Exercises for chapter 1

I. Prove in natural deduction for minimal logic (that is, with introduction and elimination rules for the connectives except $\bot E$):

1. $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$
2. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
3. $(A\&B \supset C) \supset\subset (A \supset (B \supset C))$
4. $A\&B \supset\subset B\&A$ (commutativity)
   $A \vee B \supset\subset B \vee A$
5. $A\&(B\&C) \supset\subset (A\&B)\&C$ (associativity)
   $A \vee (B \vee C) \supset\subset (A \vee B) \vee C$
6. $A \supset\subset A\&A$ (idempotence)
   $A \supset\subset A \vee A$
7. $(A\&(A \vee B) \supset\subset A$ (absorption)
   $(A \vee (A\&B) \supset\subset A$
8. $A\&(B \vee C) \supset\subset (A\&B) \vee (A\&C)$ (distributivity)
   $A \vee (B\&C) \supset\subset (A \vee B)\&(A \vee C)$
9. $(A \supset B) \supset ({\sim}B \supset {\sim}A)$ (contraposition)
10. ${\sim}(A\& {\sim}A)$
11. $A \supset {\sim}{\sim}A$
12. ${\sim}{\sim}{\sim}A \supset\subset {\sim}A$
13. ${\sim}{\sim}(A \vee {\sim}A)$
14. ${\sim}(A \vee B) \supset\subset ({\sim}A\& {\sim}B)$ (de Morgan's laws)
    ${\sim}A \vee {\sim}B \supset {\sim}(A\&B)$
15. $A \vee B \supset {\sim}({\sim}A\& {\sim}B)$
16. ${\sim}(A\&B) \supset\subset (A \supset {\sim}B)$

II. Prove in natural deduction for intuitionistic logic (that is, without excluded middle or its special case *reductio ad absurdum*):

1. $(A \& \sim A) \supset B$
2. $\sim (A \supset B) \supset \sim\sim A \& \sim B$
3. $\sim A \vee B \supset (A \supset B)$
4. $A \vee B \supset (\sim A \supset B)$

III. Prove in natural deduction for classical logic:

1. $\sim (\sim A \& \sim B) \supset A \vee B$
2. $\sim (A \& B) \supset \sim A \vee \sim B$
3. $A \& B \supset\subset \sim (\sim A \vee \sim B)$
4. $\sim (A \& \sim B) \supset\subset (A \supset B)$
5. $(\sim B \supset \sim A) \supset (A \supset B)$
6. $(A \supset B) \supset\subset (\sim A \vee B)$
7. $\sim\sim A \supset A$
8. $\sim (A \supset B) \supset A \& \sim B$
9. $((A \supset B) \supset A) \supset A$ (Peirce's law)
10. $(A \supset B \vee C) \supset (A \supset B) \vee (A \supset C)$ (disjunction property under hypothesis)

IV. (Gödel brain test) Prove in natural deduction for minimal logic, in less than 2 minutes for each direction

$\sim\sim (A \& B) \supset\subset \sim\sim A \& \sim\sim B$