Structural proof analysis for mathematical theories

Axiom-as-rules:
Formulate axioms as inference rules extending **G3** calculi .

Regular rule scheme:
Only atomic formulas as active and principal.
Logical content of the axiom embodied in the geometry of the rule.
Arbitrary context in the succedent.

$$\frac{Q_1, \Gamma \Rightarrow \Delta \quad \dots \quad Q_n, \Gamma \Rightarrow \Delta}{P_1, \dots, P_m, \Gamma, \Rightarrow \Delta} \mathit{Reg}$$

Classical logic: All quantifier-free axioms can be formulated as rules following the regular rule-scheme.

Intuitionistic logic: All *regular* axioms can be formulated as rules following the regular rule-scheme.

Handling contraction

Principal formulas repeated in the premisses

$$\frac{Q_1, P_1, \ldots, P_m, \Gamma \Rightarrow \Delta \ \ \ldots \ \ Q_n, P_1, \ldots, P_m, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_m, \Gamma, \Rightarrow \Delta} Reg$$

Closure condition: Given a system with non-logical rules, if it has a rule where a substitution instance in the atoms produces a rule of form

$$\frac{Q_1, P_1, \ldots, P_{m-2}, P, P, \Gamma \Rightarrow \Delta \ldots Q_n, P_1, \ldots, P_{m-2}, P, P, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_{m-2}, P, P, \Gamma \Rightarrow \Delta}$$

then it also has to contain the rule

$$\frac{Q_1, P_1, \ldots, P_{m-2}, P, \Gamma \Rightarrow \Delta \ldots Q_n, P_1, \ldots, P_{m-2}, P, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_{m-2}, P, \Gamma \Rightarrow \Delta}$$

Results

Theorem:

All the structural rules (weakening, contraction, and cut) are admissible in the extensions of the G3 calculi with rules following the regular rule scheme.

Weakening and contraction are admissible and *height-preserving*.

Geometric theories

Definition:
A formula in the language of first-order logic is called *geometric* if it does not contain $\supset$ or $\forall$.

A *geometric implication* is a sentence of the form

$$\forall \overline{x}(A \supset B)$$

where A and B are geometric formulas.

A *geometric theory* is a theory axiomatized by geometric implications.

Canonical form for geometric implications:
Any geometric implication can be reduced to a conjunction of formulas of the form

$$\forall \overline{x}(P_1 \& \ldots \& P_m \supset \exists y_1 M_1 \vee \ldots \vee \exists y_n M_n) \qquad GA$$

where all the P_i are atomic formulas and all the M_j are conjunctions of atomic formulas and the variables y_j are not free in the P_i.

The geometric rule scheme

$$\frac{\overline{Q}_1(y_1/x_1), \overline{P}, \Gamma \Rightarrow \Delta \quad \ldots \quad \overline{Q}_n(y_n/x_n), \overline{P}, \Gamma \Rightarrow \Delta}{\overline{P}, \Gamma \Rightarrow \Delta} \; GRS$$

The variables x_i are called the *replaced variables* of the scheme, and the variables y_i the *proper variables*

Condition: The proper variables are not free in $\overline{P}, \Gamma, \Delta$.

Elimination of structural rules

Definition: Let T be a geometric theory and let **G3cT** (**G3imT**) be the Gentzen system obtained by adding to **G3c**(**G3im**) the geometric rule-schemes corresponding to the geometric axioms of T, together with the rules arising from the closure condition.

Equivalence of axiomatic and rule system: A geometric axiom is derivable from the corresponding geometric rule-scheme.

Conversely, a geometric rule is derivable from the corresponding geometric axiom in **G3imT**+*contr*+*cut*.

Inversion lemmas: All the inversion lemmas for the *propositional* rules that hold for **G3c** and **G3im** hold for their geometric extension as well.

Substitution lemma: Given a derivation of $\Gamma \Rightarrow \Delta$ in **G3cT** (**G3imT**, resp.), with x a free variable in Γ, Δ and a term t free for x in Γ, Δ not containing any of the variables of the geometric rules in the derivation, we can find a derivation of $\Gamma(t/x) \Rightarrow \Delta(t/x)$ in **G3cT** (**G3imT**, resp.) with the same height.

Inversion lemma for quantifier rules.

(i) If $\vdash_n \exists x A, \Gamma \Rightarrow \Delta$ and y is not among the variables of the geometric rules in the derivation, then $\vdash_n A(y/x), \Gamma \Rightarrow \Delta$.

(ii) If $\vdash_n \Gamma \Rightarrow \Delta, \forall x A$ and y is not among the variables of the geometric rules in the derivation, then $\vdash_n \Gamma \Rightarrow \Delta, A(y/x)$.

Disjointness condition. In a derivation in **G3cT** (**G3imT**) the sets of proper variables of the geometric rules are pairwise disjoint.

Theorem: The rules of weakening

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} RW$$

are admissible and height preserving in **G3cT** and in **G3imT**.

Theorem: The rules of contraction

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} LC \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A} RC$$

are admissible and height-preserving in **G3cT** and in **G3imT**.

Theorem: The cut rule

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} Cut$$

is admissible in **G3cT** and in **G3imT**.

Examples of geometric theories

- Robinson's arithmetic
- Ordered fields, Real closed fields
- Constructive projective geometry

But... Classical projective geometry is not geometric

a. Robinson arithmetic

Language: constant 0, unary successor function s, binary functions $+$ and $\cdot$, relation $=$. Atomic formulas of the form $a = b$, for arbitrary terms a and b.

1. $\neg\ s(x) = 0$
2. $s(x) = s(y) \supset x = y$
3. $x = 0 \vee \exists y\ x = s(y)$
4. $x + 0 = x$
5. $x + s(y) = s(x + y)$
6. $x \cdot 0 = 0$
7. $x \cdot s(y) = x \cdot y + x$

The classically equivalent axiomatization with *3.* replaced by

$3'$. $\neg\ x = 0 \supset \exists y\ x = s(y)$

is not geometric because it has an implication ($x = 0 \supset \perp$) in the antecedent of an implication.

b. Ordered fields

I. Axioms for nondegenerate linear order

1. $x \leqslant x$
2. $x \leqslant y \vee y \leqslant x$
3. $x \leqslant y \ \& \ y \leqslant z \supset x \leqslant z$
4. $\neg\ 1 \leqslant 0$

II. Axioms for ordered additive group

5. $(x + y) + z = x + (y + z)$
6. $x + y = y + x$
7. $x + 0 = x$
8. $\exists y\ x + y = 0$
9. $x \leqslant y \supset x + z \leqslant y + z$

III. Axioms for multiplication

10. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
11. $x \cdot y = y \cdot y$
12. $x \cdot 1 = x$
13. $x = 0 \vee \exists y\ x \cdot y = 1$
14. $x \cdot (y + z) = x \cdot y + x \cdot z$
15. $x \leqslant y \ \& \ 0 \leqslant z \supset x \cdot z \leqslant y \cdot z$

The classically equivalent axiomatization with

$$13'.\ \neg\ x = 0 \supset \exists y\ x \cdot y = 1$$

in place of *13.* is not geometric.

Real closed fields: Add the axioms stating the existence of square roots and zeroes of polynomials of odd degree

16. $0 \leqslant x \supset \exists y\ x = y \cdot y$
17. $a_{2n+1} = 0 \vee \exists x\ a_{2n+1} \cdot x^{2n+1} + a_{2n} \cdot x^{2n} + \ldots a_1 \cdot x + a_0 = 0$

The classically equivalent axiomatization with *17* replaced by

$17'$. $\neg\ a_{2n+1} = 0 \supset \exists x\ a_{2n+1} \cdot x^{2n+1} + a_{2n} \cdot x^{2n} + \ldots a_1 \cdot x + a_0 = 0$

is not geometric.

Classical projective geometry with constructions: Basic concepts equality and incidence: $a = b$, $l = m$, $a \in l$. **Constructions:** $ln(a, b)$, $pt(l, m)$.

I. **Axioms for equivalence relations**

$$a = a, \quad a = c \,\&\, b = c \supset a = b$$
$$l = l, \quad l = n \,\&\, m = n \supset l = m$$

II. **Axioms of incidence:**

$$a = b \vee a \in ln(a, b), \qquad a = b \vee b \in ln(a, b)$$
$$l = m \vee pt(l, m) \in l, \qquad l = m \vee pt(l, m) \in m$$

III. **Uniqueness axiom:**

$$a \in l \,\&\, b \in l \,\&\, a \in m \,\&\, b \in m \supset a = b \vee l = m$$

IV. **Substitution axioms:**

$$a \in l \,\&\, a = b \supset b \in l$$
$$a \in l \,\&\, l = m \supset a \in m$$

V. **Existence of noncollinear points:**

$$\exists x \exists y \exists z (\sim x = y \,\&\, \sim z \in ln(x, y))$$

Not geometric.

Constructive projective geometry with constructions: Basic concepts: $a \neq b, \quad l \neq m, \quad a \notin l$
Constructions: $ln(a, b), \quad pt(l, m)$

I. **Axioms for apartness relations:**

$$a = a \supset \bot, \quad a \neq b \supset a \neq c \vee b = c$$
$$l = l \supset \bot, \quad l \neq m \supset l \neq n \vee m = n$$

II. **Axioms of incidence:**

$$a \neq b \;\&\; a \notin ln(a, b) \supset \bot$$
$$a \neq b \;\&\; b \notin ln(a, b) \supset \bot$$
$$l \neq m \;\&\; pt(l, m) \notin l \supset \bot$$
$$l \neq m \;\&\; pt(l, m) \notin m \supset \bot$$

III. **Uniqueness axiom:**

$$a \neq b \;\&\; l \neq m \supset a \notin l \vee b \notin l \vee a \notin m \vee b \notin m$$

IV. **Substitution axioms:**

$$a \notin l \supset a \neq b \vee b \notin l$$
$$a \notin l \supset l \neq m \vee a \notin m$$

V. **Existence of noncollinear points:**

$$\exists x \exists y \exists z (x \neq y \;\&\; z \notin ln(x, y))$$

Geometric.

Barr's theorem

Topos Theory: For any Grothendieck topos $\mathbf{E}$, there is a Boolean topos $\mathbf{B}$ and a geometric morphism $\gamma : \mathbf{B} \rightarrow \mathbf{E}$ such that γ^* is faithful (cf. Johnstone (1977), Mac Lane and Moerdijk (1992)).

Logic: If a geometric implication is classically derivable in a geometric theory, then it is intuitionistically derivable.

Palmgren (1998) proved the result using the Dragalin-Friedman translation.

Our statement: Let T be a geometric theory, and let A be a geometric implication. If $\mathbf{G3cT} \vdash \Rightarrow A$, *then* $\mathbf{G3imT} \vdash \Rightarrow A$.

Our proof: Nothing to prove!

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