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# Proof systems for lattice theory

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**Abstract:** A formulation of lattice theory as a system of rules added to sequent calculus is given. The analysis of proofs for the contraction-free calculus of classical predicate logic known as *G3c* extends to derivations with the mathematical rules of lattice theory. It is shown that minimum-height derivations of quantifier-free sequents enjoy a subterm property: all terms in such derivations are terms in the endsequent.

An alternative formulation of lattice theory as a system of rules in natural deduction style is given, both with explicit meet and join constructions and as a relational theory with existence axioms. A subterm property for the latter extends the standard decidable classes of quantificational formulas of pure predicate calculus to lattice theory.

## 1. Introduction

Gentzen's analysis of the structure of proofs in pure logic can be extended to mathematical theories, in the first place to theories that permit a quantifier-free axiomatization. The axioms are converted into **mathematical rules** of proof by which a suitable formulation of predicate logic is extended. The latter is a sequent calculus that needs none of the standard structural rules of weakening, contraction, or cut. These extensions were found first in Negri (1999) and Negri and von Plato (1998).

In Negri, von Plato, and Coquand (2001), a crucial property of formal derivations with mathematical rules was located: It happens for some theories that if a given derivation cannot be shortened, all terms in the derivation are terms in the endsequent. This **subterm** property is by no means obvious or universal, because in a proof search that starts from the conclusion, the rules can instantiate new terms in the premisses. Examples of theories that permit a sequent calculus formulation with the subterm property are partial and linear order, projective and affine geometry, and lattice theory.

In the following, we shall present lattice theory as a system of rules of sequent calculus that act on the left (assumption or antecedent) part of sequents. An alternative formulation is given as a system of rules that act on the right (conclusion or succedent) part, with the remarkable simplification that the number of possible cases in the succedent can be limited to one. This latter system gives a solution to the derivability of atomic formulas $a \leqslant b$ in lattice theory in a number of steps bounded by the length of the terms $a, b$.

The general problem of derivability of a quantifier-free sequent $\Gamma \to \Delta$ is equivalent, by the invertibility of the logical rules of the sequent calculus we use, to the derivability of a finite number of sequents $\Gamma_i \to \Delta_i$ with only atoms in $\Gamma_i, \Delta_i$. By the subformula property of sequent calculus, these sequents are derived by only the rules of lattice theory. There is no need to consider parts of derivations with logical rules, and we shall follow this and assume all formulas to be atomic unless otherwise stated. For the logical rules, we refer to Troelstra and Schwichtenberg (2000) or Negri and von Plato (2001).

The classical sequent calculus **G3c** has proved to be expecially suited for proof analysis. It has the remarkable property of height-preserving admissibility of the rule of contraction. In terms of root-first proof search, this property inhibits those instantiations of rules that produce a duplication of a formula in a premiss.

In the formulation with left rules, the derivations in lattice theory are linear: Each rule has at most one premiss. All terms in a minimum-height derivation of $\Gamma \to \Delta$ are subterms of the terms of $\Gamma, \Delta$. The number of distinct atomic formulas $a \leqslant b$ with $a, b$ such subterms gives an upper bound for the height of derivation of $\Gamma \to \Delta$. In the formulation with right rules, the derivations can have a single formula in the succedent. These rules can be readily translated into a system of rules in natural deduction style, with intuitionistic logic as a basis. We shall study such systems for lattice theory in sections 5 and 6. The subterm property is proved by showing that the rule of transitivity can be permuted up relative to the other lattice rules.

It has been useful to consider both left and right rule systems. In establishing properties of axiomatic systems through proof analysis, the combinatorial possibilities for formulating systems of rules are numerous, and it is difficult to tell what path will lead to results.

The subterm property shows derivability of universal formulas in lattice theory to be decidable. There is no systematic corpus of results on decidable classes of formulas for predicate logic with functions. The presence of functions permits the instantiation of ever new terms different from the previously introduced ones, with no bound on proof search. Lattice theory can be formulated as a relational theory, with two additional basic relations and existential axioms instead of functions (constructions). From the subterm property for the corresponding system of rules follows that most of the standard decidable classes of quantificational formulas of pure predicate logic extend to lattice theory, similarly to the case of partial order.

Derivability in lattice theory was studied by Thoralf Skolem in a forgotten paper of 1920 (see Burris 1995 and Freese et al. 1995 for its rediscovery by lattice theorists). His main theorem gives the decidability of universal formulas of lattice theory, as a theory with existential axioms instead of explicit constructions. The corresponding result for a formulation with constructions is the first aim of our paper.

In a proof-theoretical approach, lattices are defined axiomatically and their properties established by analyzing the structure of formal proofs. The methods of this paper lead, in principle, to a full control over the structure of possible proofs in lattice theory.

## 2. Axioms and rules

**2.1. Notation: Sequents** $\Gamma \to \Delta$ have finite, possibly empty, multisets of formulas as **antecedent** $\Gamma$ and as **succedent** $\Delta$. The proof-theoretical meaning of a sequent $\Gamma \to \Delta$ is that $\Delta$ gives the **open cases** that are **derivable** under the **open assumptions** $\Gamma$. An empty succedent represents the **impossible** case.

Arbitrary formulas are denoted by $A, B, C, \dots$ and atomic formulas (**atoms**) by $P$, $Q$, $R$, $P_1$, $Q_1$, $R_1, \dots$. Sequent calculus derivations are trees with **initial sequents** of the form $P, \Gamma \to \Delta, P$ as leaves. An atom that makes a sequent an initial sequent is called a **responsible atom**. For the logical rules, we use the classical sequent calculus **G3c**.

**2.2. Universal theories:** Given a universal axiom $\forall \dots \forall A$, consider the conjunctive normal form of its propositional matrix $A$. Each conjunct is a disjunction of atoms and negations of atoms. We may write these conjuncts in the equivalent form $P_1 \& \dots \& P_m \supset Q_1 \vee \dots \vee Q_n$. If each of the $P_i$ follows from some open assumptions $\Gamma$, the cases under $\Gamma$ are $Q_1, \dots, Q_n$. This mode of inference is formalized by the **right rule-scheme:**

$$\frac{\Gamma \to \Delta, Q_1, \dots, Q_n, P_1 \quad \dots \quad \Gamma \to \Delta, Q_1, \dots, Q_n, P_m}{\Gamma \to \Delta, Q_1, \dots, Q_n}\,{\scriptstyle Rule}$$

For full generality, this rule-scheme permits arbitrary additional open cases $\Delta$ in the succedent. Formulas $Q_1, \dots, Q_n$ in the conclusion are the **principal** formulas of the rule. Each formula $P_i$ in a premiss is a **removed atom**. The principal formulas $Q_1, \dots, Q_n$ are repeated in the premisses (see below).

There is a dual **left rule-scheme** that has the same deductive strength as the right scheme, with principal formulas $P_1, \ldots, P_m$:

$$\frac{Q_1, P_1, \ldots, P_m, \Gamma \to \Delta \quad \ldots \quad Q_n, P_1, \ldots, P_m, \Gamma \to \Delta}{P_1, \ldots, P_m, \Gamma \to \Delta}\,\scriptstyle{Rule}$$

In words, if $\Delta$ follows from each of the cases $Q_j$ (and $\Gamma$), it already follows from the $P_i$ together (and $\Gamma$). As in the right scheme, each instance of a rule removes exactly one atom $Q_j$ from each premiss. As a limiting case, if $n = 0$, a rule has no premisses. Its conclusion then acts as a topsequent by which a derivation branch can start. This situation is not encountered in lattice theory. The other limiting case is when $m = 0$ and the rule has no principal formulas.

Derivations by a left or right rule system are equivalent to derivations that start with initial sequents and **basic sequents** $P_1, \ldots, P_m \to Q_1, \ldots, Q_n$ corresponding to the rules, and use **cuts on atoms**

$$\frac{\Gamma \to \Delta, P \quad P, \Gamma \to \Delta}{\Gamma \to \Delta}\,\scriptstyle{Cut}$$

as the only rule of inference.

Initial sequents of the form $P, \Gamma \to \Delta, P$, instead of $P \to P$, are used for obtaining **height-preserving admissibility** of the rules of left and right **weakening**:

$$\frac{\Gamma \to \Delta}{A, \Gamma \to \Delta}\,\scriptstyle{LW} \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, A}\,\scriptstyle{RW}$$

Height-preserving admissibility of a rule in a given system of rules means that if the premiss of the rule is derivable with a derivation the maximum branch of which has no more than $n$ steps (**height** of derivation $\leqslant n$), the conclusion also is derivable in the system with a height of derivation $\leqslant n$.

The principal formulas, $Q_1, \ldots, Q_n$ in the right scheme and $P_1, \ldots, P_m$ in the left one, are repeated in the premisses in order to achieve height-preserving admissibility of the rules of left and right **contraction**:

$$\frac{A, A, \Gamma \to \Delta}{A, \Gamma \to \Delta}\,\scriptstyle{LC} \qquad \frac{\Gamma \to \Delta, A, A}{\Gamma \to \Delta, A}\,\scriptstyle{RC}$$

Thus, if a derivation has a rule instance that removes one of two identical atoms of some premiss, the rule can be deleted and the derivation shortened by height-preserving contraction.

The repetition of the principal formulas $P_1, \ldots, P_m$ in the left rule-scheme can be justified by noting that if the assumptions $P_1, \ldots, P_m$ are permitted in the conclusion, it does no harm to permit the use of these assumptions elsewhere in the derivation, and similarly for the right rule-scheme.

It can happen that a rule has instances in which two identical principal atoms $Q, Q$ in the conclusion:

$$\frac{\Gamma \to \Delta, Q_1, \ldots, Q, Q, \ldots, Q_n, P_1 \quad \ldots \quad \Gamma \to \Delta, Q_1, \ldots, Q, Q, \ldots, Q_n, P_m}{\Gamma \to \Delta, Q_1, \ldots, Q, Q, \ldots, Q_n}\,\scriptstyle{Rule}$$

In this case, the rule with duplications contracted to $Q$,

$$\frac{\Gamma \to \Delta, Q_1, \ldots, Q, \ldots, Q_n, P_1 \quad \ldots \quad \Gamma \to \Delta, Q_1, \ldots, Q, \ldots, Q_n, P_m}{\Gamma \to \Delta, Q_1, \ldots, Q, \ldots, Q_n}\,\scriptstyle{Rule^*}$$

has to be added to the system in order to have height-preserving contraction, and similarly for systems of left rules. A rule system thus completed is said to satisfy the **closure condition**. There can be only a bounded number of contracted forms of rules to be added in order to satisfy the condition.

**Theorem 2.1.** *The structural rules of left and right weakening and contraction are height-preserving admissible and the rule of cut admissible in extensions of* **G3c** *with rules following the right rule-scheme and the left rule-scheme and satisfying the closure condition.*

A proof for the left rule-scheme is given in Negri and von Plato (1998). The right rule-scheme has a dual proof.

Height-preserving admissibility of contraction is a key property of extensions with mathematical rules, which permits to prove results by arguments based on minimum-height derivations. Instantiations of rules that give a duplication of an atom in some premiss are not permitted.

The invertibility of the logical rules of the calculus **G3c** also holds for the calculus **G3c** extended with rules following the rule-scheme (Negri and von Plato 1998). This property has as consequence the separation of derivations into initial parts with mathematical rules followed by a part with logical rules. For this reason we shall consider only derivations of sequents that contain no compound formulas.

**2.3. Theories with existence axioms:** In Negri (2003), it is shown how axioms with a quantificational structure can be turned into mathematical rules, if these axioms are what are known as “geometric implications” in categorical terms. For the present purposes, it is sufficient to say that the axioms of lattice theory in a relational formulation and with existential axioms instead of constructions fall under the “geometric rule-scheme.” An existential axiom that replaces a construction and postulated properties of constructed objects has the form $\forall x \dots \forall y \exists z A(x, \dots, y, z)$. It corresponds to the construction of some $z$ from any given $x, \dots, y$ such that $A(x, \dots, y, z)$ holds. Constructions can have conditions, such as in elementary geometry where an intersection point of two lines, say, can be constructed only if the lines are convergent. We do not meet this situation in lattice theory, in the existential axioms of which the propositional part is an atomic formula $P(x, y, z)$. The rule corresponding to such an existential axiom is, with parameters $a, b$ in place of the universally quantified variables:

$$\frac{P(a, b, z), \Gamma \to \Delta}{\Gamma \to \Delta}\,{\scriptstyle E\text{-}Rule}$$

The rule has the **variable restriction** that the **eigenvariable** $z$ must not be free in the conclusion. Assuming the premiss of the rule, application of the logical rule $L\exists$ followed by $L\forall$ twice gives as conclusion

$$\forall x \forall y \exists z P(x, y, z), \Gamma \to \Delta$$

so that the rule has the same force as the existential axiom.

By the general theorem of Negri (2003), weakening and contraction are height-preserving and, together with the rule of cut, admissible in extensions of **G3c** with rules following the geometric rule-scheme.

**2.4. Single succedent rules in natural deduction style:** The axioms of lattice theory are all **Harrop formulas**, i.e., they do not contain any essential disjunction. Therefore the left rule system has linear derivations. In terms of the rule-schemes, lattice theory has rules with $n = 1$, and for this reason it is possible to give a single succedent rule system for lattice theory. Furthermore, these rules can be written "in natural deduction style," meaning that the open assumptions appear at the leaves of derivation trees, instead of being collected together on each line. In general, an axiom of the form $P_1 \& \dots \& P_m \supset Q$ becomes the rule

$$\frac{P_1 \quad \dots \quad P_m}{Q}$$

From a comparison of this form of rule with a single succedent left rule scheme, it is seen that the root-first construction of a derivation with left rules corresponds to a direct derivation from assumptions with natural deduction style rules. Duplication of a formula in a derivation with left rules corresponds to **looping**, i.e., to having a natural deduction style derivation tree with a branch in which the same formula is concluded twice.

## 3. Lattice axioms and rules

We shall give the standard axioms of lattice theory with the meet and join constructions, and a corresponding system of left rules.

**3.1. Partial order:** The axioms of a partial order are

$$a \leqslant a, \quad Ref, \qquad a \leqslant b \;\&\; b \leqslant c \supset a \leqslant c, \quad Trans.$$

These axioms lead to the system:

**1. Left rules for partial order**

$$\frac{a \leqslant a, \Gamma \to \Delta}{\Gamma \to \Delta}{\scriptstyle Ref} \qquad \frac{a \leqslant c, a \leqslant b, b \leqslant c, \Gamma \to \Delta}{a \leqslant b, b \leqslant c, \Gamma \to \Delta}{\scriptstyle Trans}$$

There is an instance of *Trans* with a duplication in the conclusion, when $a, b$, and $c$ are syntactically **identical**, to be written $a \equiv b, b \equiv c$. This instance,

$$\frac{a \leqslant a, a \leqslant a, a \leqslant a, \Gamma \to \Delta}{a \leqslant a, a \leqslant a, \Gamma \to \Delta}{\scriptstyle Trans}$$

is also an instance of rule *Ref* so that the closure condition is met without the addition of any rules.

We use a defined notion of **equality**, with the obvious definition

$$a = b \equiv \; a \leqslant b \;\&\; b \leqslant a.$$

**3.2. Lattice operations and laws:** The following can be considered a standard axiomatization of lattice theory with meet and join operations $a \wedge b$ (the meet of $a$ and $b$), and $a \vee b$ (the join of $a$ and $b$):

$$a \wedge b \leqslant a \quad (L\wedge_1), \qquad a \leqslant a \vee b \quad (R\vee_1),$$

$$a \wedge b \leqslant b \quad (L\wedge_2), \qquad b \leqslant a \vee b \quad (R\vee_2),$$

$$c \leqslant a \ \&\ c \leqslant b \supset c \leqslant a \wedge b \quad (R\wedge), \qquad a \leqslant c \ \&\ b \leqslant c \supset a \vee b \leqslant c \quad (L\vee).$$

The substitution of equals in the lattice operations,

$$b = c \supset a \wedge b = a \wedge c, \qquad b = c \supset a \vee b = a \vee c,$$

can be proved, because equality is defined through the partial order relation.

The above axiom system corresponds to the system of rules:

**2. Left rules for lattice theory**

$$\frac{a \wedge b \leqslant a, \Gamma \to \Delta}{\Gamma \to \Delta}\, L\wedge_1 \qquad \frac{a \leqslant a \vee b, \Gamma \to \Delta}{\Gamma \to \Delta}\, R\vee_1$$

$$\frac{a \wedge b \leqslant b, \Gamma \to \Delta}{\Gamma \to \Delta}\, L\wedge_2 \qquad \frac{b \leqslant a \vee b, \Gamma \to \Delta}{\Gamma \to \Delta}\, R\vee_2$$

$$\frac{c \leqslant a \wedge b, c \leqslant a, c \leqslant b, \Gamma \to \Delta}{c \leqslant a, c \leqslant b, \Gamma \to \Delta}\, R\wedge \qquad \frac{a \vee b \leqslant c, a \leqslant c, b \leqslant c, \Gamma \to \Delta}{a \leqslant c, b \leqslant c, \Gamma \to \Delta}\, L\vee$$

The mnemonics $L\wedge_1$, etc., indicate on which side of the removed atom the lattice operation is.

Before considering the closure condition, we give an example of a derivation in the calculus for lattice theory, namely the substitution law $b = c \supset a \wedge b = a \wedge c$. We first decompose root-first the logical part:

$$\cfrac{\cfrac{\cfrac{b \leqslant c, c \leqslant b \to a \wedge b \leqslant a \wedge c \quad b \leqslant c, c \leqslant b \to a \wedge c \leqslant a \wedge b}{b \leqslant c, c \leqslant b \to a \wedge b \leqslant a \wedge c \ \&\ a \wedge c \leqslant a \wedge b}\, R\&}{b \leqslant c \ \&\ c \leqslant b \to a \wedge b \leqslant a \wedge c \ \&\ a \wedge c \leqslant a \wedge b}\, L\&}{\to b \leqslant c \ \&\ c \leqslant b \supset a \wedge b \leqslant a \wedge c \ \&\ a \wedge c \leqslant a \wedge b}\, R\supset$$

Next the basic sequents $b \leqslant c, c \leqslant b \to a \wedge b \leqslant a \wedge c$ and $b \leqslant c, c \leqslant b \to a \wedge c \leqslant a \wedge b$ are derived by lattice rules. We show only the first:

$$\cfrac{\cfrac{\cfrac{\cfrac{a \wedge b \leqslant a \wedge c, a \wedge b \leqslant a, a \wedge b \leqslant c, a \wedge b \leqslant b, b \leqslant c, c \leqslant b \to a \wedge b \leqslant a \wedge c}{a \wedge b \leqslant a, a \wedge b \leqslant c, a \wedge b \leqslant b, b \leqslant c, c \leqslant b \to a \wedge b \leqslant a \wedge c}\, R\wedge}{a \wedge b \leqslant c, a \wedge b \leqslant b, b \leqslant c, c \leqslant b \to a \wedge b \leqslant a \wedge c}\, L\wedge_1}{a \wedge b \leqslant b, b \leqslant c, c \leqslant b \to a \wedge b \leqslant a \wedge c}\, Trans}{b \leqslant c, c \leqslant b \to a \wedge b \leqslant a \wedge c}\, L\wedge_2$$

Having chosen rule $L\wedge_2$ as the downmost step, we find that rule *Trans* matches the conclusion, and, after an instance of $L\wedge_1$, rule $R\wedge$, with its removed atom the responsible atom. Thus, a problem of proof search is encountered when no rule with a principal formula in the conclusion applies, and one of the four rules $L\wedge_1, L\wedge_2, R\vee_1, R\vee_2$, or *Ref*, has to be instantiated.

The uniqueness rules for the meet and join constructions can have instances with a duplication in the premiss and conclusion:

$$\frac{c \leqslant a \wedge a, c \leqslant a, c \leqslant a, \Gamma \to \Delta}{c \leqslant a, c \leqslant a, \Gamma \to \Delta}\, R\wedge$$

and similarly for join. The rules with the duplication contracted in both the premiss and conclusion are added to the system to meet the closure condition:

**3. Closure for left lattice rules**

$$\frac{c \leqslant a \wedge a, c \leqslant a, \Gamma \rightarrow \Delta}{c \leqslant a, \Gamma \rightarrow \Delta}{\scriptstyle R\wedge^*} \qquad \frac{a \vee a \leqslant c, a \leqslant c, \Gamma \rightarrow \Delta}{a \leqslant c, \Gamma \rightarrow \Delta}{\scriptstyle L\vee^*}$$

As shown in Negri and von Plato (2001, p. 148), the contracted rules are admissible and contraction admissible even in the system without the explicit addition of such rules. However, in order to guarantee *height-preserving* admissibility of contraction, the rules resulting from the closure condition are needed. The whole system of extension of **G3c** by the rules of partial order and the left lattice rules and their closures (**1.–3.** above) is denoted by **G3LT**. We conclude by theorem 2.1:

**Theorem 3.1.** *The structural rules of left and right weakening and contraction are height-preserving admissible and the rule of cut admissible in* **G3LT**.

Derivations with the rules of lattice theory are linear, with just one premiss, and all topsequents are initial sequents. The succedent remains the same throughout. Therefore no sequent with an empty succedent is derivable and we obtain by this simple proof analysis the

**Corollary 3.2.** *Lattice theory is consistent.*

## 4. The subterm property

A derivation that cannot be shortened by the deletion of a rule is of **minimum size**. Such shortenings, if possible, are produced as follows:

*1.* There is a duplication $P, P$ in a premiss of a rule and $P$ is the removed atom. The conclusion is obtained by deleting the rule and applying height-preserving contraction to the premiss.

*2.* The responsible atom in the initial sequent is not the removed atom in the first step. Now the conclusion is also an initial sequent and the first step can be deleted.

In both cases, the shortened derivation is a proper subtree of the original one. The second case is propagated down the derivation tree: The removed atom in a second step of inference must be a principal atom of the first step, etc. Call the principal atoms of a rule atoms **activated** by the removed atom of the rule and consider the transitive closure of the activation relation. We now have:

**Observation 4.1.** *Each removed atom must be in the transitive closure of atoms activated by the responsible atom in the initial sequent, or else the derivation can be shortened.*

**Theorem 4.2. Subterm property.** *All terms in a minimum-height derivation of* $\Gamma \rightarrow \Delta$ *in* **G3LT** *are subterms of* $\Gamma, \Delta$.

**Proof:** The derivation of $\Gamma \rightarrow \Delta$ starts with an initial sequent

$$a \leqslant c, \Gamma' \rightarrow \Delta', a \leqslant c$$

in which $\Delta', a \leqslant c$ is equal to $\Delta$. The first step removes $a \leqslant c$, else the derivation can be shortened. If the first step is one of *Ref*, $L\wedge_1, L\wedge_2, R\vee_1$, or $R\vee_2$, the conclusion follows in one step from the initial sequent, and the claim holds. Else the first step is $R\wedge$, $L\vee$, or *Trans*. For each occurrence of these rules we consider the atoms activated by the rule. By observation 4.1, at least one of them has to be the removed atom in the following step. Inspecting the rules, we see that the terms in the activated atoms are subterms of $a, c$ and therefore subterms in $\Delta$, except for rule *Trans*. Therefore, if the derivation contains a term $b_i$ which is not a subterm of the conclusion, call it a **new term**, the atom in which it occurs is activated by *Trans*. Consecutive applications of *Trans* produce chains of activated atoms

$$d \leqslant b_0, b_0 \leqslant b_1, \ldots, b_n \leqslant e$$

in which $d, e$ are subterms of the conclusion and the $b_i$ are new terms. Since all the atoms of the chain contain new terms, they have to be removed further down in the derivation.

We show that it is not restrictive to suppose that $d \leqslant b_0$ is not removed by a left rule: If $d \leqslant b_0$ were removed by $L\wedge_1$ or $L\wedge_2$, then $b_0$ would be a subterm of $d$, hence of the conclusion, contrary to the assumption. If $d \leqslant b_0$ were removed by $L\vee$, then $d \equiv d_1 \vee d_2$ and we have the activated atoms $d_1 \leqslant b_0$, $d_2 \leqslant b_0$, so the chain could be replaced by either of the chains

$$d_i \leqslant b_0, b_0 \leqslant b_1, \ldots, b_n \leqslant e, \quad \text{for } i = 1, 2$$

In the case of $L\vee^*$, $d_1 \equiv d_2$, and a similar replacement in the chain is done. Since the number of lattice operations in the left end of the chain is decreased, the replacement eventually leads to a chain with the desired property. We show in a similar way that it is not restrictive to assume that the last atom in the chain, $b_n \leqslant e$, is not removed by a right rule.

In the chain $d \leqslant b_0, b_0 \leqslant b_1, \ldots, b_n \leqslant e$ there is a contiguous pair of atoms that are removed by the rules $R\wedge$, $L\wedge_i$ or $R\vee_i$, $L\vee$ (or by pairs with one of the contracted rules, $R\wedge^*$, $L\wedge_i$ or $R\vee_i$, $L\vee^*$): Start with $d \leqslant b_0$. If the outermost lattice operation of $b_0$ is $\wedge$, it can be removed only by $R\wedge$. Whenever an atom is removed by *Ref* we analyze the next atom, so we need not list it among the possibilities. Then $b_0 \leqslant b_1$ can be removed by $R\vee_i$, $R\wedge$ or $L\wedge_i$. In the last case we are done, else we continue along the chain with the case analysis: If the first case had occurred, $b_1 \leqslant b_2$ is removed by $L\vee$, $R\vee_i$, or $R\wedge$; if the second, it is removed by $R\vee_i$, $R\wedge$, or $L\wedge_i$. In the last case we have the conclusion, else we continue the case analysis until we find that $b_{n-1} \leqslant b_n$ is removed by $R\vee_i$ or $R\wedge$. But then $b_n \leqslant e$ is removed by $L\vee$ or $L\wedge_i$, respectively, since by assumption it is not removed by a right rule.

We prove the existence of a contiguous pair in a similar way if the outermost lattice operation of $b_0$ is $\vee$.

Let two contiguous atoms $b \leqslant f\wedge g$ and $f\wedge g \leqslant f$ be removed by $R\wedge$, $L\wedge_1$. Then the topsequent contains the atoms $b \leqslant f$ and $b \leqslant g$. Replace the two atoms $b \leqslant f\wedge g$ and $f\wedge g \leqslant f$ with the single atom $b \leqslant f$, and continue the derivation as before except for deleting the instances of *Trans* in which the two atoms were active and the two steps $R\wedge$, $L\wedge_1$. In this way the derivation is shortened. A similar simplification is performed for contiguous pairs removed by the rules $R\vee_i$, $L\vee$. In case the contiguous pair is removed by one of the contracted rules, say $R\wedge^*$, $L\wedge_i$, the simplification is a special case of the above: The chain contains the atoms $b \leqslant f \wedge f$ and $f \wedge f \leqslant f$, which are replaced by the single atom $b \leqslant f$. QED.

**Corollary 4.3. Conservativity of lattice theory over partial order.** *If $\Gamma \rightarrow \Delta$ is derivable in* **G3LT** *and $\Gamma$ and $\Delta$ do not contain lattice operations, then $\Gamma \rightarrow \Delta$ is derivable by rules Ref and Trans.*

**Proof.** By the subterm property, no terms in a minimum-height derivation contain lattice operations, thus the derivation contains no lattice rules. QED.

A proof of the conservativity theorem was given in Negri and von Plato (2001, theorem 6.6.5). The above proof of the subterm property for lattice theory is a generalization of that proof.

**Corollary 4.4. Word problem.** *The derivability in lattice theory of a sequent of the form $\rightarrow a \leqslant b$ is decidable.*

**Proof.** By the subterm property, there is a bounded number $k$ of distinct terms that can be instantiated. The number of distinct atoms in $k$ terms is $k^2$ and it gives an upper bound for the number of steps in a duplication-free proof search. QED

The derivability of an atom when a finite number of atoms is assumed given is known as the "word problem for finitely presented lattices" (see Freese et al. 1995, p. 249):

**Corollary 4.5. Word problem for finitely presented lattices.** *The derivability in lattice theory of a sequent of the form $a_1 \leqslant b_1, \ldots, a_m \leqslant b_m \rightarrow a \leqslant b$ is decidable.*

The number of subterms of a term is the length of the term, that is, the number of lattice operations in the term $+1$. Proof search for a sequent $\Gamma \rightarrow \Delta$ can be effected as follows: First observe that five rules have no principal formulas and can be permuted last. Therefore, in root-first proof search, these rules can be instantiated first. The number of instances for *Ref* is the total number $n$ of subterms in $\Gamma, \Delta$, and the number for $L\wedge_1$ the number of subterms of the form $a \wedge b$, and similarly for $L\wedge_2, R\vee_1, R\vee_2$. With no duplications permitted, these five rules give altogether $< 5n$ formulas to be added to the antecedent $\Gamma$, to obtain $\Gamma'$. Next, if there is a match in $\Gamma'$ with the two principal atoms of the remaining three rules, $\Gamma'$ is extended by the corresponding removed atoms. No duplications are permitted. This procedure is repeated until a match with an atom in $\Delta$ is obtained, or until no new formulas appear.

**Corollary 4.6. Decidability of $\mathbf{\Pi}_1$-formulas.** *The derivability in lattice theory of sequents of the form $\rightarrow \forall \ldots \forall A$, with $A$ quantifier free, is decidable.*

**Proof.** Assume that $A$ is in conjunctive normal form. Each conjunct $A_k$ is equivalent to one of the form $P_1 \& \ldots \& P_m \supset Q_1 \vee \cdots \vee Q_n$, with $P_i, Q_j$ atoms. The lattice axioms are all Harrop formulas and therefore, by the disjunction property under Harrop assumptions, $A_k$ is derivable if and only if $P_1 \& \ldots \& P_m \supset Q_j$ is derivable for some $j$. Apply corollary 4.5 to each of the $Q_j$. QED.

## 5. Lattice theory with existence axioms

We study a system of rules of lattice theory that corresponds to Skolem's original work (1920). The rules are given in a single succedent formulation in natural deduction style, following the idea in section 2.4.

**5.1. Relational axioms and rules for lattice theory:** The axiomatization of lattice theory uses existence axioms for meets and joins instead of explicit meet and join operations. We assume an infinity of parameters $a, b, c, \dots$ and variables $x, y, z, \dots$. There is a binary partial order relation $a \leqslant b$ and two ternary relations $M(a,b,c)$ and $J(a,b,c)$ ("$c$ is the meet of $a$ and $b$," and "$c$ is the join of $a$ and $b$"). We call atoms of these forms **O-atoms**, **M-atoms**, and **J-atoms**. Equality is partial order in both directions. In the substitution rules below, we abbreviate the two premisses $a \leqslant b$ and $b \leqslant a$ by $a = b$. The first rule has zero premisses. In rules III–IV, the mnemonic letters $L$ and $R$ indicate that the meet and join terms appear as left resp. right members of the order relation in the conclusion. The rules for lattice theory are, with the Roman numerals giving the correspondence with Skolem (1920):

**Rules for relational lattice theory ReLT**

**I–II.** *Rules for partial order:*

$$\frac{}{a \leqslant a}\,Ref \qquad \frac{a \leqslant b \quad b \leqslant c}{a \leqslant c}\,Trans$$

**III–IV.** *Rules for Meet and Join:*

$$\frac{M(a,b,c)}{c \leqslant a}\,LM_1 \qquad \frac{M(a,b,c)}{c \leqslant b}\,LM_2 \qquad \frac{J(a,b,c)}{a \leqslant c}\,RJ_1 \qquad \frac{J(a,b,c)}{b \leqslant c}\,RJ_2$$

$$\frac{M(a,b,c) \quad d \leqslant a \quad d \leqslant b}{d \leqslant c}\,RM \qquad \frac{J(a,b,c) \quad a \leqslant d \quad b \leqslant d}{c \leqslant d}\,LJ$$

**V.** *Substitution of equals in Meet and Join:*

$$\frac{M(a,b,c) \quad a = d \quad b = e \quad c = f}{M(d,e,f)}\,SM \qquad \frac{J(a,b,c) \quad a = d \quad b = e \quad c = f}{J(d,e,f)}\,SJ$$

**VI.** *Existential rules for Meet and Join:*

$$\frac{\begin{array}{c}[M(a,b,x)^n]\\ \vdots \\ C\end{array}}{C}\,EM \qquad \frac{\begin{array}{c}[J(a,b,x)^n]\\ \vdots \\ C\end{array}}{C}\,EJ$$

A derivation can begin with any $O$-, $M$-, or $J$-atoms as assumptions. The existential rules indicate by square brackets the **discharged** assumptions that can have been made any number of times $n \geqslant 1$. Those assumptions that have not been discharged in a derivation are the **open** assumptions of the derivation. The existential rules have the

**variable restriction** that the eigenvariable $x$ must not occur free in the conclusion $C$ nor in any open assumption $C$ depends on, except in the $M$- or $J$-atoms indicated as discharged. We assume that the eigenvariable of a rule appears only in the subderivation down to that rule. It follows that all the eigenvariables of existential rules in a derivation are distinct.

The existential rules are equivalent to the existence axioms for meet and join, namely $\forall x \forall y \exists z M(x,y,z)$ and $\forall x \forall y \exists z J(x,y,z)$. If the latter are assumed, the logical rules of universal and existential quantifier elimination lead to the conclusions of the existential lattice rules; In the other direction, the existence axioms are derivable by universal and existential quantifier introduction and the existential lattice rules:

$$\dfrac{\dfrac{\forall x \forall y \exists z M(x,y,z)}{\exists z M(a,b,z)}{\scriptstyle \forall E, \forall E} \qquad \begin{matrix}[M(a,b,v)] \\ \vdots \\ C\end{matrix}}{C}{\scriptstyle \exists E} \qquad \dfrac{\dfrac{\dfrac{[M(a,b,v)]}{\exists z M(a,b,z)}{\scriptstyle \exists I}}{\exists z M(a,b,z)}{\scriptstyle EM}}{\forall x \forall y \exists z M(x,y,z)}{\scriptstyle \forall I, \forall I}$$

From the left derivation, we observe that an existence axiom turns into a corresponding existential rule of inference by the deletion of the existential premiss and its derivation. A general theory of existential rules is given in Negri (2003). In Skolem (1920), rules I–V are treated formally, but existence axioms and their variable restrictions are handled somewhat intuitively.

We consider only derivations with atoms as assumptions and conclusion, because the logical rules permute down with respect to the mathematical rules. Derivation trees have assumptions and instances of rule *Ref* as leaves.

**5.2. Permutability of rules:** The order of application of lattice rules can be permuted by suitable local transformations:

**Lemma 5.1.** (i) *Instances of rules $EM, EJ$ permute down with respect to all the other rules of* **ReLT**. (ii) *Instances of rules $SM, SJ$ permute down with respect to all the rules except for $EM, EJ$. If the conclusion is an $O$-atom, no instance of $SM, SJ$ is needed, and otherwise just one instance of $SM, SJ$ is sufficient.*

**Proof:** (i) If $EM$ or $EJ$ concludes an atom $C$ and $C$ is a premiss of a lattice rule $R$ concluding $D$, rule $R$ is applied to the premiss $C$ of $EM$ or $EJ$, and then $EM$ or $EJ$ is applied to $D$. By the conditions on eigenvariables, this can be always done. (ii) Consider a substitution on $a$ in $M(a,b,c)$. We can leave out the superfluous premisses $b = b$ and $c = c$ and have the instance

$$\dfrac{M(a,b,c) \quad a \leqslant d \quad d \leqslant a}{M(d,b,c)}{\scriptstyle SM}$$

The conclusion $M(d,b,c)$ can be a premiss in $LM_1, LM_2$, and $RM$. In the first case, make the conversion

$$\dfrac{\dfrac{M(a,b,c) \quad a \leqslant d \quad d \leqslant a}{M(d,b,c)}{\scriptstyle SM}}{c \leqslant d}{\scriptstyle LM_1} \quad \rightsquigarrow \quad \dfrac{\dfrac{M(a,b,c)}{c \leqslant a}{\scriptstyle LM_1} \quad a \leqslant d}{c \leqslant d}{\scriptstyle Trans}$$

In the case of $LM_2$ we convert as follows:

$$\frac{\dfrac{M(a,b,c) \quad a \leqslant d \quad d \leqslant a}{M(d,b,c)}\,SM}{c \leqslant b}\,LM_2 \quad \rightsquigarrow \quad \frac{M(a,b,c)}{c \leqslant b}\,LM_2$$

If $M(d,b,c)$ is a premiss in $RM$, the conversion is

$$\frac{\dfrac{M(a,b,c)\ a \leqslant d\ d \leqslant a}{M(d,b,c)}\,SM \qquad e \leqslant d \quad e \leqslant b}{e \leqslant c}\,RM \quad \rightsquigarrow \quad \frac{M(a,b,c) \qquad \dfrac{e \leqslant d \quad d \leqslant a}{e \leqslant a}\,Trans \quad e \leqslant b}{e \leqslant c}\,RM$$

Other cases of substitutions are variants of these three, until when permuting down substitution another substitution is met. We have, again assuming substitutions on the first argument:

$$\frac{\dfrac{M(a,b,c) \quad a \leqslant d \quad d \leqslant a}{M(d,b,c)}\,SM \qquad d \leqslant e \quad e \leqslant d}{M(e,b,c)}\,SM$$

This is converted into transitivities and one substitution:

$$\frac{M(a,b,c) \qquad \dfrac{a \leqslant d \quad d \leqslant e}{a \leqslant e}\,Trans \qquad \dfrac{e \leqslant d \quad d \leqslant a}{e \leqslant a}\,Trans}{M(e,b,c)}\,SM$$

No variable restrictions are violated by the above proof transformations, so that the transformations give a correct derivation of the original conclusion. In the end, if the conclusion is an $O$-atom, no substitutions are needed, and otherwise there is at most one substitution as a last rule. QED.

Lemma 5.1 corresponds to lemma 2 in Skolem (1920). Rules *SM,SJ,EM,EJ* are the only ones that conclude $M$- or $J$-atoms. If existential rules are permuted down and if the conclusion of the derivation is an $O$-atom, no substitutions are needed down to the derivation of the premiss of the first existential rule, and therefore no substitutions at all. We show later that derivations that conclude $M$- or $J$-atoms can be reduced to derivations concluding $O$-atoms so that, by lemma 5.1, we do not need to consider rules *SM,SJ*.

**Definition 5.2.** *A derivation tree in* **ReLT** *is* **loop free** *if it has no branches in which the same atom occurs more than once, except as a premiss and conclusion of an existential rule, and atoms of the form $a \leqslant a$ appear only as leaves. A term in a derivation tree that is not a term in an open assumption or the conclusion is a* **new term**.

**Lemma 5.3.** *In a loop-free derivation of an O-atom with no instances of rules $EM, EJ$, there are no new terms in the derivation.*

**Proof:** We may assume by lemma 5.1 that there are no instances of rules $SM, SJ$. Then $M$- and $J$-atoms are never conclusions so that terms in them remain terms in open assumptions. Rule *Trans* is the only one that can remove a new term, say $b$:

$$\frac{a \leqslant b \quad b \leqslant c}{a \leqslant c}\,Trans$$

Trace up atoms with the new term. First occurrences of $b$ cannot be in any $M$- or $J$-atoms or other assumptions. Thus, the new term must appear first in instances $b \leqslant b$ of rule *Ref.* Such an instance is not a premiss of *Trans* because the conclusion would be equal to the other premiss and the derivation would have a loop. Therefore $b \leqslant b$ is a premiss of $RM$ or $LJ$, say

$$\frac{M(b,e,f) \quad b \leqslant b \quad b \leqslant e}{b \leqslant f}{\scriptstyle RM} \qquad \frac{J(b,e,f) \quad b \leqslant b \quad e \leqslant b}{f \leqslant b}{\scriptstyle LJ}$$

Now the new term is in an $M$- or $J$-atom, contrary to assumption, and similarly if $b \leqslant b$ is the last premiss of $RM$ or $LJ$. QED.

**Lemma 5.4.** *In a loop-free derivation of an O-atom with one existential rule EM or EJ as last step and discharged atom $M(a,b,v)$ or $J(a,b,v)$, first occurrences of the eigenvariable $v$ are not in instances of rule Ref.*

**Proof:** By lemma 5.1, the derivation of the $O$-atom premiss of rule $EM$ does not need rules $SM, SJ$. Assume there is a leaf in the derivation tree that begins with $v \leqslant v$. It is not a premiss in *Trans* or there is a loop. By the variable restriction on rule $EM$, $v$ is not in any open assumption. Therefore $v \leqslant v$ is not a premiss in rule $LJ$. So $v \leqslant v$ is a premiss in $RM$, but then the first or second argument in $M(a,b,v)$ is $v$ and the conclusion of $RM$ is the same as the premiss $v \leqslant v$. The proof for $EJ$ is dual to above. QED.

**Theorem 5.5. Subterm property.** *If an O-atom is derivable from atomic assumptions in* **ReLT**, *it has a derivation with no new terms.*

**Proof:** We may assume the derivation is loop free. If there are no instances of $EM$ or $EJ$, the result is given in lemma 5.3. We show that derivations with existential rules transform through suitable permutations into ones with loops: Assume the derivation has instances of $EM$ or $EJ$. By lemma 5.1, these can be permuted last, and each of them concludes the $O$-atom that is the conclusion of the whole derivation. Consider the subderivation down to a first instance of an existential rule, say $EM$ that discharges $M(a,b,v)$. By lemma 5.1, rules $SM, SJ$ can be assumed absent so that all $M$- and $J$-atoms in the derivation are assumptions. The eigenvariable $v$ is a new term and by lemma 5.4, all topmost occurrences of $v$ are in the discharged assumptions $M(a,b,v)$. We transform the derivation into another one that has the same terms and show that either it has a loop or else it has the subterm property. The transformation consists in permuting up instances of rule *Trans.*

As in the proof of lemma 5.3, only rule *Trans* can remove the new term $v$ from the derivation. Consider an instance such that $v$ does not appear anywhere below in the derivation:

$$\frac{c \leqslant v \quad v \leqslant d}{c \leqslant d}{\scriptstyle Trans} \tag{1}$$

If the premiss $c \leqslant v$ is concluded by $LM_1$ or $LM_2$, then $c$ is identical to $v$. The left premiss of *Trans* is $v \leqslant v$, but then the right premiss is identical to the conclusion and there is a loop. Rules $RJ_1, RJ_2$ cannot conclude $c \leqslant v$ or else $v$ is in a $J$-atom. The remaining cases are that $c \leqslant v$ has been concluded by *Trans*, $LJ$, or $RM$. With *Trans*, we permute up the *Trans* removing $v$:

$$\dfrac{\dfrac{c \leqslant e \quad e \leqslant v}{c \leqslant v}{\scriptstyle Trans} \qquad v \leqslant d}{c \leqslant d}{\scriptstyle Trans} \quad \rightsquigarrow \quad \dfrac{c \leqslant e \qquad \dfrac{e \leqslant v \quad v \leqslant d}{e \leqslant d}{\scriptstyle Trans}}{c \leqslant d}{\scriptstyle Trans} \qquad (2)$$

With $LJ$, there is some premiss of the form $J(e, f, c)$ and *Trans* (abbr. *Tr*) is permuted up as follows:

$$\dfrac{\dfrac{J(e,f,c) \quad e \leqslant v \quad f \leqslant v}{c \leqslant v}{\scriptstyle LJ} \qquad v \leqslant d}{c \leqslant d}{\scriptstyle Tr} \rightsquigarrow \dfrac{J(e,f,c) \qquad \dfrac{e \leqslant v \quad v \leqslant d}{e \leqslant d}{\scriptstyle Tr} \qquad \dfrac{f \leqslant v \quad v \leqslant d}{f \leqslant d}{\scriptstyle Tr}}{c \leqslant d}{\scriptstyle LJ} \qquad (3)$$

The permutation of *Trans* removing $v$ as in (2) and (3) is repeated until the left premiss has been concluded by $RM$. We then have some term $c'$ such that

$$\dfrac{\dfrac{M(a,b,v) \quad c' \leqslant a \quad c' \leqslant b}{c' \leqslant v}{\scriptstyle RM} \qquad v \leqslant d}{c' \leqslant d}{\scriptstyle Trans} \qquad (4)$$

Now consider the right premiss $v \leqslant d$ of (4). Rules $RJ_1, RJ_2$, and $LJ$ would give a $J$-atom with term $v$, so the possible rules are $LM_1, LM_2, RM$, and *Trans*. With *Trans* we permute similarly to (2):

$$\dfrac{c' \leqslant v \qquad \dfrac{v \leqslant g \quad g \leqslant d}{v \leqslant d}{\scriptstyle Trans}}{c' \leqslant d}{\scriptstyle Trans} \quad \rightsquigarrow \quad \dfrac{\dfrac{c' \leqslant v \quad v \leqslant g}{c' \leqslant g}{\scriptstyle Trans} \qquad g \leqslant d}{c' \leqslant d}{\scriptstyle Trans} \qquad (5)$$

With $RM$, there is some premiss of the form $M(g, h, d)$ and *Trans* is permuted up as follows:

$$\dfrac{c' \leqslant v \qquad \dfrac{M(g,h,d) \quad v \leqslant g \quad v \leqslant h}{v \leqslant d}{\scriptstyle RM}}{c' \leqslant d}{\scriptstyle Tr} \rightsquigarrow \dfrac{M(g,h,d) \qquad \dfrac{c' \leqslant v \quad v \leqslant g}{c' \leqslant g}{\scriptstyle Tr} \qquad \dfrac{c' \leqslant v \quad v \leqslant h}{c' \leqslant h}{\scriptstyle Tr}}{c' \leqslant d}{\scriptstyle RM} \qquad (6)$$

The permutation of *Trans* removing $v$ as in (5) and (6) is repeated until for some term $d'$ an atom $v \leqslant d'$ has been concluded by $LM_1$ or $LM_2$. Then $d'$ is identical to $a$ or to $b$ and step (4) has become one of:

$$\dfrac{\dfrac{M(a,b,v) \quad c' \leqslant a \quad c' \leqslant b}{c' \leqslant v}{\scriptstyle RM} \qquad \dfrac{M(a,b,v)}{v \leqslant a}{\scriptstyle LM_1}}{c' \leqslant a}{\scriptstyle Trans} \qquad \dfrac{\dfrac{M(a,b,v) \quad c' \leqslant a \quad c' \leqslant b}{c' \leqslant v}{\scriptstyle RM} \qquad \dfrac{M(a,b,v)}{v \leqslant b}{\scriptstyle LM_2}}{c' \leqslant b}{\scriptstyle Trans}$$

Both derivations have a loop. Deletion of the part of derivation between the two occurrences of the same formula deletes also the assumption $M(a, b, v)$. Thus, in the end there is no new term $v$ in a transformed loop-free derivation, and therefore no instance of rule $EM$. The conclusion now follows by lemma 5.3. QED.

Theorem 5.5 proves the conservativity of existential rules: If an $O$-atom is derivable from given atoms in lattice theory, it is derivable without rules $EM, EJ$. This is the main theorem of Skolem (1920).

**5.3. Decidability of universal formulas:** We first reduce the derivability of arbitrary atoms to the derivability of $O$-atoms and then apply the subterm property to conclude Skolem's theorem on the decidability of universal formulas.

**Lemma 5.6.** *Derivability in* **ReLT** *of an M-atom* $M(a,b,c)$ *from assumptions* $\Gamma$ *reduces to the derivability of two O-atoms, and the same for J-atoms.*

**Proof:** Let $v$ be a fresh variable. We show that $M(a,b,c)$ is derivable from assumptions $\Gamma$ if and only if $v \leqslant c$ and $c \leqslant v$ are derivable from $M(a,b,v)$ and $\Gamma$.

If $M(a,b,c)$ is derivable from $\Gamma$, we have

$$\dfrac{M(a,b,v) \quad \dfrac{\dfrac{\begin{matrix}\Gamma\\ \vdots\end{matrix}}{M(a,b,c)}}{c \leqslant a}{\scriptstyle LM_1} \quad \dfrac{\dfrac{\begin{matrix}\Gamma\\ \vdots\end{matrix}}{M(a,b,c)}}{c \leqslant b}{\scriptstyle LM_2}}{c \leqslant v}{\scriptstyle RM} \qquad \dfrac{\begin{matrix}\Gamma\\ \vdots\\ M(a,b,c)\end{matrix} \quad \dfrac{M(a,b,v)}{v \leqslant a}{\scriptstyle LM_1} \quad \dfrac{M(a,b,v)}{v \leqslant b}{\scriptstyle LM_2}}{v \leqslant c}{\scriptstyle RM}$$

In the other direction, assuming $v \leqslant c$ and $c \leqslant v$ derivable from $M(a,b,v)$ and $\Gamma$, we have

$$\dfrac{\dfrac{[M(a,b,v)] \quad \begin{matrix}[M(a,b,v)],\Gamma\\ \vdots\\ v \leqslant c\end{matrix} \quad \begin{matrix}[M(a,b,v)],\Gamma\\ \vdots\\ c \leqslant v\end{matrix}}{M(a,b,c)}{\scriptstyle SM}}{M(a,b,c)}{\scriptstyle EM}$$

Since $v$ was chosen fresh, the variable restriction in rule $EM$ is met. QED.

**Theorem 5.7. Derivability of an atom from given atoms.** *The derivability of an atom from given atoms in* **ReLT** *is decidable.*

**Proof.** By lemma 5.6, we can assume the conclusion to be an $O$-atom. By the proof of theorem 5.5, we can assume that there are no existential rules. By the subterm property, only a bounded number of terms need be used in instances of rules. Therefore the number of loop-free derivations ending with the atom to be derived is also bounded. QED.

**5.4. Further decidable classes of formulas:** The standard decidable classes of formulas of pure predicate calculus include the quantifier prefix classes $\forall \dots \forall \exists \dots \exists$ and $\forall \dots \forall \exists \forall \dots \forall$, their degenerate cases, etc. The formulation of lattice theory with existential axioms makes it a theory expressible in the language of pure predicate calculus, that is, without constants or functions. Consider those prefix classes that have a bounded Herbrand expansion. By the subterm property, proof search terminates for these classes, and the following result is obtained:

**Theorem 5.8. Standard decidable classes.** *Let* $QA$ *be a formula in prenex form, with a quantifier prefix* $Q$ *such that the corresponding Herbrand disjunction is bounded. Then derivability of* $QA$ *in* **ReLT** *is decidable.*

## 6. A single succedent calculus with meet and join

We give briefly a formulation of a rule system in natural deduction style with explicit meet and join constructions and prove the Whitman conditions.

**6.1. Rules in natural deduction style with meet and join:** The axioms of lattice theory with meet and join lead to the following system of rules, some of which have zero premisses:

**Rules for lattice theory with meet and join**

$$\frac{}{a \leqslant a}\,Ref \qquad \frac{a \leqslant b \quad b \leqslant c}{a \leqslant c}\,Trans$$

$$\frac{}{a \wedge b \leqslant a}\,L\wedge_1 \qquad \frac{}{a \wedge b \leqslant b}\,L\wedge_2 \qquad \frac{c \leqslant a \quad c \leqslant b}{c \leqslant a \wedge b}\,R\wedge$$

$$\frac{}{a \leqslant a \vee b}\,R\vee_1 \qquad \frac{}{b \leqslant a \vee b}\,R\vee_2 \qquad \frac{a \leqslant c \quad b \leqslant c}{a \vee b \leqslant c}\,L\vee$$

These rules are designated by **NDLT**. Their permutability properties and the subterm property were established in Negri and von Plato (2002).

Term $b$ in rule *Trans* is called a **middle term**. An inspection of the rules shows that middle terms in *Trans* are the only terms in premisses that need not be also terms in a conclusion. Derivation trees have assumptions and instances of zero-premiss rules as leaves.

**Definition 6.1.** *A derivation tree in* **NDLT** *is* **loop free** *if it has no branches in which the same atom occurs more than once and formulas that match the conclusion of a zero-premiss rule appear only as leaves. A* **new term** *in a derivation tree is a term that is not a term or a subterm in an assumption or the conclusion.*

A given derivation can be made loop free by deleting parts between repeated formulas and above those formulas that can be concluded by zero-premiss rules. The rule of transitivity can be permuted up relative to most lattice rules, by which the subterm property can be concluded directly for the rules in natural deduction style:

**Theorem 6.2. Subterm property for NDLT.** *If an atom is derivable from atomic assumptions in* **NDLT**, *it has a derivation with no new terms.*

**Proof:** The proof is similar in its main outlines to the proof of the subterm property for the calculus **ReLT**. For details, see Negri and von Plato (2002). QED.

The decision method described after corollary 4.5 applies to the derivability in **NDLT** of an atom $a \leqslant b$ from a set of atomic assumptions $\Gamma$: First add to $\Gamma$ all instances of zero-premiss rules with terms subterms in $a, b, \Gamma$, then apply repeatedly the remaining three rules until $a \leqslant b$ appears or no new formulas are produced. The duplication of formulas in the left rule system corresponds to looping in **NDLT**.

**6.2. The Whitman conditions:** Whitman's solution of the word problem for lattices in (1941) established the following condition: If $a \wedge b \leqslant c \vee d$ is derivable, then one of $a \leqslant c$, $a \leqslant d$, $b \leqslant c$, $b \leqslant d$ is derivable. We give a decision method for the derivability of atoms in lattice theory that has this result as an immediate corollary.

We first modify axioms $L\wedge, R\vee$ slightly, into

$$a \leqslant c \supset a \wedge b \leqslant c, \quad c \leqslant a \supset c \leqslant a \vee b,$$

$$b \leqslant c \supset a \wedge b \leqslant c, \quad c \leqslant b \supset c \leqslant a \vee b.$$

With $c \equiv a$ and $c \equiv b$, respectively, the old axioms follow by reflexivity. In the other direction, if $a \wedge b \leqslant a$ and $a \leqslant c$, then $a \wedge b \leqslant c$ follows by transitivity.

The modified axioms convert into the following variant of **NDLT**:

$$\frac{a \leqslant c}{a \wedge b \leqslant c}\,L\wedge_1 \qquad \frac{b \leqslant c}{a \wedge b \leqslant c}\,L\wedge_2 \qquad \frac{a \leqslant c \quad b \leqslant c}{a \vee b \leqslant c}\,L\vee$$

$$\frac{c \leqslant a}{c \leqslant a \vee b}\,R\vee_1 \qquad \frac{c \leqslant b}{c \leqslant a \vee b}\,R\vee_2 \qquad \frac{c \leqslant a \quad c \leqslant b}{c \leqslant a \wedge b}\,R\wedge$$

To these rules we add the rules of partial order.

**Theorem 6.3.** *The variant of system* **NDLT**, *with rule Trans excluded, is complete for the derivability of atoms* $a \leqslant b$.

**Proof:** We shall prove rule *Trans* admissible for derivations of atoms $a \leqslant b$, by which completeness follows. Recall that admissibility requires that if the premisses of *Trans* are derivable with the other rules, also its conclusion is derivable by these rules. The proof is by induction on the sum of the heights of the derivations of the premisses of *Trans*. If one of them is obtained by *Ref*, the other premiss gives the conclusion. If the left premiss is derived by $L\wedge_1$ we have a derivation of the form

$$\frac{\dfrac{a \leqslant c}{a \wedge b \leqslant c}\,L\wedge_1 \qquad c \leqslant d}{a \wedge b \leqslant d}\,\mathit{Trans}$$

which is transformed into

$$\frac{\dfrac{a \leqslant c \quad c \leqslant d}{a \leqslant d}\,\mathit{Trans}}{a \wedge b \leqslant d}\,L\wedge_1$$

with premisses of *Trans* of smaller derivation height. A similar conversion applies if the left premiss is derived by $L\wedge_2$.

If the left premiss is derived by $R\wedge$, we analyze the derivation of the right premiss. If the right premiss is derived by a left rule, say $L\wedge_1$, we have the conversion

$$\frac{\dfrac{c \leqslant a \quad c \leqslant b}{c \leqslant a \wedge b}\,R\wedge \qquad \dfrac{a \leqslant d}{a \wedge b \leqslant d}\,L\wedge_1}{c \leqslant d}\,\mathit{Trans} \quad \rightsquigarrow \quad \frac{c \leqslant a \quad a \leqslant d}{c \leqslant d}\,\mathit{Trans}$$

If the right premiss is derived by a right rule, then the term $a \wedge b$ is found also in the premiss(es) of the right premiss, and transitivity is permuted up to these premisses. If the rule used to derive the right premiss is $R\wedge$, we have

$$\frac{\dfrac{c \leqslant a \quad c \leqslant b}{c \leqslant a \wedge b}\,R\wedge \qquad \dfrac{a \wedge b \leqslant d \quad a \wedge b \leqslant e}{a \wedge b \leqslant d \wedge e}\,R\wedge}{c \leqslant d \wedge e}\,\mathit{Trans}$$

which is transformed into

$$\frac{\dfrac{c \leqslant a \wedge b \quad a \wedge b \leqslant d}{c \leqslant d}\,\mathit{Trans} \qquad \dfrac{c \leqslant a \wedge b \quad a \wedge b \leqslant e}{c \leqslant e}\,\mathit{Trans}}{c \leqslant d \wedge e}\,R\wedge$$

All the other conversions are similar to the above ones with $L\wedge_1$ and $R\vee$: If the left premiss is derived by a left rule, transitivity is permuted, and similarly if the right premiss is derived by a right rule. If the left premiss is derived by a right rule and the right premiss by a left rule, transitivity with middle term $b$ is replaced by a transitivity with middle term given by a subterm of $b$ and smaller sum of derivation heights. QED.

Transitivity permutes up until it reaches leaves in the derivation tree and disappears. With the rule of transitivity left out, a complete system of rules for lattice theory for the derivation of atoms is still obtained. An inspection of the remaining rules shows that each premiss has one lattice operation less than the conclusion, so a decision procedure for the derivability of atomic formulas in lattice theory is obtained: Starting with the atom to be concluded, match it as a conclusion to each of the rules, then repeat until no lattice operations are left. If a derivation with reflexivities as leaves is found, the atom is derivable, otherwise it is not derivable. By the completeness of this method we have in particular:

**Corollary 6.4. Whitman conditions.** *The atom $a \wedge b \leqslant c \vee d$ is derivable only if one of $a \leqslant c$, $a \leqslant d$, $b \leqslant c$, $b \leqslant d$ is derivable.*

The result obviously generalizes to arbitrary finite meets and joins instead of the binary ones.

## 7. Concluding remarks

The form of the lattice axioms is such that it is possible to present derivations in a "logic-free" manner in a single succedent sequent calculus and its natural deduction style equivalent. However, if the order relation is linear, it is essential to use a multisuccedent calculus. With the linearity postulate $a \leqslant b \vee b \leqslant a$ added to the axioms of partial order, a **linear** lattice is obtained. The corresponding rule to be added to the left rule system **G3LT** is:

$$\frac{a \leqslant b, \Gamma \rightarrow \Delta \quad b \leqslant a, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta}{\scriptstyle Lin}$$

Should $\Delta$ consist of a single formula, the linearity rule would be superfluous, as is shown by the conservativity results of Negri, von Plato, and Coquand (2001). The analysis of linear order of that paper can be extended into the case of linear lattices.

There exists an implementation of systems of sequent calculus that supports extension by axioms, the PESCA system of Aarne Ranta. It automatically translates universal axioms into rules that are added to a chosen sequent calculus. Formal proofs can be constructed interactively or by brute force. A description of the system can be found in Ranta (2001).

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