Conservativity of apartness over equality, revisited

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1 Introduction

In our earlier work [N] we showed that the theory of apartness is conservative over the theory of equality, where equality is defined as the negation of apartness, for the fragment of formulas where all atoms are negated. A similar result was proved for theories based on a relation of *excess* (or *positive partial order*) and theories based on a relation of weak partial oder, defined as the negation of excess.

The starting point of the conservativity result was a cut-free system for the theories of apartness and excess obtained by adding to the system **G3ip** the axioms specific to the theory under consideration in the form of *nonlogical rules*. The approach to axiomatic systems via cut-free systems with nonlogical rules was then extended in [NvP]. In the latter work it was proved that cut elimination works for the theories with nonlogical rules of a certain form, called *rule-scheme*. Having a cut-free system then permits to prove that every derivation concluding a sequent with all atoms negated in the theory of apartness (resp. in a theory of excess) can be explicitly transformed into a derivation of the same endsequent in the theory of defined equality (resp. in the corresponding theory of weak partial order).

The proof was elementary, in the sense that only simple arguments based on induction on derivation height and formula weight were used. However, as is often the case with elementary proofs, it was combinatorially rather involved.

A simplification of the proof is obtained by considering, as a logical base calculus, the calculus **G4ip** rather than the calculus **G3ip**. The former calculus has rules for implication that allow a better control on derivations. Such an approach was used in a earlier proof of conservativity. It was abandoned, however, since the proof of cut elimination for the system **G4ip** extended with nonlogical rules was still missing.

In [DN] a proof of cut elimination for **G4ip** was given. It was then extended to **G4ip** plus nonlogical rules following the rule-scheme (in particular the theories of apartness and excess) in [DN1].

Now that we have such proof of completeness we can give the details of our earlier proof of conservativity.

We refer to the earlier works for general motivations, references, definitions, and all the details which are skipped here.

2 Extension with rules for apartness

The (free variable) theory of apartness can be obtained by adding to **G3ip** the axioms of apartness in the form of sequent calculus rules. Our language is no longer based on propositional variables but on atoms of the form $a \neq b$, read as "a is *apart* from b". Equality can then be defined using $(a = b) =_{def} \neg(a \neq b)$.

In [N] it is proved that by adding to **G3ip** the sequent calculus rules

$$a \neq a, \Gamma \Rightarrow C \ \textit{irref}$$

$$\frac{a \neq b, a \neq c, \Gamma \Rightarrow C \qquad a \neq b, b \neq c, \Gamma \Rightarrow C}{a \neq b, \Gamma \Rightarrow C} \ \textit{split}$$

a calculus **G3AP** is obtained in which weakening, contraction and cut are admissible.

We can change the logical part of the calculus for the theory of apartness in favour of **G4ip** in order to obtain a calculus with better features regarding proof analysis, to be called **G4AP**. The next step is to prove that in **G4AP** contraction and cut are admissible. To this purpose, since contraction and cut have been proved to be admissible in **G3AP**, it would be enough to prove the equivalence between **G3AP** and **G4AP** that extends the equivalence between **G3ip** and **G4ip**.

The "old" proof of equivalence between **G3ip** and **G4ip** (cf. [D]) relies on the property of rules of **G4ip** of being strictly monotonic (when read from premisses to conclusion) with respect to the weight of sequents. This monotonicity is destroyed by adding a rule like *split*, where the premisses are of greater weight than the conclusion. As a consequence, the old proof of equivalence between the logical calculi cannot be extended to a proof of equivalence between **G3AP** and **G4AP**. This specific problem originally led to the demand for a direct syntactic proof of admissibility of cut for **G4ip**. Such a proof is given in [DN] for **G4**-based intuitionistic calculi and in [DN1] for **G4** extended with rule-scheme.

We recall from [DN1]:

Theorem 2.1 *The structural rules of weakening, contraction and cut are admissible in the system* **G4AP**.

It follows that all the rules of **G3AP** are admissible in **G4AP**. The converse follows from admissibility of cut in **G3AP**; so the two calculi are equivalent.

3 Conservativity revisited

In [N] a proof of conservativity for apartness over equality is given. The proof is based on the sequent calculus **G3AP**.

By using **G4AP** rather than **G3AP** an alternative, somewhat easier, proof of conservativity can be given. The proof follows the same pattern of the proof done using **G3AP**, and we just recall here the basic definitions and sketch the essential steps, with particular emphasis on the places where the proof changes, referring to [N] for the details common to the two proofs.

We say that a formula A is *negatomic* if all its atoms are negated. In particular, $\perp$ is vacuously negatomic. A sequent is negatomic if all its formulas are negatomic. Conservativity of apartness over equality here means that whenever we prove a negatomic sequent in the theory of apartness then we can prove it also in the theory of equality, where equality is defined as the negation of apartness. The usual axioms of reflexivity and transitivity for equality defined as the negation of apartness assume the form

$$\begin{array}{ll} \neg\, a \neq a & (\mathit{refl}) \\ \neg\, a \neq c \;\&\; \neg\, b \neq c \supset \neg\, a \neq b & (\mathit{trans}) \end{array}$$

By *theory of equality* **G4EQ** we refer to the theory with the above two axioms (with no further properties assumed on $\neq$) added to **G4ip** as sequents with empty antecedents plus the structural rules.

We remark that in this way we do not obtain a calculus with structural rules admissible, and we conjecture this is not possible due to the form of the second axiom, *transitivity.* But admissibility of structural rules is not needed for **G4EQ**: We only need a tight control on derivations for the source theory, not for the target theory of the conservativity theorem, so having explicit structural rules in the latter is unproblematic.

In contrast to **G3AP**, we do not have for the system **G4AP** the *weak subformula property*, stating that in a derivation only atoms and subformulas of the endsequent occur. Its consequence $(*)$ is however valid also **G4AP**, as can be seen by inspection of the rules:

$(*)$ *A derivation of a negatomic sequent* $\Gamma \Rightarrow A$ *in* **G4AP** *can only contain atomic or negatomic formulas.*

Next, given a derivation of a negatomic sequent in **G4AP** we transform it in two steps, namely:

1. **pruning** some branches;
2. performing a **permutation of rules**.

1. By property $(*)$ of derivations of negatomic sequents in **G4AP**, every occurrence of the $L0\supset$ rule has to be of the form

$$\frac{P, \bot, \Gamma \Rightarrow E}{P\supset\bot, P, \Gamma \Rightarrow E}$$

The premiss is an axiom, hence we can ignore how it has been derived, or, equally, prune the branch of the derivation above it while still maintaining a correct derivation. Therefore we can assume that no other rule appears above an occurrence of a $L0\supset$ rule. The result is what we call a *pruned derivation.*

2. The second part of the manipulation of derivations consists in "pulling down" applications of the rule *split*, by permuting them downwards in all possible cases: the conversions below show that the rule *split* permutes with $L\&$, $R\&$, $L\vee$, $R\vee$, $L\&\supset$, $L\vee\supset$, $L\supset\supset$, and with $R\supset$ in case the split formula is not active in $R\supset$. When we are dealing with a pruned derivation of a negatomic sequent these permutations suffice to ensure that all occurrences of split have $\bot$ as succedent.

Here follow the conversions for the $L\supset$ rules; all the other conversions are as in [N].
If *split* occurs above $L\&\supset$, the derivation

$$\cfrac{\cfrac{C\supset(D\supset B), a\neq b, a\neq c, \Gamma \Rightarrow E \qquad C\supset(D\supset B), a\neq b, b\neq c, \Gamma \Rightarrow E}{C\supset(D\supset B), a\neq b, \Gamma \Rightarrow E}\;\mathit{split}}{C\&D\supset B, a\neq b, \Gamma \Rightarrow E}\;L\&\supset$$

is transformed into the derivation

$$\cfrac{\cfrac{C\supset(D\supset B), a\neq b, a\neq c, \Gamma \Rightarrow E}{C\&D\supset B, a\neq b, a\neq c, \Gamma, \Rightarrow E}\;L\&\supset \qquad \cfrac{C\supset(D\supset B), a\neq b, b\neq c, \Gamma \Rightarrow E}{C\&D\supset B, a\neq b, b\neq c, \Gamma \Rightarrow E}\;L\&\supset}{C\&D\supset B, a\neq b, \Gamma \Rightarrow E}\;\mathit{split}$$

If *split* occurs above $L\vee\supset$, the derivation

$$\dfrac{\dfrac{C\supset B, D\supset B, a\neq b, a\neq c, \Gamma \Rightarrow E \qquad C\supset B, D\supset B, a\neq b, b\neq c, \Gamma \Rightarrow E}{C\supset B, D\supset B, a\neq b, \Gamma \Rightarrow E}\,\textit{split}}{C\vee D\supset B, a\neq b, \Gamma \Rightarrow E}\,{\scriptstyle L\vee\supset}$$

is transformed into the derivation

$$\dfrac{\dfrac{C\supset B, D\supset B, a\neq b, a\neq c, \Gamma \Rightarrow E}{C\vee D\supset B, a\neq b, a\neq c, \Gamma \Rightarrow E}\,{\scriptstyle L\vee\supset} \qquad \dfrac{C\supset B, D\supset B, a\neq b, b\neq c, \Gamma \Rightarrow E}{C\vee D\supset B, a\neq b, b\neq c, \Gamma \Rightarrow E}\,{\scriptstyle L\vee\supset}}{C\vee D\supset B, a\neq b, \Gamma \Rightarrow E}\,\textit{split}$$

If *split* occurs above $L\supset\supset$, since this rule is not symmetric there are two cases to be considered, according to whether *split* occurs above the left or right premiss. In the first case the derivation

$$\dfrac{\dfrac{D\supset B, a\neq b, a\neq c, \Gamma, C \Rightarrow D \quad D\supset B, a\neq b, b\neq c, \Gamma, C \Rightarrow D}{D\supset B, a\neq b, \Gamma, C \Rightarrow D}\,\textit{split} \qquad B, a\neq b, \Gamma \Rightarrow E}{(C\supset D)\supset B, a\neq b, \Gamma \Rightarrow E}\,{\scriptstyle L\supset\supset}$$

is transformed as follows: by applying weakening to the right premiss of $L\supset\supset$ we obtain $B, a\neq b, a\neq c, \Gamma \Rightarrow E$ and by $L\supset\supset$ with $D\supset B, a\neq b, a\neq c, \Gamma, C \Rightarrow D$ we get

$$(C\supset D)\supset B, a\neq b, a\neq c, \Gamma \Rightarrow E\ .$$

Similarly (but weakening instead with $b\neq c$) we obtain $(C\supset D)\supset B, a\neq b, b\neq c, \Gamma \Rightarrow E$. Application of *split* gives the conclusion.

In the second case the derivation

$$\dfrac{D\supset B, a\neq b, \Gamma, C \Rightarrow D \qquad \dfrac{B, a\neq b, a\neq c, \Gamma \Rightarrow E \quad B, a\neq b, b\neq c, \Gamma \Rightarrow E}{B, a\neq b, \Gamma \Rightarrow E}\,\textit{split}}{(C\supset D)\supset B, a\neq b, \Gamma \Rightarrow E}\,{\scriptstyle L\supset\supset}$$

is transformed as follows: by weakening, from $D\supset B, a\neq b, \Gamma, C \Rightarrow D$ we get

$$D\supset B, a\neq b, a\neq c, \Gamma, C \Rightarrow D\ ,$$

hence by $L\supset\supset$ with $B, a\neq b, a\neq c, \Gamma \Rightarrow E$ we obtain $(C\supset D)\supset B, a\neq b, a\neq c, \Gamma \Rightarrow E$. Similarly we obtain $(C\supset D)\supset B, a\neq b, b\neq c, \Gamma \Rightarrow E$ and *split* gives the conclusion.

We will call a pruned derivation, in which all occurrences of split have $\perp$ on the right of the turnstile, a derivation with *split reduction.*

We have:

Proposition 3.1 *Any negatomic sequent* $\Gamma \Rightarrow A$ *derivable in* **G4AP** *has a derivation with split reduction.*

Proof: Let $\mathcal{D}$ be a pruned derivation of $\Gamma \Rightarrow A$ and suppose that $\mathcal{D}$ contains n occurrences of the split rule with succedent different from $\perp$. We show by induction on n that $\mathcal{D}$ can be transformed into a derivation $\mathcal{D}'$ with split reduction. If $n=0$, $\mathcal{D}$ is already a derivation with split reduction. If $n>0$ we consider among the n splits with succedent different from $\perp$ one with none of these splits below it. By the above conversions, we can permute this split downwards in all possible cases, and its conclusion eventually becomes either the premiss of a $R\supset$ rule with the split formula active in it, or the premiss of a split with $\bot$ as succedent. (Observe that it cannot become the last rule in the derivation, since the endsequent is negatomic). In the former case the principal formula of $R\supset$ is of form $a \neq b \supset B$ and, by property $(*)$, B has to be $\bot$, therefore the split thus transformed has $\bot$ as succedent. Also in the latter case the conclusion of split has $\bot$ as succedent. The derivation thus obtained has $n-1$ splits with succedent different form $\bot$ and by inductive hypothesis it can be transformed into a derivation with split reduction. $\Box$

The following lemma is the core of the proof of conservativity. We denote by $a \equiv b$ the *syntactic identity* of the expressions a and b.

Lemma 3.2 *Let Γ and A be negatomic, and assume we have a derivation with split reduction in* **G4AP** *of*

$$\Gamma, a_1 \neq b_1, \ldots, a_n \neq b_n \Rightarrow A \ .$$

Then either $a_i \equiv b_i$ for some i, or we have a derivation in **G4EQ** *of*

$$\Gamma, \neg\neg a_1 \neq b_1, \ldots, \neg\neg a_n \neq b_n \Rightarrow A \ .$$

Proof: Proceed as in the proof of lemma 6.6 in [N], except for the new cases of the left implication rules.

The $L\&\supset$, $L\vee\supset$ and $L\supset\supset$ rules are dealt with easily since they do not introduce (when read bottom up) new atomic formulas in the premisses.

If the last step is $L\supset 0$ the context Γ is of the form $\Gamma', \neg(a_i \neq b_i)$ where Γ' is negatomic and i is in $\{1, \ldots, n\}$. Let us assume i equal to 1. Then this instance of the rule has the form:

$$\frac{\Gamma', a_1 \neq b_1, \bot, a_2 \neq b_2, \ldots, a_n \neq b_n \Rightarrow A}{\Gamma', \neg(a_1 \neq b_1), a_1 \neq b_1, \ldots, a_n \neq b_n \Rightarrow A}$$

The sequent $\Gamma', \neg a_1 \neq b_1, \neg\neg a_1 \neq b_1, \bot, \ldots, \neg\neg a_n \neq b_n \Rightarrow A$ is an instance of an axiom, thus it is derivable in **G4EQ**. By cutting with the derivable sequent

$$\Gamma', \neg a_1 \neq b_1, \neg\neg a_1 \neq b_1, \ldots, \neg\neg a_n \neq b_n \Rightarrow \bot$$

we obtain $\Gamma', \neg a_1 \neq b_1, \neg\neg a_1 \neq b_1, \ldots, \neg\neg a_n \neq b_n \Rightarrow A$. Thus, recalling that $\Gamma', \neg a_1 \neq b_1$ equals Γ, we have a derivation in **G4EQ** of $\Gamma, \neg\neg a_1 \neq b_1, \ldots, \neg\neg a_n \neq b_n \Rightarrow A$. $\Box$

The above proof requires additional lemmas when considering **G3ip** rather than **G4ip** as the purely logical part of the calculus. With the left implication rule of **G3ip** the left premiss of a negatomic conclusion need not be negatomic, as in

$$\frac{\neg P, \Gamma \Rightarrow P \qquad \bot, \Gamma \Rightarrow A}{\neg P, \Gamma \Rightarrow A}$$

where Γ and A are negatomic. This problem does not occur with **G4ip**, because of the repetition of the atom in the conclusion of $L0\supset$. So it is not the termination property of the calculus that is used here, but its control over the antecedent of the implication.

Then we have:

Theorem 3.3 **G4AP** *is conservative over* **G4EQ** *for negatomic sequents.*

Proof: Let $\Gamma \Rightarrow A$ be a negatomic sequent derivable in **G4AP**. Then by proposition 3.1 it has a derivation with split reduction. We prove by induction on this derivation that the sequent is also derivable in **G4EQ**.

If $\Gamma \Rightarrow A$ is an axiom it can only be a logical axiom since Γ is negatomic. So the conclusion holds.

If $\Gamma \Rightarrow A$ is derived by a &, $\vee$ or a $L\supset$ rule (the rule $L0\supset$ is excluded since Γ consists of negatomic formulas only), then the premisses are negatomic if the conclusion is. We can thus apply the inductive hypothesis to the premisses and the rule again.

The same reasoning applies to a $R\supset$ rule in case the premiss is a negatomic sequent. If it is not, the last step of the derivation has the form

$$\frac{\Gamma, a \neq b \Rightarrow \bot}{\Gamma \Rightarrow \neg a \neq b}\ .$$

By the previous lemma, either $a \equiv b$, and therefore $\Gamma \Rightarrow \neg a \neq b$ is derivable in **G4EQ**, or we have a derivation in **G4EQ** of $\Gamma, \neg\neg a \neq b \Rightarrow \bot$. From the latter we obtain the conclusion by means of logical steps.

The last rule cannot be split since Γ is negatomic, and therefore the proof is finished. $\square$

We conclude with observing that the conservativity results for the theories of order (section 7 in [N]) can be simplified along the lines shown above when **G4ip** is used in place of **G3ip** as a base logical calculus.

References

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