Proof Theory for Distributed Knowledge

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Abstract. The proof theory of multi-agent epistemic logic extended with operators for distributed knowledge is studied. Distributed knowledge of A within a group G means that A follows from the totality of what the individual members of G know. There are known axiomatizations for epistemic logics with the distributed knowledge operator, but apparently no cut-free proof system for such logics has yet been presented. A Gentzen-style contraction-free sequent calculus system for propositional epistemic logic with operators for distributed knowledge is given, and a cut-elimination theorem for the system is proved. Examples of reasoning about distributed knowledge using the calculus are given.

1 Introduction

Distributed knowledge of A within a group G means that A follows from what the members of G individually know. For instance, A is distributed knowledge in group G (denoted $\mathcal{D}_G A$) consisting of three agents of which the first one knows B, the second one knows $B \supset C$, and the third one knows $B \,\&\, C \supset A$. Reasoning about the combined information possessed by different agents is an important task in systems in which all information is not available in one central source but distributed among several agents.

In such situations, epistemic logic [1] is typically used for representing and reasoning about knowledge. In the literature concerning multi-agent epistemic logics, e.g. [2, 3], operators for distributed knowledge are often included. However, these treatments usually concentrate on the model theory of the logics, whereas the proof-theoretical part is limited to providing Hilbert-style axiomatizations. Since theorem-proving is difficult in Hilbert-style systems, we shall here study Gentzen-style sequent calculi as a step towards mechanization of proof search.

One proof-theoretical approach to reasoning about distributed knowledge is given in [4], but the approach is different because of the use of natural deduction and of a language which is not epistemic. The development of a proof system for logic of distributed knowledge has been recently posed by S. Artemov as an open problem for the system of *evidence-based knowledge* (see `http://web.cs.gc.cuny.edu/~sartemov/research_problems.html`). This paper presents a solution for ordinary multi-agent epistemic logic using the methods developed in [5, 6].

Techniques for drawing inferences from distributed knowledge may be useful in several application areas which attempt to combine knowledge of agents, such as cooperative problem solving, knowledge base merging, and judgement aggregation. In cooperative problem solving it is usually assumed that the agents are willing to provide any information they have and that all information is certain. In such situations, it is possible to combine the separate knowledge bases into one and then derive theorems from the large knowledge base. However, typically in knowledge base merging the data can contain errors (and is thus not strictly speaking knowledge). Then combining information from multiple sources may lead to an inconsistent knowledge base, and special methods have to be used for dealing with contradictory information (see e.g. [7–9]). That often involves discarding some information in order to maintain the integrity of the database.

In cases with heterogeneous information sources the knowledge modalities should not be understood as knowledge proper but rather as beliefs. In open information systems, and in situations involving strategic considerations, like in judgement aggregation or voting, agents can even provide false information on purpose, so it is not possible to infer their real beliefs from what they report, but the information they provide must be treated as claims, acceptances or just as messages with propositional content.

The introduction of the knowledge modalities and the modality for distributed knowledge into the language can be beneficial, because managing the meta-information concerning the sources of knowledge and their various combinations becomes easier. When the source of information is stored in addition to the content, also contradictory information can be dealt with: If agent 1 claims that A is the case and agent 2 claims that not-A is the case, the receiving agent should decide which piece of information to accept and which one to reject. However, when such a situation arises there may not be enough information available that could be used to resolve the conflict. If our logical language is rich enough to allow also knowledge propositions and the agents are able to reason about distributed knowledge, incoming information need not be discarded nor is it necessary to immediately judge some agents unreliable. Instead, we can store the knowledge claims without violating the integrity constraints, and we can use the stored information to find out which agents we can trust, possibly later when we have gathered more information.

Thus, the addition of the knowledge operators to the language makes it possible for the agents to perform reasoning about the distributed information possessed by various agents and groups of agents and to detect inconsistencies between claims made by agents. Also, the possibility to iterate knowledge (and distributed knowledge) operators allows for more complex reasoning tasks than reasoning from an integrated knowledge base without iterated modalities. This kind of reasoning may be used in cooperative information systems to find out which agents have useful information with respect to the task at hand.

In Section 2, we introduce the logical system and show that it can be used to derive the axioms given in complete axiomatizations for the logic of distributed knowledge. In Section 3, we show that the system has the required structural properties such as admissibility of structural rules, and discuss the relevance of these result for proof search. In Section 4 we present examples of derivations in our calculus and discuss possible application areas of our methods. We shall conclude in Section 5.

2 Logic of distributed knowledge

Our starting point is the modal sequent calculus system **G3K** [6]. (For a general introduction to Gentzen-style sequent calculus, see [5].) We replace the modal operator $\Box$ with the knowledge operators $\mathcal{K}_a$ for individual agents $a \in G$. We extend the logic with the operator $\mathcal{D}_G$ with the intended meaning for $\mathcal{D}_G A$ that A is distributed knowledge within the group G (sometimes, for ease of readability, the subscript G will be omitted when clear from the context).

In [6] the rules for $\Box$ are determined by the forcing relation of Kripke semantics

$$x \Vdash \Box A \text{ iff } \forall y(xRy \to y \Vdash A)$$

where x, y range in the set of possible worlds and R is the accessibility relation. In multi-agent epistemic logic there is an accessibility relation R_a for each agent a, and validity of $\mathcal{K}_a A$ is defined by

$$x \Vdash \mathcal{K}_a A \text{ iff } \forall y(xR_a y \to y \Vdash A).$$

The right to left direction of the equivalence gives the right rule of $\mathcal{K}_a$, the opposite, the left rule. We shall use colon ':' to stand for the forcing relation (so $x : A$ can be read as saying that A holds at world x). In general, sequents of the form $\Gamma \Rightarrow \Delta$ can be understood as saying that the disjunction of the formulas in the multiset Δ can be derived from the conjunction of formulas in the multiset Γ representing the open assumptions. The rules are

$$\frac{xR_a y, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathcal{K}_a A}\ {\scriptstyle R\mathcal{K}_a} \qquad \frac{y : A, x : \mathcal{K}_a A, xR_a y, \Gamma \Rightarrow \Delta}{x : \mathcal{K}_a A, xR_a y, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\mathcal{K}_a}$$

Rule $R\mathcal{K}_a$ has the variable condition that y must not appear in the conclusion.

Distributed knowledge is defined as follows (see e.g. [2]) w.r.t. a Kripke structure M and a world s

$$(M, s) \models \mathcal{D}_G A \text{ iff } (M, t) \models A \text{ for all } t \text{ such that } (s, t) \in \bigcap_{a \in G} R_a.$$

The rules for distributed knowledge are found accordingly

$$\frac{\{xR_a y\}_{a\in G}, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathcal{D}_G A}\ {\scriptstyle R\mathcal{D}_G} \qquad \frac{y : A, x : \mathcal{D}_G A, \{xR_a y\}_{a\in G}, \Gamma \Rightarrow \Delta}{x : \mathcal{D}_G A, \{xR_a y\}_{a\in G}, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\mathcal{D}_G}$$

Similarly to rule $R\mathcal{K}_a$, also rule $R\mathcal{D}$ has the restriction that y must not appear in the conclusion. The intended meaning of the notation $\{xR_a y\}_{a\in G}$ is that what is inside the curly brackets should be repeated for each agent $a \in G$; for instance, with a group of two agents 1 and 2, the right rule becomes

$$\frac{xR_1y, xR_2y, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathcal{D}_{\{1,2\}}A} \; {\scriptstyle R\mathcal{D}_{\{1,2\}}}$$

The rules for the calculus are given in Table 1. Observe that initial sequents are restricted to atomic formulas P. This feature, common to all **G3** systems of sequent calculus, is needed in order to ensure invertibility of the rules and other structural properties. Note also that no rules for negation nor equivalence are needed because we take $\sim A$ to be a shorthand for $A \supset \bot$ and $A \supset\subset B$ as a shorthand for $(A \supset B) \;\&\; (B \supset A)$.

Initial sequents:

$x : P, \Gamma \Rightarrow \Delta, x : P$

Propositional rules:

$$\frac{x : A, x : B, \Gamma \Rightarrow \Delta}{x : A \,\&\, B, \Gamma \Rightarrow \Delta} \; {\scriptstyle L\&} \qquad \frac{\Gamma \Rightarrow \Delta, x : A \quad \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \,\&\, B} \; {\scriptstyle R\&}$$

$$\frac{x : A, \Gamma \Rightarrow \Delta \quad x : B, \Gamma \Rightarrow \Delta}{x : A \vee B, \Gamma \Rightarrow \Delta} \; {\scriptstyle L\vee} \qquad \frac{\Gamma \Rightarrow \Delta, x : A, x : B}{\Gamma \Rightarrow \Delta, x : A \vee B} \; {\scriptstyle R\vee}$$

$$\frac{\Gamma \Rightarrow \Delta, x : A \quad x : B, \Gamma \Rightarrow \Delta}{x : A \supset B, \Gamma \Rightarrow \Delta} \; {\scriptstyle L\supset} \qquad \frac{x : A, \Gamma \Rightarrow \Delta, x : B}{\Gamma \Rightarrow \Delta, x : A \supset B} \; {\scriptstyle R\supset}$$

$$\frac{}{x : \bot, \Gamma \Rightarrow \Delta} \; {\scriptstyle L\bot}$$

Modal rules:

$$\frac{y : A, x : \mathcal{K}_a A, xR_ay, \Gamma \Rightarrow \Delta}{x : \mathcal{K}_a A, xR_ay, \Gamma \Rightarrow \Delta} \; {\scriptstyle L\mathcal{K}_a} \qquad \frac{xR_ay, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathcal{K}_a A} \; {\scriptstyle R\mathcal{K}_a}$$

$$\frac{y : A, x : \mathcal{D}_G A, \{xR_ay\}_{a \in G}, \Gamma \Rightarrow \Delta}{x : \mathcal{D}_G A, \{xR_ay\}_{a \in G}, \Gamma \Rightarrow \Delta} \; {\scriptstyle L\mathcal{D}_G} \qquad \frac{\{xR_ay\}_{a \in G}, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathcal{D}_G A} \; {\scriptstyle R\mathcal{D}_G}$$

Table 1. System $\mathbf{G3KE}_D$

In addition to these rules, the properties of the agents' accessibility relations can be chosen by adding to the system suitable rules corresponding to desired properties, as explained in [6]. The common choices in the case of epistemic logic are reflexivity (which guarantees that the actual world is always taken to be epistemically possible so that nothing false can be known) and transitivity (which gives the property of positive introspection: if an agent knows something then she knows that she knows). These together yield **S4**, or **G3S4** in the terminology of [6]. In the case of doxastic logic, that is, the logic of belief, reflexivity is abandoned to allow the possibility of false beliefs. Sometimes also symmetry (which together with transitivity gives negative introspection: if an agent does not know something, she knows that she does not know it) is added, in which case the accessibility relations are equivalence relations and the system is known as **S5** or **G3S5**. The rules corresponding to reflexivity, transitivity and symmetry for agent a are, respectively

$$\frac{xR_ax, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ {\scriptstyle Ref_a} \quad \frac{xR_az, xR_ay, yR_az, \Gamma \Rightarrow \Delta}{xR_ay, yR_az, \Gamma \Rightarrow \Delta}\ {\scriptstyle Trans_a} \quad \frac{yR_ax, xR_ay, \Gamma \Rightarrow \Delta}{xR_ay, \Gamma \Rightarrow \Delta}\ {\scriptstyle Sym_a}$$

Observe that the rules have active and principal formulas in the antecedents of sequents, so they correspond to implication from the atoms in the conclusion to those in premisses.

Our system is modular in the sense that one need not be committed to a particular set of properties for the accessibility relations but the results given in this paper hold for accessibility relations with any combinations of these properties. Also other properties can be used as explained in [6]. It is also possible to have several modalities in one system without losing the good structural properties of the system. For example, knowledge and belief can be treated simultaneously by adding suitable rules for the belief operators and the doxastic accessibility relation for each agent. The relationship between modalities may require new rules, like in this case a rule for ensuring that the doxastic accessibility relation is included in the epistemic accessibility relation corresponding to the idea that knowledge entails belief. Temporal modalities can be added in a similar fashion. In the examples presented in this paper, we shall not combine different information attitudes so we can just use one type of modal operator $\mathcal{K}_a$ (specific to each agent a) to stand for whichever modality is appropriate in the situation. Similarly, the operator $\mathcal{D}_G$ is taken to mean a distributed version of the $\mathcal{K}$-modality, be it knowledge, belief, or something else.

As shown in [6], the standard axiomatic sequents $x : A, \Gamma \Rightarrow \Delta, x : A$, for arbitrary, not just atomic, A, and the characteristic axioms of the standard modal logics are derivable in the respective sequent calculus systems, and the necessitation rule is admissible. These results extend to multi-agent epistemic logic with knowledge generalization rules for each agent $a \in G$

$$\frac{\Rightarrow x : A}{\Rightarrow x : K_aA}$$

The addition of distributed knowledge operator requires new axioms. A sound and complete axiomatization for epistemic logic S5 with distributed knowledge is provided in [10], [11], and in [3]. The axioms to be added to standard axiomatizations of epistemic logics would be the following:

$$\mathcal{K}_aA \supset \mathcal{D}_GA, \text{ for each agent } a \in G \tag{1}$$

and

$$(\mathcal{D}_GA \mathbin{\&} \mathcal{D}_G(A \supset B)) \supset \mathcal{D}_GB. \tag{2}$$

In order to demonstrate the completeness of our calculus, we first show that these axioms are derivable.

Proposition 1. *$\mathcal{K}_a A \supset \mathcal{D}_G A$ is derivable for each agent $a \in G$ in* ***G3KE***$_D$.

Proof. For each agent $a \in G$, the derivation goes as follows:

$$\cfrac{\cfrac{\cfrac{y : A, \{xR_a y\}_{a \in G}, x : \mathcal{K}_a A \Rightarrow y : A}{\{xR_a y\}_{a \in G}, x : \mathcal{K}_a A \Rightarrow y : A}\ {\scriptstyle L\mathcal{K}_a}}{x : \mathcal{K}_a A \Rightarrow x : \mathcal{D}_G A}\ {\scriptstyle R\mathcal{D}_G}}{\Rightarrow x : \mathcal{K}_a A \supset \mathcal{D}_G A}\ {\scriptstyle R\supset}$$

where the uppermost sequent is derivable.

Proposition 2. *$(\mathcal{D}_G A \,\&\, \mathcal{D}_G(A \supset B)) \supset \mathcal{D}_G B$ is derivable in* ***G3KE***$_D$.

Proof. The derivation is:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{y : A, \ldots \Rightarrow y : B, y : A \qquad y : B, \ldots \Rightarrow y : B}{y : A \supset B, y : A, \{xR_a y\}_{a \in G}, x : \mathcal{D}_G A, x : \mathcal{D}_G(A \supset B) \Rightarrow y : B}\ {\scriptstyle L\supset}}{y : A, \{xR_a y\}_{a \in G}, x : \mathcal{D}_G A, x : \mathcal{D}_G(A \supset B) \Rightarrow y : B}\ {\scriptstyle L\mathcal{D}_G}}{\{xR_a y\}_{a \in G}, x : \mathcal{D}_G A, x : \mathcal{D}_G(A \supset B) \Rightarrow y : B}\ {\scriptstyle L\mathcal{D}_G}}{x : \mathcal{D}_G A, x : \mathcal{D}_G(A \supset B) \Rightarrow x : \mathcal{D}_G B}\ {\scriptstyle R\mathcal{D}_G}}{x : \mathcal{D}_G A \,\&\, \mathcal{D}_G(A \supset B) \Rightarrow x : \mathcal{D}_G B}\ {\scriptstyle L\&}}{\Rightarrow x : (\mathcal{D}_G A \,\&\, \mathcal{D}_G(A \supset B)) \supset \mathcal{D}_G B}\ {\scriptstyle R\supset}$$

where the uppermost sequents are derivable.

Completeness with respect to the mentioned Hilbert-style system further requires closure under modus ponens and under the necessitation rules for epistemic operators. These properties are shown in the following section.

In some applications it is useful to be able to reason about shared knowledge, that is, something that all the agents know. It is straightforward to add to the calculus an operator $\mathcal{E}$ for shared knowledge: Since $\mathcal{E}_G A$ means that everyone in group G knows that A, it can be used as a short-hand expression for the conjunction $\mathcal{K}_{a_1} A \,\&\, \ldots \,\&\, \mathcal{K}_{a_n} A$ where $G = \{a_1, \ldots, a_n\}$. The right and left rules for shared knowledge are thus not required for the calculus but can be derived from the K_i rules. These are as follows

$$\cfrac{\{xR_a y_a, \Gamma \Rightarrow \Delta, y_a : A\}_{a \in G}}{\Gamma \Rightarrow \Delta, x : \mathcal{E}_G A}\ {\scriptstyle R\mathcal{E}_G} \qquad \cfrac{y_a : A, x : \mathcal{E}_G A, xR_a y_a, \Gamma \Rightarrow \Delta}{x : \mathcal{E}_G A, xR_a y_a, \Gamma \Rightarrow \Delta}\ {\scriptstyle L\mathcal{E}_G \quad (a \in G)}$$

with the variable condition in $R\mathcal{E}_G$ that no variable y_a appears in the conclusion.

3 Structural properties

We shall now proceed with the structural properties of our system. The use of variables referring to possible worlds requires that we define substitution and prove a substitution lemma as in [6]. Substitution is defined as follows:

$xR_ay(z/w) \equiv xR_ay$ if $w \neq x$ and $w \neq y$,
$xR_ay(z/x) \equiv zR_ay$ if $x \neq y$,
$xR_ay(z/y) \equiv xR_az$ if $x \neq y$,
$xR_ax(z/x) \equiv zR_az$,
$x : A(z/y) \equiv x : A$ if $x \neq y$,
$x : A(z/x) \equiv z : A$

for all $a \in G$. Extension to multisets is obvious.

Lemma 1 (Substitution lemma). *If $\Gamma \Rightarrow \Delta$ is derivable in* $\boldsymbol{G3KE}_D$*, then also $\Gamma(y/x) \Rightarrow \Delta(y/x)$ is derivable, with the same derivation height.*

Proof. The proof is by induction on the height n of the derivation of $\Gamma \Rightarrow \Delta$ as in [6]. If $n = 0$ and the substitution y/x is not vacuous, the sequent $\Gamma \Rightarrow \Delta$ is either an initial sequent or conclusion of $L\bot$. In either case $\Gamma(y/x) \Rightarrow \Delta(y/x)$ is also an initial sequent of the same form or conclusion of $L\bot$. Suppose then that the claim holds for derivations of length n and consider the last rule applied in the derivation. If the last rule is a propositional rule or a modal rule without variable conditions, apply the inductive hypothesis to the premisses and then apply the rule. If the last rule is a rule with a variable condition ($R\mathcal{K}_a$ or $R\mathcal{D}_G$), we must be careful with the the cases in which either x or y is the eigenvariable of the rule, because a straightforward substitution may result in a violation of the restriction. In those cases we must apply the inductive hypothesis to the premiss and replace the eigenvariable with a fresh variable that does not appear in the derivation. The details are omitted here but similar cases are considered in [6, Lemma 4.3].

Theorem 1 (Height-preserving weakening). *The rules of weakening*

$$\frac{\Gamma \Rightarrow \Delta}{x : A, \Gamma \Rightarrow \Delta}\ LW \qquad \frac{\Gamma \Rightarrow \Delta}{xR_ay, \Gamma \Rightarrow \Delta}\ LW_{R_a} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x : A}\ RW$$

are height-preserving admissible in $\boldsymbol{G3KE}_D$.

Proof. The proof is by induction on the height of the derivation of the premiss. The cases with propositional rules and the modal and nonlogical rules without variable conditions are straightforward. As in [6], if the last step is a rule with a variable condition ($R\mathcal{K}_a$ or $R\mathcal{D}$), we need to apply the substitution lemma to the premisses of the rule in order to avoid a clash with the variables in $x : A$ or xR_ay. The conclusion is then obtained by applying the inductive hypothesis and the rule in question.

Theorem 2. *The necessitation rules*

$$\frac{\Rightarrow x : A}{\Rightarrow x : \mathcal{K}_a A} \qquad \frac{\Rightarrow x : A}{\Rightarrow x : \mathcal{D}_G A}$$

are admissible in $\boldsymbol{G3KE}_D$.

Proof. Suppose we have a derivation of $\Rightarrow x : A$. By the substitution lemma we obtain a derivation of $\Rightarrow y : A$ and, by admissibility of weakening, of $xR_a y \Rightarrow y : A$, and $\{xR_a y\}_{a \in G} \Rightarrow y : A$. By $R\mathcal{K}_a$ and $R\mathcal{D}$, respectively, we have $\Rightarrow x : \mathcal{K}_a A$ and $\Rightarrow x : \mathcal{D}_G A$.

Theorem 3. *The rules of **G3KE**$_D$ are height-preserving invertible.*

Proof. For the propositional rules, the proof is exactly as the proof of height-preserving invertibility of the rules of **G3c** in [5, Theorem 3.1.1]. For the $\mathcal{K}$-rules and rules for the accessibility relations, the proof is similar to [6, Proposition 4.11]. Invertibility of LD_G is immediate because the premiss can be obtained from the conclusion by (height-preserving) weakening.

Invertibility of $R\mathcal{D}_G$ is proved by induction on the height n of the derivation of the conclusion $\Gamma \Rightarrow \Delta, x : \mathcal{D}_G A$. If $n = 0$, it is an axiom or conclusion of $L\bot$ and so is the premiss $\{xR_a y\}_{a \in G}, \Gamma \Rightarrow \Delta, y : A$. If $n > 0$ and $\Gamma \Rightarrow \Delta, x : \mathcal{D}_G A$ is concluded by a rule other than $R\mathcal{K}_a$ or $R\mathcal{D}_G$ (which have a variable condition), we apply the inductive hypothesis to the premiss(es) and the rule. If the rule is $R\mathcal{K}_a$, we have a derivation ending with

$$\frac{xR_a w, \Gamma \Rightarrow \Delta, x : \mathcal{D}_G A, w : A}{\Gamma \Rightarrow \Delta, x : \mathcal{D}_G A, x : \mathcal{K}_a A} \; {\scriptstyle R\mathcal{K}_a}$$

We can assume that the eigenvariable w is different from y, otherwise we can apply the substitution lemma. Now the inductive hypothesis applied to the premiss gives a derivation of the same height ending with

$$\frac{\{xR_a y\}_{a \in G}, xR_a w, \Gamma \Rightarrow \Delta, w : A, y : A}{\{xR_a y\}_{a \in G}, \Gamma \Rightarrow \Delta, x : \mathcal{K}_a A, y : A} \; {\scriptstyle R\mathcal{K}_a}$$

The case in which the conclusion was derived using $R\mathcal{D}_G$ and the principal formula is in Δ is similar. In case the principal formula was $x : \mathcal{D}_G A$ itself, the premiss is already the sequent we wanted to prove derivable, except for the possibly different eigenvariable, which can be changed by height-preserving substitution.

Invertibility of the rules is useful for theoretical purposes because it simplifies some other proofs but it is crucial for the practical reason that root-first proof search requires no backtracking mechanism if all the rules are invertible: If we find out in our current branch of a proof tree that a sequent is not derivable we can immediately see that the proof search has failed and can be terminated because, by invertibility, the conclusion cannot be derivable either.

Theorem 4. *The rules of contraction*

$$\frac{x : A, x : A, \Gamma \Rightarrow \Delta}{x : A, \Gamma \Rightarrow \Delta} \; {\scriptstyle LCtr} \quad \frac{xR_a y, xR_a y, \Gamma \Rightarrow \Delta}{xR_a y, \Gamma \Rightarrow \Delta} \; {\scriptstyle LCtr_{R_a}} \quad \frac{\Gamma \Rightarrow \Delta, x : A, x : A}{\Gamma \Rightarrow \Delta, x : A} \; {\scriptstyle RCtr}$$

*are height-preserving admissible in **G3KE**$_D$.*

Proof. By simultaneous induction on the height of derivation for left and right contractions. In the base case, observe that an initial sequent stays initial if two occurrences of a formula are contracted into one. For the inductive step, two cases are distinguished: The case with none of the contraction formulas principal in the last rule, and the case with one principal. In the former, apply inductive hypothesis to the premiss of the rule, then the rule. In the latter, apply the matching height-preserving inversion to the premiss(es) of the rule, the inductive hypothesis, and the rule.

Also admissibility of contraction is useful for the practical reason that it guarantees that we need not multiply formulas in sequents during the proof search. If a sequent can be derived using contraction, it can be derived without using it. In addition, height-preserving admissibility of contraction permits the restriction of the search space also with respect to other rules: Whenever application of a rule, root-first, produces a duplication, by height-preserving admissibility of contraction the conclusion of the rule can be obtained in one step less. The possible applicable rule can thus be discarded if we reasonably assume that the derivation we are looking for is a minimal one, i.e. one that does not admit any local shortening through the elimination of contraction steps.

Theorem 5. *The cut rule*

$$\dfrac{\Gamma \Rightarrow \Delta, C \quad C, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ {\scriptstyle Cut}$$

is admissible in $\boldsymbol{G3KE}_D$.

Proof. The proof proceeds by induction on the structure of the cut formula C with subinduction on the *cut-height*, that is, the sum of the heights of the derivations of the premisses. The proof is to a large extent similar to the cut-elimination proofs in [5] (e.g. Theorem 3.2.3) so we shall consider in detail only the case in which the cut formula is $\mathcal{D}_G A$ and is principal in both premisses:

$$\dfrac{\dfrac{\{xR_a y\}_{a\in G}, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathcal{D}_G A}\ {\scriptstyle R\mathcal{D}} \qquad \dfrac{z : A, x : \mathcal{D}_G A, \{xR_a z\}_{a\in G}, \Gamma' \Rightarrow \Delta'}{x : \mathcal{D}_G A, \{xR_a z\}_{a\in G}, \Gamma' \Rightarrow \Delta'}\ {\scriptstyle L\mathcal{D}}}{\Gamma, \{xR_a z\}_{a\in G}, \Gamma' \Rightarrow \Delta, \Delta'}\ {\scriptstyle Cut}$$

Let n be the height of the derivation of the left premiss and m the height of the second. Then the cut-height is $n+1+m+1$. This derivation can be transformed into the following:

$$\dfrac{\dfrac{\dfrac{\{xR_a y\}_{a\in G}, \Gamma \Rightarrow \Delta, y : A}{\{xR_a z\}_{a\in G}, \Gamma \Rightarrow \Delta, z : A}\ {\scriptstyle Subst} \quad \dfrac{\dfrac{\{xR_a y\}_{a\in G}, \Gamma \Rightarrow \Delta, y : A}{\Gamma \Rightarrow \Delta, x : \mathcal{D}_G A}\ {\scriptstyle R\mathcal{D}} \quad z : A, x : \mathcal{D}_G A, \{xR_a z\}_{a\in G}, \Gamma' \Rightarrow \Delta'}{z : A, \Gamma, \{xR_a z\}_{a\in G}, \Gamma' \Rightarrow \Delta, \Delta'}\ {\scriptstyle Cut}}{\{xR_a z\}_{a\in G}, \Gamma, \Gamma, \{xR_a z\}_{a\in G}, \Gamma' \Rightarrow \Delta, \Delta, \Delta'}\ {\scriptstyle Cut}}{\Gamma, \{xR_a z\}_{a\in G}, \Gamma' \Rightarrow \Delta, \Delta'}\ {\scriptstyle Ctr^*}$$

Note that the height-preserving substitution in the derivation of the left premiss of the second cut has no effect on Γ or Δ because, by the variable restriction of rule $R\mathcal{D}$ used in the original derivation, y does not appear free in Γ or Δ. The derivation has two cuts, the first of which has lower height and the second smaller size of the cut formula.

As a consequence of admissibility of cut, it follows that our system is closed under modus ponens, and therefore it is complete with respect to the known Hilbert-type systems for the logic of distributed knowledge.

In [3], an alternative Hilbert-type system is presented; the system is obtained by adding to the standard axiomatizations of **T**, **S4**, or **S5**, the rule

$$\frac{A_1\&\dots\&A_m \supset B}{\mathcal{K}_{a_1}A_1\&\dots\&\mathcal{K}_{a_m}A_m \supset \mathcal{D}B}$$

where $a_1,\dots,a_m$ are the agents in G. The rule is shown admissible in our system as follows:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Rightarrow x : A_1\&\dots\&A_m \supset B}{x : A_1,\dots,x : A_m \Rightarrow x : B}\,{\scriptstyle L\&\text{-}Inv,R\supset\text{-}Inv}}{y : A_1,\dots,y : A_m \Rightarrow y : B}\,{\scriptstyle Subst}}{\{xR_{a_i}y\}_{a_i\in G}, y : A_1,\dots,y : A_m, x : \mathcal{K}_{a_1}A_1,\dots,x : \mathcal{K}_{a_m}A_m \Rightarrow y : B}\,{\scriptstyle LW^*}}{\{xR_{a_i}y\}_{a_i\in G}, x : \mathcal{K}_{a_1}A_1,\dots,x : \mathcal{K}_{a_m}A_m \Rightarrow y : B}\,{\scriptstyle L\mathcal{K}_{a_1},\dots,L\mathcal{K}_{a_m}}}{x : \mathcal{K}_{a_1}A_1,\dots,x : \mathcal{K}_{a_m}A_m \Rightarrow x : \mathcal{D}B}\,{\scriptstyle R\mathcal{D}}}{x : \mathcal{K}_{a_1}A_1\&\dots\&\mathcal{K}_{a_m}A_m \Rightarrow x : \mathcal{D}B}\,{\scriptstyle L\&^*}}{x : \mathcal{K}_{a_1}A_1\&\dots\&\mathcal{K}_{a_m}A_m \supset \mathcal{D}B}\,{\scriptstyle R\supset}$$

where *L&-Inv*, *R⊃-Inv* denote the (admissible) invertibilities of $L\&$ and $R\supset$, respectively, *Subst* the admissible rule of substitution, and the asterisk indicates possibly repeated applications of a rule.

Admissibility of cut is crucial for delimiting the space of proof search, because it guarantees that no arbitrary new formulas need to be constructed during the search. However, our system does not enjoy a full subformula property because some rules remove atoms, but a *weak form of subformula property*, that is, all formulas in a derivation are either subformulas of (formulas in) the endsequent or atomic formulas of the form xRy. By considering minimal derivations, that is, derivations in which shortenings are not possible, the weak subformula property can be strengthened by restricting the labels that can appear in the relational atoms to those in the conclusion or to eigenvariables (*subterm property*). This property, together with height-preserving admissibility of contraction, ensures the consequences of the full subformula property and has been used for establishing decidability through terminating proof search for the system **G3K** and several extensions in [6]. The proofs are involved so we shall not consider the issue for $\mathbf{G3KE}_D$ here but leave it to future work. However, we do not expect problems from the addition of the rules for distributed knowledge.

4 Examples

As a simple example of reasoning about distributed knowledge, consider the case of three agents mentioned in the beginning of the article. Suppose we have encoded the initial situation to a knowledge base *KB* so that it consists of the formulas $x : \mathcal{K}_1 B, x : \mathcal{K}_2(B \supset C), x : \mathcal{K}_3(B \mathbin{\&} C \supset A)$. We can now ask whether $x : \mathcal{D}_{\{1,2,3\}} A$ can be derived from the knowledge base. (We shall use $\mathcal{D}$ instead of $\mathcal{D}_{\{1,2,3\}}$ for clarity). Proceeding in a root-first fashion starting from the bottom we get the following derivation in which the uppermost sequents have the same formula on both sides of the sequent arrow and are thus clearly derivable:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\ldots, y : B, \ldots \Rightarrow y : A, y : B \qquad \dfrac{\dfrac{y : C, y : B \ldots \Rightarrow \ldots, y : B \quad y : C \ldots \Rightarrow \ldots, y : C}{y : C, y : B \ldots \Rightarrow y : A, y : B \mathbin{\&} C}{\scriptstyle R\&} \quad y : A, y : C, \ldots \Rightarrow y : A}{y : C, y : B \mathbin{\&} C \supset A, y : B, \ldots \Rightarrow y : A}{\scriptstyle L\supset}}{y : B \mathbin{\&} C \supset A, y : B \supset C, y : B, \ldots \Rightarrow y : A}{\scriptstyle L\supset}}{y : B \supset C, y : B, xR_1y, xR_2y, xR_3y, x : K_1B, x : \mathcal{K}_2(B \supset C), x : \mathcal{K}_3(B \mathbin{\&} C \supset A) \Rightarrow y : A}{\scriptstyle L\mathcal{K}_3}}{y : B, xR_1y, xR_2y, xR_3y, x : K_1B, x : \mathcal{K}_2(B \supset C), x : \mathcal{K}_3(B \mathbin{\&} C \supset A) \Rightarrow y : A}{\scriptstyle L\mathcal{K}_2}}{xR_1y, xR_2y, xR_3y, x : K_1B, x : \mathcal{K}_2(B \supset C), x : \mathcal{K}_3(B \mathbin{\&} C \supset A) \Rightarrow y : A}{\scriptstyle L\mathcal{K}_1}}{x : K_1B, x : \mathcal{K}_2(B \supset C), x : \mathcal{K}_3(B \mathbin{\&} C \supset A) \Rightarrow x : \mathcal{D}A}\;{\scriptstyle R\mathcal{D}}
$$

Next we shall consider the derivation of the birthday case mentioned e.g. in [12]: In any group it is distributed knowledge whether two agents have the same birthday. The assumption is, of course, that everyone knows one's own birthday. Should these pieces of individual knowledge be combined, it would be easy to verify whether the birthdays of any two agents are identical. We shall here consider only a group of two people but the extension would be straightforward.

Take $P(i,t)$ to be the proposition that agent i's birthday is t and consider the proposition that it is distributed knowledge in group G whether two agents in G have the same birthday. This could be expressed as follows:

$$\mathcal{D}_G \exists i, j, t\ (P(i,t) \mathbin{\&} P(j,t) \mathbin{\&} i \neq j) \vee \mathcal{D}_G \sim\!\exists i, j, t\ (P(i,t) \mathbin{\&} P(j,t) \mathbin{\&} i \neq j).$$

Note that although we have been concerned with propositional epistemic logic, we shall here use first-order notation for ease of exposition. The example can be cast in propositional logic as well, for example, by using standard propositionalization techniques, see e.g. [13, pages 274–275]. To avoid having to use a large number of propositional symbols, we shall use notation that looks like first-order notation, but we shall assume that everything is encoded in propositional logic. We can thus suppose that our knowledge base *KB* consists of sentences such as the following: $x : P(i,t) \supset \mathcal{K}_i P(i,t)$ for all agents i and all dates t. In propositional logic, this would look something like:

$$\mathit{Birthday\_Agent1\_January01} \supset \mathcal{K}_1(\mathit{Birthday\_Agent1\_January01}) \text{ etc.}$$

KB now encodes the assumptions that each agent, here 1 and 2, knows one's own birthday.

Instead of putting the assumptions in *KB* to the left hand side of the sequent arrow as in the previous example, we shall here employ the method of converting axioms to rules as in [14, 5]. Thus, each axiom in KB will be replaced by a corresponding rule of the form:

$$\frac{x : \mathcal{K}_i P(i,t), x : P(i,t), \Gamma \Rightarrow \Delta}{x : P(i,t), \Gamma \Rightarrow \Delta}\ {\scriptstyle KB_{i,t}}$$

Observe that the addition of a rule of this form maintains all the structural properties of the system: Admissibility of contraction is guaranteed by the repetition of the principal formula $x : P(i,t)$ in the premiss. Further, there is no interference with the process of cut elimination because the principal formula $x : P(i,t)$ is atomic. Also invertibility of all the rules is for the same reason unaffected by the addition.

To make the example manageable, we shall prove only the part saying that if two agents have the same birthday, it will be distributed knowledge that they have the same birthday. (Proving the other part saying that if their birthdays differ, they will know that they differ, would require stating in addition that each agent can only have one birthday.) By making a further simplification and treating t as a constant here, we can express the claim as follows:

$$P(1,t) \,\&\, P(2,t) \supset \mathcal{D}_{\{1,2\}}(P(1,t) \,\&\, P(2,t)).$$

Without this simplification we would have to use the more complete expression, that is, a conjunction of all implications of the above kind for all possible values of t.

We get root-first the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{y : P(1,t), x : \mathcal{K}_1 P(1,t), xR_1y, xR_2y, x : P(1,t), x : P(2,t) \Rightarrow y : P(1,t)}\ {\scriptstyle Ax}}{x : \mathcal{K}_1 P(1,t), xR_1y, xR_2y, x : P(1,t), x : P(2,t) \Rightarrow y : P(1,t)}\ {\scriptstyle L\mathcal{K}_1}}{xR_1y, xR_2y, x : P(1,t), x : P(2,t) \Rightarrow y : P(1,t)}\ {\scriptstyle KB_1}}{xR_1y, xR_2y, x : P(1,t) \,\&\, P(2,t) \Rightarrow y : P(1,t)}\ {\scriptstyle L\&} \qquad \vdots\ (\text{similarly})}{xR_1y, xR_2y, x : P(1,t) \,\&\, P(2,t) \Rightarrow y : P(1,t) \,\&\, P(2,t)}\ {\scriptstyle R\&}}{x : P(1,t) \,\&\, P(2,t) \Rightarrow x : \mathcal{D}_{\{1,2\}}(P(1,t) \,\&\, P(2,t))}\ {\scriptstyle R\mathcal{D}}}{\Rightarrow x : P(1,t) \,\&\, P(2,t) \supset \mathcal{D}_{\{1,2\}}(P(1,t) \,\&\, P(2,t))}\ {\scriptstyle R\supset}$$

Consider next an example in which agents possess information that put together leads to a contradiction. Suppose that agent 1 knows that $A \supset B$ and $B \supset C$. She then receives a message from agent 2 claiming that A and a message from agent 3 claiming that $\sim B \,\&\, (A \vee C)$. Agent 1 stores the information so her knowledge base KB contains the following formulas: $x : \mathcal{K}_1(A \supset B)$, $x : \mathcal{K}_1(B \supset C)$, $x : \mathcal{K}_2 A, x : \mathcal{K}_3(\sim B \,\&\, (A \vee C))$. To find out whether all incoming information can be safely believed, agent 1 should check whether the believed contents lead to a contradiction, as it indeed does:

$$
\dfrac{y : A, \ldots \Rightarrow y : \bot, y : A \qquad \dfrac{\dfrac{\dfrac{y : B, \ldots \Rightarrow y : \bot, y : B \qquad \dfrac{}{y : \bot, \ldots \Rightarrow y : \bot}\,{\scriptstyle L\bot}}{y : B \supset \bot, y : A \vee C, y : B, y : A, xR_1y, xR_2y, xR_3y, KB \Rightarrow y : \bot}\,{\scriptstyle L\supset}}{y : (B \supset \bot)\ \&\ (A \vee C), y : B, y : A, xR_1y, xR_2y, xR_3y, KB \Rightarrow y : \bot}\,{\scriptstyle L\&}}{y : B, y : A, xR_1y, xR_2y, xR_3y, KB \Rightarrow y : \bot}\,{\scriptstyle L\mathcal{K}_3}}{\dfrac{\dfrac{\dfrac{y : A \supset B, y : A, xR_1y, xR_2y, xR_3y, KB \Rightarrow y : \bot}{y : A, xR_1y, xR_2y, xR_3y, KB \Rightarrow y : \bot}\,{\scriptstyle L\mathcal{K}_1}}{xR_1y, xR_2y, xR_3y, KB \Rightarrow y : \bot}\,{\scriptstyle L\mathcal{K}_2}}{KB \Rightarrow x : \mathcal{D}_{\{1,2,3\}}\bot}\,{\scriptstyle R\mathcal{D}_{\{1,2,3\}}}}\,{\scriptstyle L\supset}
$$

Since combining all information leads to contradiction, agent 1 must find a subset of agents such that contradiction cannot be inferred. In this particular case she may decide that either 2 or 3 is less reliable than the others, or she may even decide that her own previous beliefs should be revised in light of the new information provided by 2 and 3. Supposing that she decides to drop the information provided by agent 3, she should then check that contradiction cannot be derived from the combined knowledge of 1 and 2 as follows:

$$
\dfrac{y : A, \ldots \Rightarrow y : \bot, y : A \qquad \dfrac{\dfrac{y : B, \ldots \Rightarrow y : \bot, y : B \qquad y : C, y : B, y : A, xR_1y, xR_2y, KB \Rightarrow y : \bot}{y : B \supset C, y : B, y : A, xR_1y, xR_2y, KB \Rightarrow y : \bot}\,{\scriptstyle L\supset}}{y : B, y : A, xR_1y, xR_2y, KB \Rightarrow y : \bot}\,{\scriptstyle L\mathcal{K}_1}}{\dfrac{\dfrac{\dfrac{y : A \supset B, y : A, xR_1y, xR_2y, KB \Rightarrow y : \bot}{y : A, xR_1y, xR_2y, KB \Rightarrow y : \bot}\,{\scriptstyle L\mathcal{K}_1}}{xR_1y, xR_2y, KB \Rightarrow y : \bot}\,{\scriptstyle L\mathcal{K}_2}}{KB \Rightarrow x : \mathcal{D}_{\{1,2\}}\bot}\,{\scriptstyle R\mathcal{D}_{\{1,2\}}}}\,{\scriptstyle L\supset}
$$

The uppermost premiss on the right hand side is not derivable: It is possible to continue the derivation by re-applying the left knowledge rules applied to formulas in KB or using the reflexivity and symmetry (and later transitivity) rules for the accessibility relation. After that left knowledge rules can also be applied with the expression xR_1x (or xR_2x) to yield $x : A$, $x : B$ and $x : C$ on the left hand side. Eventually only duplicates of existing formulas will be produced and the search can be terminated.

Agent 1 can now conclude that it is safe to reason about distributed knowledge among herself and agent 2 (because not everything can be inferred). Then she can find out, for instance, that together they can conclude that C holds:

$$
\dfrac{y : A, \ldots \Rightarrow y : C, y : A \qquad \dfrac{\dfrac{y : B, \ldots \Rightarrow y : C, y : B \qquad y : C, y : B, y : A, xR_1y, xR_2y, KB \Rightarrow y : C}{y : B \supset C, y : B, y : A, xR_1y, xR_2y, KB \Rightarrow y : C}\,{\scriptstyle L\supset}}{y : B, y : A, xR_1y, xR_2y, KB \Rightarrow y : C}\,{\scriptstyle L\mathcal{K}_1}}{\dfrac{\dfrac{\dfrac{y : A \supset B, y : A, xR_1y, xR_2y, KB \Rightarrow y : C}{y : A, xR_1y, xR_2y, KB \Rightarrow y : C}\,{\scriptstyle L\mathcal{K}_1}}{xR_1y, xR_2y, KB \Rightarrow y : C}\,{\scriptstyle L\mathcal{K}_2}}{KB \Rightarrow x : \mathcal{D}_{\{1,2\}}C}\,{\scriptstyle R\mathcal{D}_{\{1,2\}}}}\,{\scriptstyle L\supset}
$$

This is actually the same derivation as the previous one just with $\bot$ replaced by C, but now all the premisses are derivable.

Instead of having decided to trust agent 2, agent 1 could have decided that agent 3 is more reliable. Then she would have had to check that $\mathcal{D}_{\{1,3\}}\perp$ cannot be derived and then use the distributed knowledge between 1 and 3 as the basis of her reasoning. In general, reasoning and decision-making of an agent a can be based on the distributed knowledge $\mathcal{D}_{T_a}$ where $T_a \subseteq G$ is the set of agents currently held reliable by agent a. The choice of which agents to trust can later be retracted: If it turns out that some of the agents provide information that is clearly false, these agents can then be dropped from the subset T_a. The main point is that storing the source of information as well as the information content in the databases and using proof methods for reasoning about distributed knowledge provides a flexible way to deal with possibly erroneous multi-source information in a controlled fashion.

Note, however, that reasoning about distributed knowledge should not be seen as an alternative to existing information merging methods but rather as a tool for recognizing inconsistencies and making inferences from combined knowledge bases. This is because distributed knowledge is defined as whatever follows from the totality of what a collection of agents know. Thus, distributed knowledge does not directly support the idea of taking just one part of an agent's knowledge and reject another part that causes contradictions, as is often done in belief base merging. Reasoning about distributed knowledge requires either including everything an agent knows or excluding the agent altogether. Certainly the approach can be modified by using a more fine-grained conception of agency: Instead of labelling everything agent a has claimed under $\mathcal{K}_a$ we can use, for instance, occasion-based labels or topic-based labels, as in $\mathcal{K}_{a\_on\_Thursday}$ or $\mathcal{K}_{Berlusconi\_about\_Prodi}$. Then only certain parts of an agent's total knowledge can be taken into consideration.

Similarly, it is not possible to infer from contradictory reports that their disjunction must hold, as is done in e.g. [7]. If one witness claims that a seen car was black and another says it was red, it is often inferred that it must have been either black or red, but not white, for instance. This inference is not directly supported in our approach but must be implemented as a meta-level principle: If it is distributed knowledge within one consistent subset of agents that the car is black and within another that the car is red, we may want to conclude that it is either black or red. In a similar fashion meta-level principles are required for implementing other conflict-resolution methods like accepting a view supported by a majority of agents.

Consider then another application area, cooperative problem solving, where it is assumed that all information is correct and the agents work together to solve theoretical or practical reasoning problems. Then distributed knowledge can be used to identify the collection of agents needed to provide a solution to a problem. Suppose, for instance, that the agents are asked to find out whether B holds. Suppose that we have in our use the following sentences obtained from agents 1 and 2: $\mathcal{K}_1(A \supset\subset B)$, $\mathcal{K}_2(\mathcal{K}_3 A \vee \mathcal{K}_3 \sim A)$. We are interested in the truth of B, and the first agent knows that another proposition, A, is equivalent with B, and the second agent knows that a third agent knows whether this proposition

A holds. Now agents 1,2 and 3 distributively know whether B holds but, in fact, after getting from agent 2 information concerning agent 3's knowledge, agent 2 is not needed anymore, because it is actually distributed knowledge between 1 and 3 alone whether B is the case or not, as can be seen from the derivation below:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y : A, \dots \Rightarrow \dots, y : A \qquad y : B, \dots \Rightarrow \dots, y : B}{y : A, y : A \supset B, y : B \supset A, xR_1y, xR_3y, x : \mathcal{K}_3 A, \dots \Rightarrow x : \mathcal{D}_{\{1,3\}} {\sim}B, y : B}\,{\scriptstyle L\supset}}{y : A \supset B, y : B \supset A, xR_1y, xR_3y, x : \mathcal{K}_3 A, \dots \Rightarrow x : \mathcal{D}_{\{1,3\}} {\sim}B, y : B}\,{\scriptstyle L\mathcal{K}_3}}{y : A \supset\subset B, xR_1y, xR_3y, x : \mathcal{K}_3 A, \dots \Rightarrow x : \mathcal{D}_{\{1,3\}} {\sim}B, y : B}\,{\scriptstyle L\&}}{xR_1y, xR_3y, x : \mathcal{K}_3 A, xR_2x, x : \mathcal{K}_1(A \supset\subset B), \dots \Rightarrow x : \mathcal{D}_{\{1,3\}} {\sim}B, y : B}\,{\scriptstyle L\mathcal{K}_1}}{x : \mathcal{K}_3 A, xR_2x, x : \mathcal{K}_1(A \supset\subset B), x : \mathcal{K}_2(\mathcal{K}_3 A \vee \mathcal{K}_3 {\sim}A) \Rightarrow x : \mathcal{D}_{\{1,3\}} B, x : \mathcal{D}_{\{1,3\}} {\sim}B}\,{\scriptstyle R\mathcal{D}_{\{1,3\}}}}{x : \mathcal{K}_3 A, xR_2x, x : \mathcal{K}_1(A \supset\subset B), x : \mathcal{K}_2(\mathcal{K}_3 A \vee \mathcal{K}_3 {\sim}A) \Rightarrow x : \mathcal{D}_{\{1,3\}} B \vee \mathcal{D}_{\{1,3\}} {\sim}B}\,{\scriptstyle R\vee} \qquad \vdots}{x : \mathcal{K}_3 A \vee \mathcal{K}_3 {\sim}A, xR_2x, x : \mathcal{K}_1(A \supset\subset B), x : \mathcal{K}_2(\mathcal{K}_3 A \vee \mathcal{K}_3 {\sim}A) \Rightarrow x : \mathcal{D}_{\{1,3\}} B \vee \mathcal{D}_{\{1,3\}} {\sim}B}\,{\scriptstyle L\vee}}{xR_2x, x : \mathcal{K}_1(A \supset\subset B), x : \mathcal{K}_2(\mathcal{K}_3 A \vee \mathcal{K}_3 {\sim}A) \Rightarrow x : \mathcal{D}_{\{1,3\}} B \vee \mathcal{D}_{\{1,3\}} {\sim}B}\,{\scriptstyle L\mathcal{K}_2}}{x : \mathcal{K}_1(A \supset\subset B), x : \mathcal{K}_2(\mathcal{K}_3 A \vee \mathcal{K}_3 {\sim}A) \Rightarrow x : \mathcal{D}_{\{1,3\}} B \vee \mathcal{D}_{\{1,3\}} {\sim}B}\,{\scriptstyle Ref_2}$$

The branch marked with dots derives the sequent $x : \mathcal{K}_3 \sim A, xR_2x, x : \mathcal{K}_1(A \supset\subset B), x : \mathcal{K}_2(\mathcal{K}_3 A \vee \mathcal{K}_3 {\sim}A) \Rightarrow x : \mathcal{D}_{\{1,3\}} B \vee \mathcal{D}_{\{1,3\}} {\sim}B$, which is slightly more complicated due to the negation but is derivable as well. In this example, information provided by a collection of agents was used to find out another collection capable of providing an answer to the original query. The agents in the first group do not know the answer (it is not the case that $\mathcal{D}_{\{1,2\}} B \vee \mathcal{D}_{\{1,2\}} {\sim}B$), but they know who knows the answer (it is the case that $\mathcal{D}_{\{1,2\}}(\mathcal{D}_{\{1,3\}} B \vee \mathcal{D}_{\{1,3\}} {\sim}B)$). Although, agent 2's knowledge was crucial for finding out the set of agents needed to solve the original problem, the actual query can now be given as a task for agents 1 and 3 to cooperatively solve. Thus, distributed knowledge gives a way of reasoning about informants and their knowledge without requiring that the reasoning agent possesses the actual knowledge. It may be enough if the agent can find out who has the actual knowledge, as is often the case in real life situations.

5 Conclusions

We have here presented a sequent calculus system for formal reasoning in multi-agent epistemic logic with operators for distributed knowledge. Our system enjoys the structural properties that support proof search starting from the conclusion to be derived. Because the rules are invertible, there is no need for a backtracking mechanism, since if the conclusion is derivable also the premisses are guaranteed to be derivable. Admissibility of the contraction rules guarantees that rules that only produce duplications of the existing formulas need not be considered in the proof search. Finally, admissibility of cut is crucial for delimiting the space of the proof search, because it guarantees that no arbitrary new formulas need to be constructed during the search.

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