Proof-theoretical analysis of order relations

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Abstract: A proof-theoretical analysis of elementary theories of order relations is effected through the formulation of order axioms as mathematical rules added to contraction-free sequent calculus. Among the results obtained are proof-theoretical formulations of conservativity theorems corresponding to Szpilrajn's theorem on the extension of a partial order into a linear one. Decidability of the theories of partial and linear order for quantifier-free sequents is shown by giving terminating methods of proof-search.

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1. Introduction

Previous work (Negri and von Plato 1998) gave a proof-theoretical way of treating axiomatic systems. The axioms are converted into rules of inference added to suitable systems of logical rules. The particular logical systems used were what are known as contraction-free sequent calculi. These calculi have the remarkable feature that none of the standard structural rules, weakening, contraction, exchange, change of bound variable, or cut, need be used in derivations. Our previous work showed that mathematical axioms can be converted into rules that operate on atomic formulas of antecedents of sequents, with no structural rules. Moreover, these *mathematical rules* commute with the rules of classical logic, so that, permuting down the latter, it suffices to consider derivations with only mathematical rules.

The main advantage of the present approach is that well-developed methods of proof analysis from sequent calculi for pure logic become applicable to mathematical theories. Systems of mathematical rules can be found for all those axiomatic theories that permit of a quantifier-free formulation. In typical cases such as lattice theory or plane affine geometry, free parameters and constructions are used in place of Π^0_2-axioms. (See Negri and von Plato 2001 for these applications.)

We first present the general form of rules that can be added to contraction-free sequent calculi as rules that operate on the antecedent parts of sequents. A new dual form is found with rules that act symmetrically on the succedent parts. Some theories permit a formulation of the latter type with the extra restriction that there is just one formula in the succedent, with a greatly simplified proof analysis.

The main result about order relations is a conservativity theorem relating partial and linear order. Such conservativity results are proof-theoretical counterparts to extension results, and we obtain as a corollary a proof-theoretical version of Szpilrajn's theorem on the extension of a partial order into a linear one.

In root-first proof search, mathematical rules instantiate new terms in premisses. We prove that derivability with the rules of partial and linear order is decidable by giving a terminating method of proof search. Termination follows from the *subterm property:* the new terms instantiated by mathematical rules can be restricted to terms in the conclusion.

Section 7, by the third author, gives an alternative approach to Szpilrajn's theorem and the decidability of the quantifier-free theory of linear order.

2. A duality for mathematical rules

For the notation and sequent calculus rules, we refer to Negri and von Plato (1998), §2. Mathematical axioms were there added to classical and intuitionistic multisuccedent sequent calculi **G3c** and **G3im** as rules of the form

$$\frac{Q_1, P_1, \dots, P_m, \Gamma \Rightarrow \Delta \quad \dots \quad Q_n, P_1, \dots, P_m, \Gamma \Rightarrow \Delta}{P_1, \dots, P_m, \Gamma \Rightarrow \Delta}\; \scriptstyle{Rule}$$

Here the Q_i are the *removed* atoms, and the $P_1, \dots, P_m$ the *activated* atoms.

The *left rule-scheme* has a dual formulation as a *right rule-scheme:*

$$\frac{\Gamma \Rightarrow \Delta, Q_1, \dots, Q_n, P_1 \quad \dots \quad \Gamma \Rightarrow \Delta, Q_1, \dots, Q_n, P_m}{\Gamma \Rightarrow \Delta, Q_1, \dots, Q_n}\; \scriptstyle{Rule}$$

Thus, an axiom of the form $P_1 \& \dots \& P_m \supset Q_1 \vee \dots \vee Q_n$ becomes the rule: If each of P_i follows from Γ, the cases under Γ are $Q_1, \dots, Q_n$. The atoms Q_i are repeated in the premisses in order to obtain admissibility of right contraction. As for the left rule-scheme, if the atoms in a rule have an instance that makes two atoms in the conclusion identical, the rule with these atoms contracted into one atom has to be included in the system of rules. This requirement is called the *closure condition.*

Theorem 2.1. *The structural rules of left and right weakening and contraction and the rule of cut are admissible in extensions of* **G3c** *and* **G3im** *with rules following the right rule-scheme and satisfying the closure condition.*

Proof: Dual to the proofs of admissibility in Negri and von Plato (1998). QED.

In extensions of **G3c** and **G3im** with rules following the above schemes, weakening and contraction are admissible and *height preserving*, that is, whenever their premisses are derivable, then also their conclusion is derivable with equal or smaller height. The elimination of these rules from a derivation gives actually a subtree of the original derivation tree.

Height-preserving admissibility of structural rules is a key property of extensions with mathematical rules, because it permits to prove results by arguments based on minimum-size derivations. These proof methods are central in the developments of Sections 4–6.

The invertibility of the logical rules of the classical calculus **G3c** has as a consequence the separation of derivations into initial parts with mathematical rules followed by a part with logical rules. For this reason we shall consider only derivations of sequents which are not conclusions of logical rules, i.e., sequents of the form $P_1, \ldots, P_m \Rightarrow Q_1, \ldots, Q_n, \bot, \ldots, \bot$ with P_i, Q_j atomic formulas. Logical rules will be conservative over the mathematical ones.

3. Harrop systems

We consider theories that can be formulated as systems of mathematical rules with just one formula as a succedent.

Definition 3.1. *The class of* Harrop formulas *is defined by*

1. *Atoms $P, Q, R, \ldots$, and $\bot$ are Harrop formulas,*
2. *If A and B are Harrop formulas, then $A \& B$ is a Harrop formula,*
3. *If B is a Harrop formula, then $A \supset B$ is a Harrop formula.*

A *Harrop theory* is one the axioms of which consist of Harrop formulas. A *left Harrop system* is the system of mathematical rules obtained from the axioms of a Harrop theory by using the left rule-scheme, and similarly for a *right Harrop system.* The observation is immediate that the rules of a left Harrop system have *at most one* premiss, thus the derivations are linear, which has the following consequence.

Theorem 3.2. *If a sequent $\Gamma \Rightarrow \Delta$ is derivable in a left Harrop system, then $\Gamma \Rightarrow P$ is derivable for some atom P. If Δ contains atoms, the atom P can be chosen from Δ.*

Proof: Consider a derivation $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$. If the topsequent is a logical axiom $P, \Gamma' \Rightarrow \Delta', P$, with $\Delta = \Delta', P$, the succedent can be changed into P. If the topsequent is a zero-premiss mathematical rule, any atom P can be put as the succedent and the derivation with the new succedent continued as with Δ. QED.

We translate single succedent left rules with no more than one premiss into single succedent right rules as follows:

$$\frac{Q, P_1, \ldots, P_m, \Gamma \Rightarrow C}{P_1, \ldots, P_m, \Gamma \Rightarrow C}\,R \quad \rightsquigarrow \quad \frac{\Gamma \Rightarrow P_1 \quad \ldots \quad \Gamma \Rightarrow P_m}{\Gamma \Rightarrow Q}\,R$$

With $m = 0$ we have as a limiting case a zero-premiss right rule.

If the formula Q is repeated in the succedents of the premisses of the translated rule, it becomes an instance of the right rule scheme. To conclude the admissibility of structural rules for the single succedent formulation without repetition, we note that right contraction in the conclusion does not arise and that cut permutes up as usual. The rules are interderivable by cuts and contractions: Set $C = Q$ in the left rule, and cuts of the conclusion $P_1, \ldots, P_m, \Gamma \Rightarrow Q$ with the premisses of the right rule followed by left contractions give the conclusion $\Gamma \Rightarrow Q$. In the other direction, apply the right rule to the premisses $P_1, \ldots, P_m, \Gamma \Rightarrow P_i$, $0 \leqslant i \leqslant m$ to conclude $P_1, \ldots, P_m, \Gamma \Rightarrow Q$, then cut away Q with the premiss of the left rule to obtain the conclusion of the left rule, modulo contractions. Thus we have:

Proposition 3.3. *A left Harrop system and its corresponding single succedent right system derive the same single succedent sequents.*

In a single succedent rule system for axioms that are not Harrop formulas, as in Negri's (1999) treatment of apartness relations, a disjunction is needed for expressing the presence of genuine cases, with the consequence that mathematical rules are mixed with instances of rule $R\vee$.

4. Partial order

The axioms of partial order are

PO1. $a \leqslant a$,
PO2. $a \leqslant b \;\&\; b \leqslant c \supset a \leqslant c$.

Equality is defined by $a = b \equiv \; a \leqslant b \;\&\; b \leqslant a$. (Thus, we are working with what are sometimes called quasiorderings.) It follows that equality is an equivalence relation. Further, since equality is defined in terms of partial order, the principle of substitution of equals for the latter is provable.

The axioms of partial order determine by the rule-scheme the mathematical rules *Ref* and *Trans*:

$$\frac{a \leqslant a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\textit{Ref} \qquad \frac{a \leqslant c, a \leqslant b, b \leqslant c, \Gamma \Rightarrow \Delta}{a \leqslant b, b \leqslant c, \Gamma \Rightarrow \Delta}\,\textit{Trans}$$

This system of rules is designated **GPO**. The closure condition arises when $a \equiv b$ and $b \equiv c$. Then the premiss of rule *Trans* in fact becomes

$$a \leqslant a, a \leqslant a, a \leqslant a, \Gamma \Rightarrow \Delta$$

The conclusion follows by rule *Ref* so that the closure condition is satisfied. We conclude by theorem 4.1 of Negri and von Plato (1998):

Theorem 4.1. *The rules of weakening, contraction, and cut are admissible in* **GPO**. *Weakening and contraction are admissible and height preserving.*

There are essentially two kinds of derivations in **GPO**. To see what they are, we assume that derivations contain no unnecessary detours: a derivation is of *minimum size* if no proper subtree of the given derivation is a derivation. Reasoning as in the proof of theorem 3.2, if $\Gamma \Rightarrow \Delta$ is derivable, the topsequent has the form $P, \Gamma' \Rightarrow \Delta', P$ with $\Delta', P = \Delta$, and we can delete Δ' from the topsequent. The two kinds of derivations are:

1. Reflexivity derivations: P in the topsequent is a *reflexivity atom* of the form $a \leqslant a$. The conclusion $\Gamma \Rightarrow a \leqslant a$ follows from the logical axiom $a \leqslant a, \Gamma \Rightarrow a \leqslant a$ with one application of rule *Ref*:

$$\frac{a \leqslant a, \Gamma \Rightarrow a \leqslant a}{\Gamma \Rightarrow a \leqslant a}\,\scriptstyle{Ref}$$

The context Γ is superfluous and can be deleted, thus, the conclusion becomes $\Rightarrow a \leqslant a$.

2. Transitivity derivations: The topsequent is of the form $a_1 \leqslant a_n, \Gamma' \Rightarrow a_1 \leqslant a_n$. The atom $a_1 \leqslant a_n$ must be the removed atom in a first step of transitivity or else the derivation can be shortened: If some other atom P were removed, with $\Gamma' \equiv P, \Gamma''$, the derivation could be shortened by starting with $a_1 \leqslant a_n, \Gamma'' \Rightarrow a_1 \leqslant a_n$ as topsequent. We note that there cannot be steps of reflexivity in this derivation: The reflexivity atom would be principal in a step of transitivity, else it could be removed tout court from the derivation with a subsequent shortening, thus there would be a step of the form

$$\frac{a \leqslant b, a \leqslant a, a \leqslant b, \Gamma \Rightarrow a_1 \leqslant a_n}{a \leqslant a, a \leqslant b, \Gamma \Rightarrow a_1 \leqslant a_n}\,\scriptstyle{Trans}$$

By height-preserving admissibility of contraction the conclusion of this step could be obtained already from the premiss, without using transitivity, in one step less.

Two atoms $a_1 \leqslant a_2, a_2 \leqslant a_n$ are activated by the step of *Trans* removing $a_1 \leqslant a_n$ so that the topsequent is of the form

$$a_1 \leqslant a_n, a_1 \leqslant a_2, a_2 \leqslant a_n, \Gamma'' \Rightarrow a_1 \leqslant a_n$$

In the second step, one of the activated atoms must become the removed atom, with two new activated atoms, say $a_2 \leqslant a_3, a_3 \leqslant a_n$, or else the derivation can be shortened. The closure of the atom $a_1 \leqslant a_n$ with respect to the activation relation gives us a *chain* $a_1 \leqslant a_2, a_2 \leqslant a_3, \ldots a_{n-1} \leqslant a_n$ in the topsequent. Deleting the atoms that have not been active in the derivation, we have a derivation of the form

$$\cfrac{\cfrac{\Gamma''', a_1 \leqslant a_2, a_2 \leqslant a_3, \ldots a_{n-1} \leqslant a_n \Rightarrow a_1 \leqslant a_n}{\vdots}\,\scriptstyle{Trans}}{a_1 \leqslant a_2, a_2 \leqslant a_3, \ldots a_{n-1} \leqslant a_n \Rightarrow a_1 \leqslant a_n}\,\scriptstyle{Trans}$$

in which Γ''' consists of the removed atoms $a_1 \leqslant a_n, \ldots$.

Thus, the two kinds of derivations in **GPO** amount to concluding a reflexivity atom $a \leqslant a$ and an ordering $a_1 \leqslant a_n$ of the extremes of a chain $a_1 \leqslant a_2, a_2 \leqslant a_3, \ldots, a_{n-1} \leqslant a_n$ the atoms of which have been assumed. We have:

Proposition 4.2. *Sequents* $\Gamma \Rightarrow \Delta$ *derivable in* **GPO** *are derivable as left and right weakenings of reflexivity and transitivity derivations.*

Proof search for a sequent $\Gamma \Rightarrow \Delta$ is effected by two controls: Does Δ contain a reflexivity atom? Does Γ contain a chain from a_1 to a_n with the atom $a_1 \leqslant a_n$ in Δ? If so, the sequent $\Gamma \Rightarrow \Delta$ is derivable, otherwise it is underivable.

Nondegenerate partial order is obtained by adding the axiom

PO3. $\sim 1 \leqslant 0$

to PO1 and PO2. The corresponding rule has zero premisses:

$$\frac{}{1 \leqslant 0, \Gamma \Rightarrow \Delta}{\scriptstyle Ndeg}$$

Derivations remain linear and theorem 3.2 applies. If the topsequent is an instance of *Ndeg*, the atom $1 \leqslant 0$ is removed by *Trans*, for it cannot be removed with *Ref.* Steps of *Trans* hide the inconsistent assumption $1 \leqslant 0$, with the general form of conclusion

$$1 \leqslant a_1, a_1 \leqslant a_2, \ldots, a_{n-1} \leqslant 0 \Rightarrow a \leqslant b$$

with the chain in the antecedent being the closure of formulas activated by $1 \leqslant 0$ and $a \leqslant b$ in the succedent an arbitrary atom.

We also need to have a condition of *nontriviality* for a partial order, in order to exclude the case that all derivable atoms are reflexivity atoms. We therefore add a fourth axiom

PO4. $0 \leqslant 1$.

The corresponding rule is

$$\frac{0 \leqslant 1, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}{\scriptstyle Ntriv}$$

This rule commutes down with instances of *Ref* and *Trans.* The only interesting case is a transitivity derivation with a chain from which atoms $0 \leqslant 1$ have been removed by *Ntriv.*

5. Linear order and Szpilrajn's theorem

The theory of linear order is here obtained by adding to partial order the *linearity axiom*

LO. $\quad a \leqslant b \vee b \leqslant a$

The corresponding rule is

$$\frac{a \leqslant b, \Gamma \Rightarrow \Delta \quad b \leqslant a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}{\scriptstyle Lin}$$

The closure condition does not arise and we conclude, as in theorem 4.1, that the structural rules are admissible.

Rule *Lin* is *prima facie* a difficult rule. Let us note first that if $a \equiv b$, axiom LO gives $a \leqslant a \vee a \leqslant a$ and rule *Lin* has correspondingly two identical premisses, so reflexivity would follow as a special case of linearity. However, it is still useful to maintain rule *Ref* as a separate rule, in order to relate linear and partial order. The system of rules for linear order is designated **GLO**. The following conservativity theorem for single succedent sequents is the key insight:

Theorem 5.1 *If* $\Gamma \Rightarrow P$ *is derivable in* **GLO**, *it is already derivable in* **GPO**..

Proof: Consider a derivation with just one instance of *Lin*, as the last rule, and assume the derivation to have minimum size. Thus, the premisses of *Lin* $c \leqslant d, \Gamma \Rightarrow P$ and $d \leqslant c, \Gamma \Rightarrow P$ are derivable in nontrivial partial order. If P is a reflexivity atom in either topsequent, $\Gamma \Rightarrow P$ is derivable in one step of *Ref*. Otherwise, with $P \equiv a \leqslant b$, there will be two transitive closures of the removed atom $a \leqslant b$ in both derivations of the two premisses of *Lin*, and let them be $a \leqslant a_1, \ldots a_{m-1} \leqslant b$ and $a \leqslant b_1, \ldots b_{n-1} \leqslant b$. If $c \leqslant d$ is not an atom in the first chain, it can be deleted and a derivation of $\Gamma \Rightarrow P$ in partial order obtained, and similarly for $d \leqslant c$ in the second chain. Thus, we have the two chains

$a \leqslant a_1, \ldots, a_i \leqslant c, c \leqslant d, d \leqslant a_{i+1}, \ldots a_{m-1} \leqslant b$
$a \leqslant b_1, \ldots, b_j \leqslant d, d \leqslant c, c \leqslant b_{j+1}, \ldots b_{n-1} \leqslant b$

We form the chain

$a \leqslant a_1, \ldots, a_i \leqslant c, c \leqslant b_{j+1}, \ldots b_{n-1} \leqslant b$

and by proposition 4.2, the sequent

$a \leqslant a_1, \ldots, a_i \leqslant c, c \leqslant b_{j+1}, \ldots b_{n-1} \leqslant b \Rightarrow a \leqslant b$

is derivable in partial order. (A second construction of a chain is possible; note also that as a limiting case, one premiss can be a logical axiom). Each atom in the antecedent is an atom in Γ so also $\Gamma \Rightarrow P$ is derivable in partial order.

For a derivation with an arbitrary number of instances of *Lin*, the above construction is repeated. QED.

The addition of rules *Ndeg* and *Ntriv* does not change much in the above picture. If $1 \leqslant 0$ is an atom removed by *Lin*, one premiss is an instance of

Ndeg with height of derivation 1 and the other is of the form

$$a \leqslant a_1, \ldots, a_i \leqslant 0, 0 \leqslant 1, 1 \leqslant a_{i+1}, \ldots a_n \leqslant b \Rightarrow a \leqslant b$$

The conclusion is instead obtained from the right premiss by rule *Ntriv* in partial order already. In other cases instances of *Ntriv* commute last and the construction of the chain in theorem 5.1 carries over to nondegenerate nontrivial linear order, so we have:

Theorem 5.2 *If $\Gamma \Rightarrow P$ is derivable in nondegenerate* **GLO**, *it is derivable in nondegenerate nontrivial* **GPO** *already.*

Note that a nondegenerate linear order is always nontrivial. Conservativity theorems can be regarded as the proof-theoretical counterparts of extension theorems. We observe that from our conservativity result one can obtain an extension algorithm:

Definition 5.3. *An ordering Σ is* inconsistent *if $\Gamma \Rightarrow 1 \leqslant 0$ is derivable for some finite subset Γ of Σ, otherwise it is* consistent.

Corollary 5.4. Szpilrajn's theorem. *Given a set Σ of atoms in a consistent nondegenerate partial ordering, it can be extended to a consistent nondegenerate linear ordering.*

Proof: Let a, b be any two elements in Σ not ordered in Σ. We claim that either $\Sigma, a \leqslant b$ or $\Sigma, b \leqslant a$ is consistent in **GPO**. Let us assume the contrary, i.e., that there exists a finite subset Γ of Σ such that both $\Gamma, a \leqslant b \Rightarrow 1 \leqslant 0$ and $\Gamma, b \leqslant a \Rightarrow 1 \leqslant 0$ are derivable in **GPO**. Application of rule *Lin* gives the conclusion $\Gamma \Rightarrow 1 \leqslant 0$ in **GLO**. By theorem 5.1, $\Gamma \Rightarrow 1 \leqslant 0$ is already derivable in **GPO**, contrary to the consistency assumption. Iteration of the procedure gives the desired extension. QED.

Whereas the proofs of the conservativity theorems 5.1 and 5.2 are constructive, the extension given by corollary 5.4 is not constructive in general, and effectivity depends on how the set Σ is given. This is a general phenomenon when classical set-theoretic extension results are reformulated as constructive proof-theoretical conservativity results. An example is the pointfree Hahn-Banach theorem of Cederquist, Coquand, and Negri (1998). In the case of Szpilrajn's theorem, classical proofs use non-constructive principles such as Zorn's lemma.

The set-theoretical notion of partial order is:

1. An arbitrary set D of any cardinality,
2. A function $f : D \times D \to \{0, 1\}$ such that
   (a) $f(x, x) = 1$,
   (b) If $f(x, y) = 1$ and $f(y, z) = 1$, then $f(x, z) = 1$.

The order relation is defined by
$x \leqslant y \equiv f(x, y) = 1.$

In a proof-theoretical, formal approach, there is assumed to be a formal language with individual parameters $a, b, c, \ldots$, variables $x, y, z, \ldots$ and a two-place propositional function $\leqslant$ over the terms. There need be no general definition of how $a \leqslant b$ can be proved. We only consider derivability in the pure theory of partial order, expressed by a sequent $\Gamma \Rightarrow \Delta$ that is the formal counterpart of the provability relation. Here Γ and Δ are finite (multi)sets of atomic formulas. Theorems about derivability such as our theorem 5.1 are proof-theoretical algorithms for the effective transformation of formal derivations.

Computational approaches to order relations assume, as a matter of course, that the basic order relation is decidable. If this assumption is not explicit, it is brought in implicitly through the application of the law of excluded middle. Our proof-theoretical approach does not impose any such requirement and therefore does not rule out a computational approach to order relations in continuous sets. The law of excluded middle is avoided by considering extensions of the intuitionistic calculus **G3im** instead of the classical one. It now happens that the rules of implication do not permute down with mathematical rules if these latter have at least two premisses. However, in the case of Harrop theories, such as partial order or lattice theory, logical rules do permute down and derivations with mathematical rules can be considered in isolation.

6. Proof search and decidability of partial and linear order

For purposes of *proof search* we look at the properties of derivations with mathematical rules from the conclusion upwards. A rule such as *Ref* in the system of left rules of partial order has the property that the reflexivity atom and its term is removed. Thus, we could try to find a derivation root first by *Ref* in as many ways as there are individuals in our set. It turns out, however, that in many theories the terms appearing in a derivation can be restricted to *terms in the conclusion.*

Definition 6.1. Subterm property. *A derivation of* $\Gamma \Rightarrow \Delta$ *in a system of mathematical rules has the* subterm property *if all terms in the derivation are terms in* Γ, Δ.

Theorem 6.2. *A minimum-size derivation of* $\Gamma \Rightarrow \Delta$ *in* **GPO** *has the subterm property.*

Proof: We observe that rule *Trans* maintains the subterm property. If rule *Ref* was used, we have by Proposition 4.2 a reflexivity derivation and the reflexivity atom and its term appears in the succedent. QED.

The decision method described after Proposition 4.2 can be put in terms of terminating proof search: Is there an instance of *Ref* concluding $\Gamma \Rightarrow \Delta$

in one step? If not, transitivity is applied root-first until no new atoms are produced. If the topsequent is a logical axiom, a derivation was found.

To prove the subterm property for the theory of linear order, we formulate it as a theory with right rules, with a rule of reflexivity added in order to contract the duplication arising from linearity when $a \equiv b$. The rules are

$$\frac{}{\Gamma \Rightarrow \Delta, a \leqslant b, b \leqslant a}{\scriptstyle Lin} \qquad \frac{}{\Gamma \Rightarrow \Delta, a \leqslant a}{\scriptstyle Ref}$$

$$\frac{\Gamma \Rightarrow \Delta, a \leqslant c, a \leqslant b, \quad \Gamma \Rightarrow \Delta, a \leqslant c, b \leqslant c}{\Gamma \Rightarrow \Delta, a \leqslant c}{\scriptstyle Trans}$$

Term b in rule *Trans* is called a *middle term.*

Theorem 6.3. *All terms in a minimum-size derivation of* $\Gamma \Rightarrow \Delta$ *in the right theory of linear order are terms in* Γ, Δ.

Proof: We first show that rule *Ref* need not be considered: If a topsequent is an instance of *Ref*, the first step must be a step of *Trans* removing a reflexivity atom $a \leqslant a$. The derivation has the form

$$\frac{\Gamma \Rightarrow \Delta, a \leqslant c, a \leqslant a, \quad \Gamma \Rightarrow \Delta, a \leqslant c, a \leqslant c}{\Gamma \Rightarrow \Delta, a \leqslant c}{\scriptstyle Trans}$$

The conclusion follows from the right premiss by height-preserving contraction, contrary to the assumption of a shortest derivation. Thus, proper derivations start with logical axioms or instances of *Lin*, followed by instances of *Trans.*

Let b be a first middle term from top that disappears from the derivation in a step of transitivity, and we may assume this to be the last step. We show that the derivation can be shortened. We have the instance

$$\frac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, a \leqslant c, a \leqslant b,\end{array} \quad \begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, a \leqslant c, b \leqslant c\end{array}}{\Gamma \Rightarrow \Delta, a \leqslant c}{\scriptstyle Trans}$$

If $a \leqslant b$ is never activated in the rightmost branch of the derivation leading to the left premiss, it can be deleted and the derivation shortened. Tracing up along the right branch from $a \leqslant b$, we find a removed atom $d \leqslant b$, and we continue tracing the atoms removed in steps of transitivity having the previously traced atom as an activated atom, until we arrive at an atom $e \leqslant b$ in a topsequent. In the case that $e \leqslant b$ is not an atom that makes the topsequent a logical axiom or an instance of *Lin*, it is deleted together with the step of *Trans* removing it. In the case of a logical axiom, the term b appears in the antecedent. In the case of *Lin*, the topsequent is of the form

$$\Gamma \Rightarrow \Delta', e \leqslant b, b \leqslant e$$

and there must be a step removing $b \leqslant e$. Because a right branch was followed, there is a step with a removed atom $f \leqslant b$ in a left premiss:

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta'', f \leqslant e, f \leqslant b,\end{array} \quad \begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta'', f \leqslant e, b \leqslant e\end{array}}{\begin{array}{c}\Gamma \Rightarrow \Delta'', f \leqslant e\\ \vdots\end{array}}{\scriptstyle Trans}$$

Tracing $f \leqslant b$ up the rightmost branch in the same way as $a \leqslant b$, we find a topsequent with an atom $g \leqslant b$. Now an argument as for the atom $e \leqslant b$ applies, and at some finite stage we find that such an atom either makes the topsequent a logical axiom and b appears in the antecedent or it is not needed in *Lin*. In the latter case it can be deleted and the derivation shortened. QED.

Corollary 6.4. *The quantifier-free theory of linear order is decidable.*

Proof: Application of rule *Trans* root first without the use of new terms can produce only a bounded number of distinct atoms in the premisses. Whenever a duplication is produced, proof search fails by the admissibility of contraction. QED.

The subterm property is provable also for the left rule system of linear order but the proof is not easy. We consider the difficulty of the decision problem for the left rule system. By theorems 5.1 and 5.2, a proper use of the linearity rule needs derivations of premisses with different atoms rendering the respective topsequents logical axioms. The simplest example is

$$\dfrac{a \leqslant b \Rightarrow a \leqslant b, b \leqslant a \quad b \leqslant a \Rightarrow a \leqslant b, b \leqslant a}{\Rightarrow a \leqslant b, b \leqslant a}{\scriptstyle Lin}$$

In general, if there are $n+1$ distinct atoms in the succedent, a minimum-size derivation has not more than n instances of linearity on top of which there are linear sequences of instances of transitivity. In the other direction, if there are n instances of *Lin* in a derivation, the succedent contains in the general case at least $n+1$ atoms or else some step of linearity is superfluous.

The first-order theory of linear order is decidable by a result announced in Ehrenfeucht (1959). The proof has not been published but it uses model-theoretic methods. Läuchli and Leonard (1966) prove decidability by showing that the set of nontheorems is recursively enumerable, thus, with no upper bound on proof search.

The decidability of linear order was posed as a problem in Janiczak (1953). Kreisel's (1954) review of this article contains a proof of decidability based on the standard method of reduction to monadic predicate calculus. However, Janiczak had died in 1951 and Kreisel's proof has gone, to our knowledge, unnoticed in the literature.

It has been written that "the decision problem of any formalized theory is as least as complex as the decision problem of the propositional calculus" (Rabin 1977, p. 600). By our method, the logical and mathematical parts of derivations can be completely separated. It seems useful to consider in isolation the intrinsic complexity of inferences with the postulates of a mathematical theory, and to compare such complexities, without the contribution of logical inferences.

Our corollary 6.4 proves decidability only for the universal fragment. In what way the general decision problem reduces to that for the universal fragment is a question we plan to study in a more general work that relates the decision problem of predicate logic to general properties of systems of mathematical rules, such as the subterm property.

7. An alternative approach

The purpose of this section is to give an alternative proof of the proof-theoretical version of Szpilrajn's theorem (Theorem 5.1) and of the decidability of the theory of linear order, by using the notion of an entailment relation of Scott (1974) and some results of Cederquist and Coquand (2000).

An *entailment relation* is a set S with a binary relation $\vdash$ between finite subsets of S such that

$$(I) \qquad s \vdash s$$

$$(M) \qquad \frac{X' \supseteq X \qquad X \vdash Y \qquad Y \subseteq Y'}{X' \vdash Y'}$$

$$(C) \qquad \frac{X \vdash s, Y \qquad s, X \vdash Y}{X \vdash Y}$$

Here we write X, Y for $X \cup Y$ and X, s for $X \cup \{s\}$.

An *interpretation* of an entailment relation is a map $m : S \to D$ from S into a distributive lattice D such that for every finite X, Y

$$X \vdash Y \Rightarrow \wedge_{x \in X} m(x) \leqslant \vee_{y \in Y} m(y)$$

A distributive lattice $\mathsf{L}(S)$ is *generated* by $(S, \vdash)$ with interpretation $m_0 : S \to \mathsf{L}(S)$ if it is universal:

$$\begin{tikzcd} S \arrow[r, "m"] \arrow[d, "m_0"'] & D \\ \mathsf{L}(S) \arrow[ur, "\exists! f"'] & \end{tikzcd}$$

The following result can be seen as an abstract form of cut-elimination.

Theorem 7.1. *Every entailment relation* $(S, \vdash)$ *generates a distributive lattice* $\mathsf{L}(S)$ *such that*

$$X \vdash Y \Leftrightarrow \wedge_{x \in X} m_0(x) \leqslant \vee_{y \in Y} m_0(y)$$

We define now an entailment relation associated to the theory of linear order. We let S be the set of atomic formulas $a \leqslant b$, and we define $X \vdash Y$ as: there exists a cycle

$$a_0, \ldots, a_n = a_0$$

where for each $i < n$, we have $a_i \leqslant a_{i+1} \in X$ or $a_{i+1} \leqslant a_i \in Y$, and at least one element of Y appears.

Theorem 7.2. *The relation* $\vdash$ *is an entailment relation. Furthermore, it is the least entailment relation on* S *such that*

- $\vdash a \leqslant a$,
- $a \leqslant b, b \leqslant c \vdash a \leqslant c$,
- $\vdash a \leqslant b, b \leqslant a$.

Proof: The only thing that is not direct is that $\vdash$ satisfies the cut-rule (C). Let us assume $X, a \leqslant b \vdash Y$ and $X \vdash a \leqslant b, Y$. We get two corresponding cycles. If one of them does not mention $a \leqslant b$ it is a witness that $X \vdash Y$. If both cycles mention $a \leqslant b$, we can glue them together along the pair a, b and we get a new cycle that is a witness for $X \vdash Y$. QED.

To look for such a cycle provides then an algorithm to see if $X \vdash Y$ is derivable in the theory of linear order. Notice that if Y is a singleton $a \leqslant b$, to have such a cycle is equivalent to having $a = b$ or a chain

$$a = a_0, \ldots, a_{n-1} = b$$

such that $a_i \leqslant a_{i+1} \in X$. Hence in this case $X \vdash Y$ is derivable without the linearity axiom, which gives us the proof-theoretic version of Szpilrajn's theorem.

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