Four approaches to extension by axioms

1. Addition of axioms A into sequent calculus in the form of sequents $\Rightarrow A$ by which derivations can start. Leads to failure of cut elimination.

2. Gentzen (1938, sec. 1.4): add "mathematical basic sequents"

$$P_1, \ldots, P_m \Rightarrow Q_1, \ldots, Q_n.$$

By Gentzen's "Hauptsatz," the cut rule can be pushed into such basic sequents.

3. Gentzen's consistency proof of elementary arithmetic (1934, sec. IV.3): treat axioms as a context Γ, and prove results of form $\Gamma \Rightarrow C$. Cut elimination applies.

4. Axioms as rules.

All these approaches are equivalent, but the fourth is the best suited for proof analysis.

$\mathcal{D}$ finite set of regular formulas.

Definition

(a) A-system *for* $\mathcal{D}$*: sequent system with* axioms, **G3ipm**+LW+RW+ LC+RC+Cut+A$\mathcal{D}$; *$A\mathcal{D}$ set of sequents* $\Rightarrow D$ *from* D *in* $\mathcal{D}$*. Derivability of a sequent* $\Gamma \Rightarrow \Delta$ *in an* A*-system where sequents from* $A\mathcal{D}$ *may appear as premisses denoted by* $\vdash_{A\mathcal{D}} \Gamma \Rightarrow \Delta$.

(b) B-system *for* $\mathcal{D}$*: sequent system with* basic sequents, **G3ipm**+LW+ RW+LC+RC+Cut+B$\mathcal{D}$, *where* $B\mathcal{D}$ *is the set of regular sequents corresponding to elements of* $\mathcal{D}$*. Derivability of a sequent* $\Gamma \Rightarrow \Delta$ *in a* B*-system, where sequents from* $B\mathcal{D}$ *may appear as premisses, denoted by* $\vdash_{B\mathcal{D}} \Gamma \Rightarrow \Delta$.

(c) C-system *for* $\mathcal{D}$*: sequent system with a* context. *Derivability of a sequent* $\Gamma \Rightarrow \Delta$ *in a* C*-system, where instances of formulas in* $C\mathcal{D}$ *appear in the antecedent, denoted by* $\vdash_{C\mathcal{D}} \Gamma \Rightarrow \Delta$.

(d) R-system *for $\mathcal{D}$: sequent system with* rules, **G3ipm**+R$\mathcal{D}$, *where $R\mathcal{D}$ is the set of rules of inference given by the regular decomposition of the formulas in $\mathcal{D}$. Derivability of a sequent* $\Gamma \Rightarrow \Delta$ *in an R-system, where rules from $R\mathcal{D}$ are permitted, denoted by* $\vdash_{R\mathcal{D}} \Gamma \Rightarrow \Delta$.

Theorem $\vdash_{A\mathcal{D}} \Gamma \Rightarrow \Delta$ *iff* $\vdash_{B\mathcal{D}} \Gamma \Rightarrow \Delta$ *iff* $\vdash_{C\mathcal{D}} \Gamma \Rightarrow \Delta$ *iff* $\vdash_{R\mathcal{D}} \Gamma \Rightarrow \Delta$.

Predicate logic with equality

Primitive relation $a = b$

$a = a$ (*reflexivity*)

$a = b \,\& A(a/x) \supset A(b/x)$ (*replacement scheme*)

In the usual treatment (B-system), add basic mathematical sequents

$$\Rightarrow a = a$$

$$a = b, P(a/x) \Rightarrow P(b/x)$$

By Gentzen's "extended Hauptsatz" cuts can be reduced to cuts on axioms, but cut elimination fails. For instance, there is no cut-free derivation of symmetry. (Weakening and) contraction must be assumed.

We restrict the replacement scheme to atomic predicates $P, Q, R, \ldots$ and convert the axioms into rules,

$$\frac{a = a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,{\scriptstyle Ref} \qquad \frac{a = b, P(a/x), P(b/x), \Gamma \Rightarrow \Delta}{a = b, P(a/x), \Gamma \Rightarrow \Delta}\,{\scriptstyle Repl}$$

G3im and **G3c** plus *Ref* and *Repl* give **intuitionistic and classical predicate logic with equality**.

Theorem *The rules of weakening, contraction and cut are admissible in predicate logic with equality.*

Lemma *The replacement axiom* $a = b, A(a/x) \Rightarrow A(b/x)$ *is derivable for arbitrary* A.

Theorem *The replacement rule*

$$\frac{a = b, A(a/x), A(b/x), \Gamma \Rightarrow \Delta}{a = b, A(a/x), \Gamma \Rightarrow \Delta}\,{\scriptstyle Repl}$$

is admissible for arbitrary predicates A.

Our cut- and contraction-free calculus is equivalent to the usual calculi. But the formulation of equality axioms as rules permits proofs by induction on height of derivation. The conservativity of predicate logic with equality over predicate logic illustrates such proofs.

To prove the conservativity, we show that *Ref* can be eliminated from the derivation.

Rule of replacement has an instance with a duplication. Closure condition is satisfied: instance where both duplications are contracted is an instance of reflexivity. For the proof of conservativity, the closure condition satisfied by adding the contracted instance of *Repl*:

$$\frac{a = b, b = b, \Gamma \Rightarrow \Delta}{a = b, \Gamma \Rightarrow \Delta}\, {\scriptstyle Repl^*}$$

Lemma *If* $\Gamma \Rightarrow \Delta$ *has no equalities and is derivable in* **G3c**+Ref+ Repl+Repl*, *no sequents in its derivation have equalities in the succedent.*

Lemma *If* $\Gamma \Rightarrow \Delta$ *has no equalities and is derivable in* **G3c**+Ref+ Repl+Repl* *it is derivable in* **G3c**+Repl+Repl*.

Proof: We show that all instances of *Ref* can be eliminated from a given derivation, by induction on the height of derivation on a topmost instance

$$\frac{a = a, \Gamma' \Rightarrow \Delta'}{\Gamma' \Rightarrow \Delta'}\,Ref$$

If the premiss is an axiom also the conclusion is, since by the above lemma Δ' contains no equality, and the same if it is a conclusion of $L\bot$. If the premiss has been concluded by a logical rule, apply the inductive hypothesis to the premisses and then the rule.

If the premiss has been concluded by *Repl* there are two cases, according to whether $a = a$ is or is not principal. In the latter case the derivation is, with $\Gamma' = P(b/x), \Gamma''$,

$$\dfrac{\dfrac{a = a, b = c, P(b/x), P(c/x), \Gamma'' \Rightarrow \Delta'}{a = a, b = c, P(b/x), \Gamma'' \Rightarrow \Delta'}\,{\scriptstyle Repl}}{b = c, P(b/x), \Gamma'' \Rightarrow \Delta'}\,{\scriptstyle Ref}$$

By permuting the two rules, the inductive hypothesis can be applied.

If $a = a$ is principal, the derivation is, with $\Gamma' = P(a/x), \Gamma''$,

$$\dfrac{\dfrac{a = a, P(a/x), P(a/x), \Gamma'' \Rightarrow \Delta'}{a = a, P(a/x), \Gamma'' \Rightarrow \Delta'}\,{\scriptstyle Repl}}{P(a/x), \Gamma'' \Rightarrow \Delta'}\,{\scriptstyle Ref}$$

By height-preserving contraction, there is a derivation of $a = a, P(a/x), \Gamma'' \Rightarrow \Delta'$ to which the inductive hypothesis applies, giving a derivation of $\Gamma' \Rightarrow \Delta'$ without rule *Ref*.

If the premiss of *Ref* has been concluded by *Repl**
with $a = a$ not principal the derivation is

$$\cfrac{\cfrac{a = a, b = c, c = c, \Gamma' \Rightarrow \Delta'}{a = a, b = c, \Gamma' \Rightarrow \Delta'}\,\mathit{Repl}^*}{b = c, \Gamma'' \Rightarrow \Delta'}\,\mathit{Ref}$$

The rules are permuted and the inductive hypothesis applied.

If $a = a$ is principal the derivation is

$$\cfrac{\cfrac{a = a, a = a, \Gamma' \Rightarrow \Delta'}{a = a, \Gamma' \Rightarrow \Delta'}\,\mathit{Repl}^*}{\Gamma' \Rightarrow \Delta'}\,\mathit{Ref}$$

and we apply height-preserving contraction and the inductive hypothesis. QED.

Theorem *If* $\Gamma \Rightarrow \Delta$ *is derivable in* **G3c**+Ref+Repl+Repl* *and if* Γ, Δ *contain no equality, then* $\Gamma \Rightarrow \Delta$ *is derivable in* **G3c**.

Herbrand's theorem for universal theories T theory with a finite number of purely universal axioms and classical logic. Turn the theory **T** into a system of nonlogical rules: first remove the quantifiers from each axiom, then convert the remaining part into nonlogical rules. The resulting system will be denoted **G3cT**.

Theorem: Herbrand's theorem: *If the sequent $\Rightarrow \forall x \exists y_1 \ldots \exists y_k A$, with A quantifier-free, is derivable in* **G3cT**, *then there are terms t_{i_j} with $i \leqslant n, j \leqslant k$ such that*

$$\bigvee_{i=1}^{n} A(t_{i_1}/y_1, \ldots, t_{i_k}/y_k)$$

is derivable in **G3cT**.

Proof: Suppose that $k = 1$. The derivation of $\Rightarrow \forall x \exists y A$ ends with

$$\cfrac{\cfrac{\Rightarrow A(z/x, t_1/y), \exists y A(z/x)}{\Rightarrow \exists y A(z/x)}\, R\exists}{\Rightarrow \forall x \exists y A}\, R\forall$$

Every sequent in the derivation is of the form

$\Gamma \Rightarrow \Delta, A(z/x, t_m/y), \ldots, A(z/x, t_{m+l}/y), \exists y A(z/x)$

where Γ, Δ consist of subformulas of $A(z/x, t_i/y)$, with $i < m$.

Consider the topsequents of the derivation. If they are axioms or conclusions of $L\perp$ they remain so after deletion of the formula $\exists y A(z/x)$. If they are conclusions of zero premiss nonlogical rules, they remain so after the deletion since the right context in these rules is arbitrary. After deletion, every topsequent in the derivation is of the form

$$\Gamma \Rightarrow \Delta, A(z/x, t_m/y), \ldots, A(z/x, t_{m+l}/y)$$

Making the propositional and nonlogical inferences as before, but without the formula $\exists y A(z/x)$ in the succedent, produces a derivation of

$$\Rightarrow A(z/x, t_1/y), \ldots, A(z/x, t_{m-1}/y),$$

$$A(z/x, t_m/y), \ldots, A(z/x, t_n/y)$$

and conclusion by applications of rule $R\vee$. QED.

If the theory **T** is empty we have

Corollary *If* $\Rightarrow \exists x A$ *is derivable in* **G3c**, *there are terms* $t_1, \ldots, t_n$ *such that* $\Rightarrow A(t_1/x) \vee \ldots \vee A(t_n/x)$ *is derivable.*

Theory of equality: Basic relation $a = b$ with the axioms

EQ1. $a = a$,

EQ2. $a = b \;\&\; a = c \supset b = c$.

Symmetry of equality follows by substituting a for c in EQ2. Transitivity is directly an instance of the replacement axiom, with A equal to $x = c$.

Addition of the rules

$$\frac{a = a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,{\scriptstyle Ref} \qquad \frac{b = c, a = b, a = c, \Gamma \Rightarrow \Delta}{a = b, a = c, \Gamma \Rightarrow \Delta}\,{\scriptstyle Trans}$$

gives a calculus **G3im**+Ref+Trans the rules of which follow the rule-scheme. A duplication in *Trans* is produced in case b is identical to c, but the corresponding contracted rule is an instance of rule *Ref*. The closure condition is satisfied and the structural rules admissible.

Occurences of atoms of the form $a = a$ disappear in *Ref*, however...

Theorem (subterm property) *All terms in a minimum size derivation of* $\Gamma \Rightarrow \Delta$ *in* **G3EQ** *are terms in the conclusion.*

Proof: Consider an atom $a = a$ active in *Ref* and trace it along the derivation. If it is never principal, it can be removed altogether, and *Ref* as well, with subsequent shortening of the derivation. If it is principal in an axiom, a is found in Δ. If if is principal in *Trans*, it has the form

$$\frac{a = c, a = b, a = c, \Gamma \Rightarrow \Delta}{a = a, a = c, \Gamma \Rightarrow \Delta}\,{\scriptstyle Trans}$$

Apply height-preserving contraction to the premiss and get a shorter derivation. QED

Theory of apartness: Basic relation $a \neq b$ with the axioms

AP1. $\sim a \neq a$,

AP2. $a \neq b \supset a \neq c \ \vee \ b \neq c$.

The rules are

$$\frac{}{a \neq a, \Gamma \Rightarrow \Delta}\,\textit{Irref}$$

$$\frac{a \neq c, a \neq b, \Gamma \Rightarrow \Delta \quad b \neq c, a \neq b, \Gamma \Rightarrow \Delta}{a \neq b, \Gamma \Rightarrow \Delta}\,\textit{Split}$$

Both rules follow the rule-scheme, the closure condition does not arise because there is only one principal formula, and therefore structural rules are admissible in **G3im**+Irref+Split.

Exercise: Prove the subterm property for the theory of apartness. Hint: Replace AP2 by

$a \neq b \supset b \neq a$

$a \neq b \supset a \neq c \ \vee \ c \neq b$

and trace up in the derivation the term removed by *Split*.

The elementary theories of equality and apartness can also be given in a single succedent formulation based on extension of the calculus **G3i**, as in Negri (1999).

Corollary: Disjunction property for the theory of apartness. *If* $\Rightarrow A \vee B$ *is derivable in the single succedent calculus for the theory of apartness, either* $\Rightarrow A$ *or* $\Rightarrow B$ *is derivable.*

Proof: Consider the last rule in the derivation. The rules for apartness cannot conclude a sequent with an empty antecedent and therefore the last rule must be rule $R\vee$ of **G3i**. QED.

Compare to the treatment of axiom systems as a context: $\Gamma \Rightarrow A \vee B$ is derivable in **G3i**. Whenever Γ contains an instance of the "split" axiom it has a formula with a disjunction in the consequent of an implication. Therefore, Γ does not consist of Harrop formulas only, so the above corollary gives a proper extension of the disjunction property under hypotheses that are Harrop formulas.

Harrop systems

Definition

1. *Atoms* $P, Q, R, \ldots,$ *and* $\perp$ *are Harrop formulas*,

2. *If* A *and* B *are Harrop formulas, then* $A \& B$ *is a Harrop formula*,

3. *If* B *is a Harrop formula, then* $A \supset B$ *is a Harrop formula.*

Harrop theory: axioms consist of Harrop formulas.

Left Harrop system: system of mathematical rules obtained from the axioms of a Harrop theory by using the left rule-scheme.

The rules of a left Harrop system have *at most one* premiss, thus the derivations are linear, and therefore

Theorem *If a sequent* $\Gamma \Rightarrow \Delta$ *is derivable in a left Harrop system, then* $\Gamma \Rightarrow P$ *is derivable for some atom* P*. If* Δ *contains atoms, the atom* P *can be chosen from* Δ*.*

Proof: Consider a derivation $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$. If the topsequent is a logical axiom $P, \Gamma' \Rightarrow \Delta', P$, with $\Delta = \Delta', P$, the succedent can be changed into P. If the topsequent is a zero-premiss mathematical rule, any atom P can be put as the succedent and the derivation with the new succedent continued as with Δ. QED.

Partial order: The axioms of partial order are

PO1. $a \leqslant a$,

PO2. $a \leqslant b \,\&\, b \leqslant c \supset a \leqslant c$.

Equality is defined by $a = b \equiv a \leqslant b \,\&\, b \leqslant a$. (Thus, we are working with what are sometimes called quasiorderings.)

Equality is an equivalence relation and satisfies the principle of substitution of equals.

As a system with rules **GPO**

$$\frac{a \leqslant a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,{\scriptstyle Ref} \qquad \frac{a \leqslant c, a \leqslant b, b \leqslant c, \Gamma \Rightarrow \Delta}{a \leqslant b, b \leqslant c, \Gamma \Rightarrow \Delta}\,{\scriptstyle Trans}$$

Closure condition arises when $a \equiv b$ and $b \equiv c$ when the premiss of rule *Trans*

$$a \leqslant a, a \leqslant a, a \leqslant a, \Gamma \Rightarrow \Delta$$

Conclusion follows by rule *Ref* so that the closure condition is satisfied. Thus the rules of weakening, contraction, and cut are admissible in **GPO**. Weakening and contraction are admissible and height preserving.

Proof analysis GPO: There are exactly 2 kinds of derivation. To see what they are, assume that derivations contain no unnecessary detours: a derivation is of *minimum size* if no proper subtree of the given derivation is a derivation.

If $\Gamma \Rightarrow \Delta$ is derivable, the topsequent has the form $P, \Gamma' \Rightarrow \Delta', P$ with $\Delta', P = \Delta$, and we can delete Δ' from the topsequent.

The two kinds of derivations are:

1. Reflexivity derivations: $P \equiv a \leqslant a$.

The conclusion $\Gamma \Rightarrow a \leqslant a$ follows in one step from the logical axiom $a \leqslant a, \Gamma \Rightarrow a \leqslant a$ with one application of rule *Ref*:

$$\frac{a \leqslant a, \Gamma \Rightarrow a \leqslant a}{\Gamma \Rightarrow a \leqslant a}\,{\scriptstyle Ref}$$

The context Γ is superfluous and can be deleted, thus, the conclusion becomes $\Rightarrow a \leqslant a$.

2. Transitivity derivations: The topsequent is $a_1 \leqslant a_n, \Gamma' \Rightarrow a_1 \leqslant a_n$.

The atom $a_1 \leqslant a_n$ must be the *removed atom* in a first step of transitivity or else the derivation can be shortened: If some other atom P were removed, with $\Gamma' \equiv P, \Gamma''$, the derivation could be shortened by starting with $a_1 \leqslant a_n, \Gamma'' \Rightarrow a_1 \leqslant a_n$ as topsequent.

There cannot be steps of reflexivity in this derivation: The reflexivity atom would be principal in a step of transitivity, else it could be removed tout court from the derivation with a subsequent shortening, thus there would be a step of the form

$$\frac{a \leqslant b, a \leqslant a, a \leqslant b, \Gamma \Rightarrow a_1 \leqslant a_n}{a \leqslant a, a \leqslant b, \Gamma \Rightarrow a_1 \leqslant a_n}\,{\scriptstyle Trans}$$

By height-preserving admissibility of contraction the conclusion of this step could be obtained already from the premiss, without using transitivity, in one step less.

Two atoms $a_1 \leqslant a_2, a_2 \leqslant a_n$ are *activated* by the step of *Trans* removing $a_1 \leqslant a_n$ so that the topsequent is of the form

$$a_1 \leqslant a_n, a_1 \leqslant a_2, a_2 \leqslant a_n, \Gamma'' \Rightarrow a_1 \leqslant a_n$$

In the second step, one of the activated atoms must become the removed atom, with two new activated atoms, say $a_2 \leqslant a_3, a_3 \leqslant a_n$, or else the derivation can be shortened. The closure of the principal atom $a_1 \leqslant a_n$ with respect to the activation relation gives us a *chain* $a_1 \leqslant a_2, a_2 \leqslant a_3, \ldots a_{n-1} \leqslant a_n$ in the topsequent. Deleting the atoms that have not been active in the derivation, we have a derivation of the form

$$\cfrac{\cfrac{\Gamma''', a_1 \leqslant a_2, a_2 \leqslant a_3, \ldots a_{n-1} \leqslant a_n \Rightarrow a_1 \leqslant a_n}{\vdots}\,{\scriptstyle Trans}}{a_1 \leqslant a_2, a_2 \leqslant a_3, \ldots a_{n-1} \leqslant a_n \Rightarrow a_1 \leqslant a_n}\,{\scriptstyle Trans}$$

in which Γ''' consists of the removed atoms $a_1 \leqslant a_n, \ldots$.

Sequents $\Gamma \Rightarrow \Delta$ derivable in **GPO** *are derivable as left and right weakenings of reflexivity and transitivity derivations.*

Proof search for a sequent $\Gamma \Rightarrow \Delta$ is effected by two controls:

Does Δ contain a reflexivity atom?

Does Γ contain a chain from a_1 to a_n with the atom $a_1 \leqslant a_n$ in Δ?

If so, the sequent $\Gamma \Rightarrow \Delta$ is derivable, otherwise it is underivable.

Nondegenerate partial order is obtained by adding the axiom

PO3. $\sim 1 \leqslant 0$

to PO1 and PO2.

The corresponding rule has zero premisses:

$$\frac{}{1 \leqslant 0, \Gamma \Rightarrow \Delta}\,{}^{Ndeg}$$

Derivations remain linear and the theorem on Harrop systems applies.

If the topsequent is an instance of *Ndeg*, the atom $1 \leqslant 0$ is removed by *Trans* (it cannot be removed by *Ref*). Steps of *Trans* hide the inconsistent assumption $1 \leqslant 0$, with the general form of conclusion

$$1 \leqslant a_1, a_1 \leqslant a_2, \ldots, a_{n-1} \leqslant 0 \Rightarrow a \leqslant b$$

with the chain in the antecedent being the closure of formulas activated by $1 \leqslant 0$ and $a \leqslant b$ in the succedent an arbitrary atom.

Nontrivial partial order has in addition

PO4. $\quad 0 \leqslant 1.$

The corresponding rule is

$$\frac{0 \leqslant 1, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, Ntriv$$

This rule commutes down with instances of *Ref* and *Trans*. The only interesting case is a transitivity derivation with a chain from which atoms $0 \leqslant 1$ have been removed by *Ntriv*.

Linear order and Szpilrajn's theorem

Theory of linear order: adding to partial order the *linearity axiom*

LO. $a \leqslant b \vee b \leqslant a$

The corresponding rule is

$$\frac{a \leqslant b, \Gamma \Rightarrow \Delta \quad b \leqslant a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, {\scriptstyle Lin}$$

The system of rules for linear order is designated **GLO**.

Conservativity theorem *If* $\Gamma \Rightarrow P$ *is derivable in* **GLO**, *it is derivable in* **GPO** *already.*

Proof: Consider a derivation with just one instance of *Lin*, as the last rule, and assume the derivation to have minimum size. Thus, the premisses of *Lin* $c \leqslant d, \Gamma \Rightarrow P$ and $d \leqslant c, \Gamma \Rightarrow P$ are derivable in partial order. If P is a reflexivity atom in either topsequent, $\Gamma \Rightarrow P$ is derivable in one step of *Ref*. Otherwise, with $P \equiv a \leqslant b$, there will be two transitive closures of the removed atom $a \leqslant b$ in both derivations of the two premisses of *Lin*, and let them be $a \leqslant a_1, \ldots a_{m-1} \leqslant b$ and $a \leqslant b_1, \ldots b_{n-1} \leqslant b$. If $c \leqslant d$ is not an atom in the first chain, it can be deleted and a derivation of $\Gamma \Rightarrow P$ in partial order obtained, and similarly for $d \leqslant c$ in the second chain. Thus, we have the two chains

$$a \leqslant a_1, \ldots, a_i \leqslant c, c \leqslant d, d \leqslant a_{i+1}, \ldots a_{m-1} \leqslant b$$

$$a \leqslant b_1, \ldots, b_j \leqslant d, d \leqslant c, c \leqslant b_{j+1}, \ldots b_{n-1} \leqslant b$$

The chain

$$a \leqslant a_1, \ldots, a_i \leqslant c, c \leqslant b_{j+1}, \ldots b_{n-1} \leqslant b$$

appears in the antecedent of the topsequent, thus the sequent

$$a \leqslant a_1, \ldots, a_i \leqslant c, c \leqslant b_{j+1}, \ldots b_{n-1} \leqslant b \Rightarrow a \leqslant b$$

is derivable in partial order. QED.

The conservativity theorem extends to nondegenerate nontrivial partial order:

Theorem *If* $\Gamma \Rightarrow P$ *is derivable in nondegenerate* **GLO**, *it is derivable in nondegenerate nontrivial* **GPO** *already.*

(Note that a nondegenerate linear order is always nontrivial.)

Extension algorithm:

Definition *An ordering* Σ *is* inconsistent *if* $\Gamma \Rightarrow 1 \leqslant 0$ *is derivable for some finite subset* Γ *of* Σ, *otherwise it is* consistent.

Corollary: Szpilrajn's theorem. *Given a set* Σ *of atoms in a consistent nondegenerate partial ordering, it can be extended to a consistent nondegenerate linear ordering.*

Proof: Let a, b be any two elements in Σ not ordered in Σ. We claim that either $\Sigma, a \leqslant b$ or $\Sigma, b \leqslant a$ is consistent in **GPO**. Let us assume the contrary, i.e., that there exists a finite subset Γ of Σ such that both $\Gamma, a \leqslant b \Rightarrow 1 \leqslant 0$ and $\Gamma, b \leqslant a \Rightarrow 1 \leqslant 0$ are derivable in **GPO**. Application of rule *Lin* gives the conclusion $\Gamma \Rightarrow 1 \leqslant 0$ in **GLO**. By the conservativity theorem, $\Gamma \Rightarrow 1 \leqslant 0$ is already derivable in **GPO**, contrary to the consistency assumption. Iteration of the procedure gives the desired extension. QED.

Constructive conservativity vs nonconstructive extension

Proof of the conservativity theorem is constructive, effectivity of the extension depends on how the set Σ is given.

General phenomenon:

Classical, nonconstructive set-theoretic extension results are reformulated as constructive proof-theoretical conservativity results.

Examples:

Pointfree constructive Hahn-Banach theorem (Cederquist, Coquand, and Negri (1998)) vs. classical nonconstructive Hahn-Banach theorem .

Constructive conservativity of linar order over partial order vs classical Szpilrajn's theorem.

Classical proofs use non-constructive principles like Zorn's lemma.

Decidability of order relation often assumed, either explicitly or through the application of the law of excluded middle.

Our approach does not impose any such requirement and therefore does not rule out a computational approach to order relations in continuous sets.

The law of excluded middle is avoided by considering extensions of the intuitionistic calculus **G3im** instead of the classical one.

Proviso: The rules of implication do not permute down with mathematical rules if these latter have at least two premisses. In the case of Harrop theories, such as partial order or lattice theory, logical rules do permute down and derivations with mathematical rules can be considered in isolation.

The right rule scheme

The *left rule-scheme* for an axiom of the form

$$P_1 \& \ldots \& P_m \supset Q_1 \vee \ldots \vee Q_n$$

$$\frac{Q_1, P_1, \ldots, P_m, \Gamma \Rightarrow \Delta \quad \ldots \quad Q_n, P_1, \ldots, P_m, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_m, \Gamma \Rightarrow \Delta}$$

has a dual formulation as a *right rule-scheme:*

$$\frac{\Gamma \Rightarrow \Delta, Q_1, \ldots, Q_n, P_1 \quad \ldots \quad \Gamma \Rightarrow \Delta, Q_1, \ldots, Q_n, P_m}{\Gamma \Rightarrow \Delta, Q_1, \ldots, Q_n}$$

Repetition of the atoms Q_i in the premisses to obtain admissibility of right contraction.

Closure condition: If the atoms in a rule have an instance that makes two atoms in the conclusion identical, contracted rule has to be added.

Theorem *The structural rules of left and right weakening and contraction and the rule of cut are admissible in extensions of* **G3c** *and* **G3im** *with rules following the right rule-scheme and satisfying the closure condition.*

Right rules for linear order

$$\frac{}{\Gamma \Rightarrow \Delta, a \leqslant b, b \leqslant a}\,{\scriptstyle Lin} \qquad \frac{}{\Gamma \Rightarrow \Delta, a \leqslant a}\,{\scriptstyle Ref}$$

$$\frac{\Gamma \Rightarrow \Delta, a \leqslant c, a \leqslant b, \quad \Gamma \Rightarrow \Delta, a \leqslant c, b \leqslant c}{\Gamma \Rightarrow \Delta, a \leqslant c}\,{\scriptstyle Trans}$$

Term b in rule *Trans* is called a *middle term.*

Theorem *All terms in a minimum-size derivation of* $\Gamma \Rightarrow \Delta$ *in the right theory of linear order are terms in* Γ, Δ.

Corollary *The quantifier-free theory of linear order is decidable.*

Proof: Application of rule *Trans* root first with middle terms chosen from the conclusion can produce only a bounded number of distinct atoms in the premisses. Whenever a duplication is produced, proof search fails by the admissibility of contraction. QED.