial for $n > 1$ is essential for the invertibility of rule *LDef*. ■

THEOREM 6. *The rules of left and right contraction*

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\, LCtr \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\, RCtr$$

are height-preserving admissible in G3LT.

Proof. By simultaneous induction on the height of derivation for left and right contractions. For $h = 0$, note that if $\varphi, \varphi, \Gamma \Rightarrow \Delta$ (resp. $\Gamma \Rightarrow \Delta, \varphi, \varphi$) is an initial sequent or a conclusion of $L\bot$, so is $\varphi, \Gamma \Rightarrow \Delta$ (resp. $\Gamma \Rightarrow \Delta, \varphi$). For $h = n + 1$, we distinguish two cases: If none of the contraction formulas is principal in the last rule, we apply the inductive hypothesis to the premiss(es) and then the rule; If one of the contraction formulas is principal, we first apply height-preserving inversion to the premiss(es), then inductive hypothesis, and last the rule; If both are principal, necessarily in a mathematical rule, by the closure condition contraction is absorbed into the contracted instance of the rule. ■

The system G3LT has mathematical rules that act on both the left- and the right-hand sides of sequents, and a measure of complexity for relational atoms is needed in the proof of cut elimination, as in [6].

DEFINITION 7. The length of a labelled formula $x : A$ is defined as the length of A. The length of relational and equality formulas is defined as follows: $l(x \prec y) = l(x \leq y) = l(x = y) = 1$ and $l(x \prec^n y) = n$ for $n \geq 1$.

THEOREM 8. *The rule of cut*

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,{\scriptstyle Cut}$$

is admissible in G3LT.

Proof. By induction on the length of the cut formula and a subinduction on the sum of the heights of the derivations of the premisses of cut. The proof has the structure of the proof of cut elimination for modal logics (see [15], Theorem 4.13). However, we have to consider here an essentially new case, because of the simultaneous presence of mathematical rules that act on both the left- and the right-hand sides of sequents. This is the case with the cut formula $x \prec^{n+1} y$ principal in both premisses of cut:

$$\frac{\dfrac{\Gamma \Rightarrow \Delta, x \prec^{n+1} y, x \prec^n z \quad \Gamma \Rightarrow \Delta, x \prec^{n+1} y, z \prec y}{\Gamma \Rightarrow \Delta, x \prec^{n+1} y}\,{\scriptstyle RDef} \quad \dfrac{x \prec^n w, w \prec y, \Gamma' \Rightarrow \Delta'}{x \prec^{n+1} y, \Gamma' \Rightarrow \Delta'}\,{\scriptstyle LDef}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,{\scriptstyle Cut}$$

This derivation is transformed as follows:

We first cut the left premiss of *RDef* with the conclusion of *LDef*

1. $$\frac{\Gamma \Rightarrow \Delta, x \prec^{n+1} y, x \prec^n z \quad \dfrac{x \prec^n w, w \prec y, \Gamma \Rightarrow \Delta}{x \prec^{n+1} y, \Gamma \Rightarrow \Delta}\,{\scriptstyle LDef}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', x \prec^n z}\,{\scriptstyle Cut}$$

thus obtaining a cut of shorter height. Then we cut the right premiss of $RDef$ with the conclusion of *LDef*

2. $$\frac{\Gamma \Rightarrow \Delta, x \prec^{n+1} y, z \prec y \quad \dfrac{x \prec^n w, w \prec y, \Gamma' \Rightarrow \Delta'}{x \prec^{n+1} y, \Gamma' \Rightarrow \Delta'}\,{\scriptstyle LDef}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta', z \prec y}\,{\scriptstyle Cut}$$

thus obtaining another cut of shorter height. Finally, we use the sequents thus obtained and the premiss of $LDef$ as follows:

$$\dfrac{\dfrac{\Gamma, \Gamma' \overset{2.}{\Rightarrow} \Delta, \Delta', z \prec y \qquad \dfrac{\Gamma, \Gamma' \overset{1.}{\Rightarrow} \Delta, \Delta', x \prec^{n-1} z \qquad \dfrac{x \prec^{n-1} w, w \prec y, \Gamma' \Rightarrow \Delta'}{x \prec^{n-1} z, z \prec y, \Gamma' \Rightarrow \Delta'}\,\textit{Hp-Subst}}{z \prec y, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta, \Delta', \Delta'}\,\textit{Cut}}{\Gamma, \Gamma, \Gamma', \Gamma', \Gamma' \Rightarrow \Delta, \Delta, \Delta', \Delta', \Delta'}\,\textit{Cut}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,\textit{Ctr}^*$$

Here the two cuts are on formulas of smaller length and *Hp-Subst* denotes a height-preserving substitution. ■

COROLLARY 9. *The following generalized rules of substitution of equals*

$$\dfrac{y \prec^n z, x = y, x \prec^n z, \Gamma \Rightarrow \Delta}{x = y, x \prec^n z, \Gamma \Rightarrow \Delta} \qquad \dfrac{z \prec^n y, x = y, z \prec^n x, \Gamma \Rightarrow \Delta}{x = y, z \prec^n x, \Gamma \Rightarrow \Delta} \qquad \dfrac{y : A, x = y, x : A, \Gamma \Rightarrow \Delta}{x = y, x : A, \Gamma \Rightarrow \Delta}$$

are admissible in G3LT*.*

Proof. Similar to the proof of admissibility of the replacement rule in predicate logic with equality (Theorem 6.5.3 in [18]). Using a cut of the premisses of the rules with the derivable sequents $x \prec^n z, x = y \Rightarrow y \prec^n z$ and $z \prec^n x, x = y \Rightarrow z \prec^n y$ and $x : A, x = y \Rightarrow y : A$, respectively, and admissibility of the rules of cut and contraction. ■

Because of the internalization of the semantics, most labelled sequents cannot be directly interpreted as temporal formulas. However, we can single out a class of sequents with a plain correspondence to their associated formulas:

DEFINITION 10. A *purely logical* sequent is a sequent that contains no relational atoms and in which every formula is labelled by the same variable x.

The Hilbert-style system for Priorean linear time logic can be embedded into our calculus: We show that the purely logical sequents corresponding to the temporal axioms and the modal rules are derivable/admissible in G3LT. Admissibility of *modus ponens* follows by cut elimination.

PROPOSITION 11. *The following purely logical sequents*

$$\begin{array}{ll}
x : \mathbf{G}(A \supset B), x : \mathbf{G}A \Rightarrow x : \mathbf{G}B & x : \mathbf{T}(A \supset B), x : \mathbf{T}A \Rightarrow x : \mathbf{T}B \\
x : \mathbf{T}\neg A \Rightarrow x : \neg\mathbf{T}A & x : \neg\mathbf{T}A \Rightarrow x : \mathbf{T}\neg A \\
x : \mathbf{G}A \Rightarrow x : A \,\&\, \mathbf{TG}A & x : A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow x : \mathbf{G}A \\
x : \mathbf{TG}A \Rightarrow x : \mathbf{GT}A & x : \mathbf{GT}A \Rightarrow x : \mathbf{TG} \\
x : \mathbf{TY}A \Rightarrow x : A & x : A \Rightarrow x : \mathbf{TY}A
\end{array}$$

and their temporal mirror images[1] *are derivable in* G3LT*.*

[1] The *temporal mirror image* of a purely logical sequent is obtained by replacing each occurrence of a future (resp. past) operator by its past (resp. future) analogue. For example the temporal mirror image of $x : \mathbf{G}P \Rightarrow x : P \& \mathbf{TG}P$ is $x : \mathbf{H}P \Rightarrow x : P \& \mathbf{YH}P$.

Proof. By root-first proof search from the sequent to be derived. Note that derivability of $x : A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow x : \mathbf{G}A$ and of its temporal mirror image require an application of T^ω. ■

PROPOSITION 12. *The necessitation rules for* **G**, **H**, **T** *and* **Y**

$$\frac{\Rightarrow x : A}{\Rightarrow x : \mathbf{G}A}\,\mathbf{G}Nec \quad \frac{\Rightarrow x : A}{\Rightarrow x : \mathbf{H}A}\,\mathbf{H}Nec \quad \frac{\Rightarrow x : A}{\Rightarrow x : \mathbf{T}A}\,\mathbf{T}Nec \quad \frac{\Rightarrow x : A}{\Rightarrow x : \mathbf{Y}A}\,\mathbf{Y}Nec$$

are admissible in G3LT.

Proof. Let us suppose that we have a derivation of $\Rightarrow x : A$. By Lemma 3 we obtain a derivation of $\Rightarrow y : A$ and by admissibility of weakening we obtain the sequents $x \le y \Rightarrow y : A$, $y \le x \Rightarrow y : A$, $x \prec y \Rightarrow y : A$, and $y \prec x \Rightarrow y : A$. We finally conclude $\Rightarrow x : \mathbf{G}A$, $\Rightarrow x : \mathbf{H}A$, $\Rightarrow x : \mathbf{T}A$, and $\Rightarrow x : \mathbf{Y}A$ by a single step of $R\mathbf{G}$, $R\mathbf{H}$, $R\mathbf{T}$ and $R\mathbf{Y}$, respectively. ■

COROLLARY 13. *The calculus* G3LT *is complete with respect to Priorean linear time logic.*

The equivalences $\mathbf{G}A \supset\subset (A \,\&\, \mathbf{TG}A)$ and $\mathbf{H}A \supset\subset (A \,\&\, \mathbf{YH}A)$ define recursively the operator **G** in terms of **T** and the operator **H** in terms of **Y**, respectively. The left-to-right directions are axioms (see Proposition 11); their converses, $(A \,\&\, \mathbf{TG}A) \supset \mathbf{G}A$ and $(A \,\&\, \mathbf{YH}A) \supset \mathbf{H}A$, are easily derivable by means of the following admissible rules:

$$\frac{x = y, x \le y, \Gamma \Rightarrow \Delta \quad x \prec z, z \le y, x \le y, \Gamma \Rightarrow \Delta}{x \le y, \Gamma \Rightarrow \Delta}\,Mix_1$$

$$\frac{x = y, x \le y, \Gamma \Rightarrow \Delta \quad x \le z, z \prec y, x \le y, \Gamma \Rightarrow \Delta}{x \le y, \Gamma \Rightarrow \Delta}\,Mix_2$$

both with the condition that z is not in the conclusion. Rules Mix_1 and Mix_2 correspond to frame properties $x \le y \supset (x = y \vee \exists z(x \prec z \,\&\, z \le y))$ and $x \le y \supset (x = y \vee \exists z(x \le z \,\&\, z \prec y))$ that permit the splitting of an interval $[x, y]$ with an immediate successor of x and an immediate predecessor of y respectively.

PROPOSITION 14. *Rules* Mix_1 *and* Mix_2 *are admissible in* G3LT.

Proof. Whenever the premisses of rules Mix_1 or Mix_2 are derivable, so are sequents $x \prec^n y, x \le y, \Gamma \Rightarrow \Delta$ for all n. An application of rule T^ω gives the desired conclusion. ■

Finally, completeness of G3LT implies the admissibility of the rules of left and right linearity:

PROPOSITION 15. *The rules of left and right linearity*

$$\frac{y \leq z, z \leq x, y \leq x, \Gamma \Rightarrow \Delta \quad z \leq y, z \leq x, y \leq x, \Gamma \Rightarrow \Delta}{z \leq x, y \leq x, \Gamma \Rightarrow \Delta}\,{\scriptstyle L\text{-}Lin}$$
$$\frac{y \leq z, x \leq z, x \leq y, \Gamma \Rightarrow \Delta \quad z \leq y, x \leq z, x \leq y, \Gamma \Rightarrow \Delta}{x \leq z, x \leq y, \Gamma \Rightarrow \Delta}\,{\scriptstyle R\text{-}Lin}$$

are admissible in G3LT*.*

Proof. By means of two applications of T^ω, with principal formulas $x \leq z$ and $x \leq y$, and derivability of $x \prec^m z, x \leq z, x \prec^n y, x \leq y, \Gamma \Rightarrow \Delta$ for every $m, n \in \mathbb{N}$, whenever the premisses of *R-Lin* are derivable. ■

3 A non-standard system for linear time

We define the system G3LT$_f$ by substituting, in the calculus G3LT, the rules T^ω, *LDef* and *RDef*, with the rules *Mix*$_1$, *Mix*$_2$, *L-Lin* and *R-Lin* as primitive.

The standard frame for linear time logic corresponds to the set of the integers $\mathbb{Z}$: Every instant greater (smaller) than x can be reached from x by finitely many iterations of the immediate successor (predecessor) relation. This condition corresponds to the infinitary rule T^ω of the calculus G3LT.

Because of the absence of T^ω, the systems G3LT$_f$ allows non-standard frames that consist of several, possibly infinite, consecutive copies of the integers, $\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$: Even though every point is the unique immediate successor of its unique immediate predecessor (and viceversa), it is not always true that between any two points x, y such that $x \leq y$, there are finitely many other points.

It is easy to verify that the system G3LT$_f$ can be embedded in G3LT: Every sequent derivable in G3LT$_f$ is derivable in G3LT.

THEOREM 16. *If* $\vdash_{G3LT_f} \Gamma \Rightarrow \Delta$*, then* $\vdash_{G3LT} \Gamma \Rightarrow \Delta$*.*

Proof. Every rule of G3LT$_f$ except *Mix*$_1$, *Mix*$_2$, *L-Lin* and *R-Lin* is a rule of G3LT, and *Mix*$_1$, *Mix*$_2$, *L-Lin* and *R-Lin* are admissible in G3LT, by Proposition 14 and Proposition 15, respectively. ■

The converse fails because of the infinitary rule: For instance, any proof search for the induction principle $x : A, x : \mathbf{G}(A \supset \mathbf{T}A) \Rightarrow x : \mathbf{G}A$ would require infinitely many applications of rule *Mix*$_1$. Nevertheless, we identify a conservative fragment for which derivability in G3LT implies derivability in G3LT$_f$. Our result is confined to purely logical sequents, but this condition is not restrictive, since, as we noticed before, only purely logical sequents can be interpreted as corresponding modal formulas.

THEOREM 17. *If a purely logical sequent* $\Gamma \Rightarrow \Delta$ *is derivable in* G3LT *and the operators* $\mathbf{G}$, $\mathbf{H}$ *do not appear in its positive part, nor* $\mathbf{F}$, $\mathbf{P}$ *in its negative part, then* $\Gamma \Rightarrow \Delta$ *is derivable without the use of the infinitary rule.*

Proof. We show that all the applications of the infinitary rule can be dispensed with. Without loss of generality, we assume that the given derivation is minimal in the sense that no shortenings that arise from applications of height preserving contraction are possible: This excludes rule instances such as transitivity with a reflexitity atom as principal. Observe that all relational atoms $x \leq y$, in particular those concluded by T^{ω}, have to disappear before the conclusion. We consider one such downmost atom and the rule that makes it disappear: Rules $R\mathbf{G}$, $R\mathbf{H}$, $L\mathbf{F}$ and $L\mathbf{P}$ are excluded because they would introduce $\mathbf{G}$, $\mathbf{H}$ in the positive part or $\mathbf{F}$, $\mathbf{P}$ in the negative part. Thus, the atom can disappear by means of *Inc* or *Ref*.

If the atom concluded by T^{ω} is removed by *Ref*, we have

$$\cfrac{\cfrac{\{x \prec^n x, x \leq x, \Gamma' \Rightarrow \Delta'\}_{n \in \mathbb{N}}}{x \leq x, \Gamma' \Rightarrow \Delta'}\, T^{\omega}}{\cfrac{\vdots}{\cfrac{x \leq x, \Gamma'' \Rightarrow \Delta''}{\Gamma'' \Rightarrow \Delta''}\, \mathit{Ref}}}$$

We take the leftmost premiss of T^{ω} and transform the derivation into the following

$$\cfrac{\cfrac{x = x, x \leq x, \Gamma' \Rightarrow \Delta'}{x \leq x, \Gamma' \Rightarrow \Delta'}\, \mathit{EqRef}}{\cfrac{\vdots}{\cfrac{x \leq x, \Gamma'' \Rightarrow \Delta''}{\Gamma'' \Rightarrow \Delta''}\, \mathit{Ref}}}$$

The application of T^{ω} is removed from the derivation.

If the atom concluded by T^{ω} is removed by Inc, we have

$$\cfrac{\cfrac{\{x \prec^n y, x \leq y, \Gamma' \Rightarrow \Delta'\}_{n \in \mathbb{N}}}{x \leq y, \Gamma' \Rightarrow \Delta'}\, T^{\omega}}{\cfrac{\vdots}{\cfrac{x \leq y, x \prec y, \Gamma'' \Rightarrow \Delta''}{x \prec y, \Gamma'' \Rightarrow \Delta''}\, \mathit{Inc}}}$$

The second premiss of T^{ω} has the form $x \prec y, x \leq y, x \prec y, \Gamma''' \Rightarrow \Delta'$, with $\Gamma' \equiv x \prec y, \Gamma'''$. By height-preserving contraction we obtain $x \leq y, \Gamma' \Rightarrow \Delta'$

and proceed with the derivation until we reach $x \prec y, x \leq y, \Gamma'' \Rightarrow \Delta''$. Then we conclude $x \prec y, \Gamma'' \Rightarrow \Delta''$ by an application of *Inc*. Note that the derivation is shortened, contrary to the assumption of minimality.

If the atom concluded by T^ω is removed by applications of *Trans* followed by applications of *Inc*, we have the derivation

$$\dfrac{\dfrac{\dfrac{\{x \prec^n y, x \leq y, z_1 \leq y, \ldots, z_{m-1} \leq y, x \leq z_1, \ldots, z_m \leq y, x \prec z_1, \ldots, z_m \prec y, \Gamma' \Rightarrow \Delta'\}_{n \in \mathbb{N}}}{x \leq y, z_1 \leq y, \ldots, z_{m-1} \leq y, x \leq z_1, \ldots, z_m \leq y, x \prec z_1, \ldots, z_m \prec y, \Gamma' \Rightarrow \Delta'}\, T^\omega}{\vdots}}{\begin{array}{c} \dfrac{x \leq y, z_1 \leq y, \ldots, z_{m-1} \leq y, x \leq z_1, \ldots, z_m \leq y, x \prec z_1, \ldots, z_m \prec y, \Gamma'' \Rightarrow \Delta''}{x \leq z_1, \ldots, z_m \leq y, x \prec z_1, \ldots, z_m \prec y, \Gamma'' \Rightarrow \Delta''}\, Trans \times m \\ \vdots \\ \dfrac{x \leq z_1, \ldots, z_m \leq y, x \prec z_1, \ldots, z_m \prec y, \Gamma''' \Rightarrow \Delta'''}{x \prec z_1, \ldots, z_m \prec y, \Gamma''' \Rightarrow \Delta'''}\, Inc \times (m+1) \end{array}}$$

These can be transformed into the following derivation

$$\begin{array}{c} \dfrac{\dfrac{x \prec^{m+1} y, x \leq y, z_1 \leq y, \ldots, z_{m-1} \leq y, x \leq z_1, \ldots, z_m \leq y, x \prec z_1, \ldots, z_m \prec y, \Gamma' \Rightarrow \Delta'}{x \prec z_1, \ldots, z_m \prec y, x \leq y, z_1 \leq y, \ldots, z_{m-1} \leq y, x \leq z_1, \ldots, z_m \leq y, x \prec z_1, \ldots, z_m \prec y, \Gamma' \Rightarrow \Delta'}\, \mathcal{I}}{x \leq y, z_1 \leq y, \ldots, z_{m-1} \leq y, x \leq z_1, \ldots, z_m \leq y, x \prec z_1, \ldots, z_m \prec y, \Gamma' \Rightarrow \Delta'}\, \mathcal{C} \\ \vdots \\ \dfrac{x \leq y, z_1 \leq y, \ldots, z_{m-1} \leq y, x \leq z_1, \ldots, z_m \leq y, x \prec z_1, \ldots, z_m \prec y, \Gamma'' \Rightarrow \Delta''}{x \leq z_1, \ldots, z_m \leq y, x \prec z_1, \ldots, z_m \prec y, \Gamma'' \Rightarrow \Delta''}\, Trans \times m \\ \vdots \\ \dfrac{x \leq z_1, \ldots, z_m \leq y, x \prec z_1, \ldots, z_m \prec y, \Gamma''' \Rightarrow \Delta'''}{x \prec z_1, \ldots, z_m \prec y, \Gamma''' \Rightarrow \Delta'''}\, Inc \times (m+1) \end{array}$$

Here $\mathcal{I}$ stands for m applications of height-preserving invertibility of rule $LDef$ and $\mathcal{C}$ for several application of height preserving contraction. Again, the derivation is shortened, contrary to the assumption.

Note that if the atom concluded by T^ω is removed by an application of $EqSubst_{At}$, we have the following derivation:

$$\begin{array}{c} \dfrac{\{x \prec^n y, z = y, x \leq y, x \leq z, \Gamma' \Rightarrow \Delta'\}_{n \in \mathbb{N}}}{z = y, x \leq y, x \leq z, \Gamma' \Rightarrow \Delta'}\, T^\omega \\ \vdots \\ \dfrac{z = y, x \leq y, x \leq z, \Gamma'' \Rightarrow \Delta''}{z = y, x \leq z, \Gamma'' \Rightarrow \Delta''}\, EqSubst_{At} \end{array}$$

It is possible to permute up rule $EqSubst_{At}$ with respect to rule T^ω. We modify each premiss of T^ω as follows:

$$
\dfrac{\dfrac{\dfrac{x \prec^n y, x \leq y, z = y, x \leq z, \Gamma' \Rightarrow \Delta'}{x \prec^n y, x \leq y, x \prec^n z, z = y, x \leq z, \Gamma' \Rightarrow \Delta'}\,{\scriptstyle LWk}}{\vdots}}{}
$$

$$
\dfrac{\dfrac{x \prec^n y, x \leq y, x \prec^n z, z = y, x \leq z, \Gamma'' \Rightarrow \Delta''}{x \prec^n y, x \prec^n z, z = y, x \leq z, \Gamma' \Rightarrow \Delta'}\,{\scriptstyle Inc_n}}{x \prec^n z, z = y, x \leq z, \Gamma'' \Rightarrow \Delta''}\,{\scriptstyle EqSubst_n}
$$

We can now apply previous modifications. The case of $EqSubst_{At}$ with active formulas $z = x, x \leq y, z \leq y$ is analogous. ■

COROLLARY 18. *If* $\vdash_{G3LT} \Gamma \Rightarrow \Delta$ *and* $\Gamma \Rightarrow \Delta$ *is as in the previous theorem, then* $\vdash_{G3LT_f} \Gamma \Rightarrow \Delta$.

Proof. By Theorem 17, $\Gamma \Rightarrow \Delta$ is derivable without using rule T^ω. ■

4 Related work

From the extensive literature on labelled and hybrid systems, we have followed the method introduced by the second author in [15]. The latter is, compared with the development within Gabbay's labelled deductive systems (see, e.g., chapter 4 in [9]), more explicitly proof-theoretic.

In Baaz *et al.* [4] first-order linear time temporal logic, with future operators $\Box$ and $\bigcirc$, that correspond to our **G** and **T**, is compared to the logic for branching time gaps, the frames of which are well-founded trees of copies of $\mathbb{N}$: Whereas in the former an infinitary rule is needed, the latter is formulated as a cut-free extension of Gentzen's system for classical predicate logic with finitary rules for temporal operators. A conservativity result is then obtained for the $\Box$-free fragment of the system.

If we drop the rule of right linearity from G3LT$_f$, we obtain a labelled sequent calculus that generalizes the propositional fragment of the system in [4]: It is easy to verify that for every propositional formula derivable in the latter system, the corresponding purely logical sequent is derivable in G3LT$_f$ and if a purely logical sequent does not contain past operators and is derivable in G3LT$_f$ without using *R-Lin*, then the corresponding formula is derivable by means of the rules in [4]. However, we can prove a stronger conservativity result: Our theorem has only the condition that endsequents do not contain **G** in the positive part (nor **F** in the negative part), whereas in [4] the modality $\Box$, corresponding to **G**, cannot appear at all in the formula to be derived.

We have identified in our work a finitary fragment of Priorean linear time logic by substituting the rule that corresponds to the reflexive and transitive closure with two weaker finitary counterparts. A somewhat related result is presented by Antonakos and Artemov [2, 3] for the different, but qualitatively similar, logic of common knowledge.

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